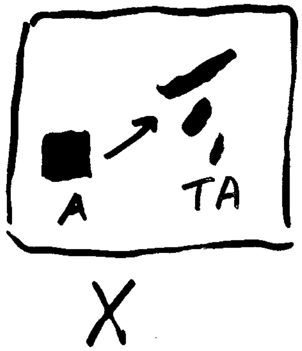
II

ERGODICITY, MIXING,

and the Long-Term Memory of DYNAMICAL SYSTEMS

Measure Preserving Systems



$(X, \mathcal{B}, \mu, T)$

X any space

$\mathcal{B}$ σ-algebra of "measurable" se

μ σ-additive measure on $\mathcal{B}$

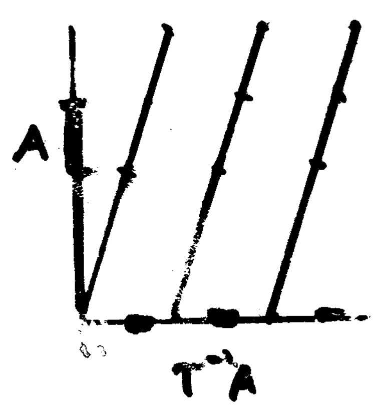
$T: X \to X$, a measure preserv map:

$$\mu(T^{-1}A) = \mu(A) \text{ for } A \in \mathcal{B}$$

Example:

$X = [0,1]$, $\mathcal{B}$ = borel sets, μ = lebesgue mea

$$Tx = 3x - [3x]$$



T need not be 1-1 in which case we may not have $\mu(TA) = \mu(A)$

We will assume that $\mu(X) = 1$

Mixing Properties of Meas. Pres. Systems

a) ERGODICITY

b) WEAK MIXING (WM)

c) (STRONG) MIXING

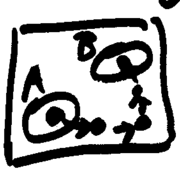
d) MIXING of all ORDERS

e) K-system

f) Bernoulli

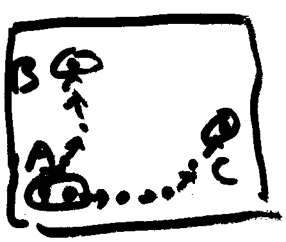
<u>Ergodicity</u>: $\forall\ A, B \in \mathcal{B}$ with $\mu(A), \mu(B) > 0$

$$\exists\, n : \mu(A \cap T^{-n} B) > 0$$



<u>Weak Mixing</u>: $\forall\ A, B, C \in \mathcal{B}$ with $\mu(A), \mu(B), \mu(C) > 0$

$$\exists\, n : \mu(A \cap T^{-n} B) > 0$$
$$\& \ \mu(A \cap T^{-n} C) > 0$$



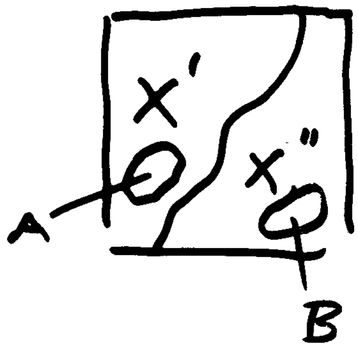
<u>Strong Mixing</u>: $\forall\ A, B \in \mathcal{B}$

$$\mu(A \cap T^{-n} B) \longrightarrow \mu(A)\mu(B)$$

asymptotic independence

Ergodicity

$(X, \mathcal{B}, \mu, T)$ is not ergodic $\Longleftrightarrow$



$X = X' \cup X''$ $\quad T: X' \to X'$

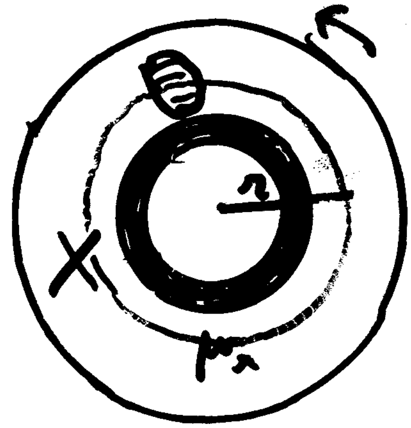
$T: X'' \to X''$

$\mu(X') > 0 \quad \mu(X'') > 0$

A permutation $\pi: (1, 2, \ldots, n) \to (1, 2, \ldots, n)$ is ergodic $\Longleftrightarrow$ it's cyclic

Every permutation can be decomposed into cyclic permutations $\quad (1\ 3\ 7)(2\ 6\ 4\ 5)$

Every meas. pres. system can be decomposed into ergodic systems



$Tz = e^{2\pi i \alpha} z$

α irrational

$\mu = \frac{1}{\pi}$ lebesgue

$\mu = \int \mu_r \, 2r\,dr \qquad d\mu_r = \frac{1}{2\pi}\, d\theta$

Let $f(x)$ be a measurable function with

$$f(x) = f(Tx)$$

then the sets $A = \{x : f(x) < t\}$

$B = \{x : f(x) \geq t\}$

give $X = A \cup B$ $\quad T(A) \subset A \quad T(B) \subset B$

Theorem: If $(X, \mathcal{B}, \mu, T)$ is ergodic, and $f(x)$ is measurable with $f(Tx) = f(x)$, then $f(x) =$ some constant a.e.

ERGODIC THEOREM for Ergodic Systems

If $(X, \mathcal{B}, \mu, T)$ is ergodic and $f \in L^1(X, \mathcal{B}, \mu)$ then

$$\frac{1}{N} \sum_{0}^{N-1} f(T^n x) \longrightarrow \int f \, d\mu$$

for a.e. x, and in L^p whenever $f \in L^p$

For, $\quad \bar{f}(x) = \lim \frac{1}{N} \sum_{0}^{N-1} f(T^n x)$ satisfies

$$\bar{f}(Tx) = \bar{f}(x), \quad \int \bar{f} \, d\mu = \int f \, d\mu$$

so $\bar{f} = \text{const} = \int \bar{f} \, d\mu = \int f \, d\mu$

<u>MIXING OF ALL ORDERS</u>:

ORDER 2: $A_0, A_1, A_2 \in \mathcal{B}$

$$\mu(A_0 \cap T^{-n_1}A_1 \cap T^{-n_1-n_2}A_2) \to \mu(A_0)\mu(A_1)\mu(A_2)$$

as $n_1, n_2 \to \infty$

ORDER 3: $A_0, A_1, A_2, A_3 \in \mathcal{B}$

$$\mu(A_0 \cap T^{-n_1}A_1 \cap T^{-n_1-n_2}A_2 \cap T^{-n_1-n_2-n_3}A_3) \to$$

$$\mu(A_0)\mu(A_1)\mu(A_2)\mu(A_3)$$

ETC.

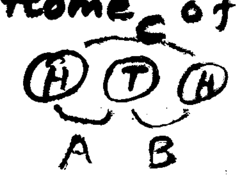
NOTE: In probability theory, we can have 3 events A, B, C with

$\mu(A \cap B) = \mu(A)\mu(B)$, $\mu(A \cap C) = \mu(A)\mu(C)$, $\mu(B \cap C) = \mu(B)\mu(C)$

but $\mu(A \cap B \cap C) \neq \mu(A)\mu(B)\mu(C)$

A, B, C pairwise independent but not independent as a triple

(the outcome of A, B affects C)



IT IS UNKNOWN WHETHER MIXING $\Rightarrow$ HIGHER ORDER MIXING !!

WE DO NOT EVEN KNOW IF MIXING $\Rightarrow$

$$\underline{\mu(A \cap T^{-n}B \cap T^{-2n}C)} \to \mu(A)\mu(B)\mu(C)$$

<u>SECOND CHARACTERIZATION of WEAK MIXING</u>

CONVERGENCE IN DENSITY:

<u>LEMMA</u>: For a bounded sequence $\{a_n\}$ and α the following are equivalent

a) $\frac{1}{N}\sum(a_n-\alpha)^2 \longrightarrow 0$

b) $\frac{1}{N}\sum|a_n-\alpha| \longrightarrow 0$

c) $\forall \varepsilon>0$ the set $\{n\in\mathbb{N} : |a_n-\alpha|>\varepsilon\}$ has density 0.

d) $\frac{1}{N}\sum a_n \longrightarrow \alpha$ <u>and</u> $\frac{1}{N}\sum a_n^2 \longrightarrow \alpha^2$

When these take place we say "a_n converges to α in density"

$$a_n \xrightarrow{\mathcal{D}} \alpha$$

THEOREM: $(X,\mathcal{B},\mu,T)$ is WM $\Leftrightarrow$ for all $A,B\in\mathcal{B}$

$$\mu(A\cap T^{-n}B) \xrightarrow{\mathcal{D}} \mu(A)\mu(B)$$

THEOREM: If $(X,\mathcal{B},\mu,T)$ is weakly mixing then for any $k=1,2,3,\ldots$ and $k+1$ sets $A_0, A_1, \ldots, A_k \in \mathcal{B}$

$$\mu(A_0 \cap T^{-n}A_1 \cap T^{-2n}A_2 \cap \cdots \cap T^{-kn}A_k) \xrightarrow{D} \mu(A_0)\mu(A_1)\cdots\mu(A_k)$$

In particular, this gives multiple recurrence for WM systems:

$$\mu(A)>0 \Rightarrow \exists\, m \text{ with } \mu(A \cap T^{-n}A \cap \cdots \cap T^{-kn}A)>0$$

THEOREM: (V. BERGELSON) If $(X,\mathcal{B},\mu,T)$ is weakly mixing $A_0, A_1, \ldots, A_k \in \mathcal{B}$ and $p_1(t), \ldots, p_k(t) \in \mathbb{Z}[t]$ with positive leading coefficents, and $\deg p_i > 0$ & $\deg(p_i - p_j) > 0$, then

$$\mu(A_0 \cap T^{-p_1(n)}A_1 \cap T^{-p_2(n)}A_2 \cap \cdots \cap T^{-p_k(n)}A_n) \xrightarrow{D} \mu(A_0)\mu(A_1)\cdots\mu(A_k)$$

$\Rightarrow$ polynomial multiple recurrence

Back to Non-conventional ergodic averages

<u>**WM Ergodic Theorem:**</u>

If $(X,\mathcal{B},\mu,T)$ is weakly mixing,

$f_1, f_2, \dots f_k \in L^\infty(X,\mathcal{B},\mu)$ bounded measurable functions

then

$$\frac{1}{N}\sum f_1(T^n x) f_2(T^{2n}x) \cdots f_k(T^{kn}x) \xrightarrow{L^2}$$

$$\int f_1(y)\,d\mu(y) \int f_2(y)\,d\mu(y) \cdots \int f_k(y)\,d\mu(y)$$

for $f_1 = 1_{A_1}$, $f_2 = 1_{A_2}$, $\dots$ $f_k = 1_{A_k}$

$$\frac{1}{N}\sum 1_{T^{-n}A_1}(x)\, 1_{T^{-2n}A_2}(x) \cdots 1_{T^{-kn}A_k}(x) \xrightarrow{L^2}$$

$$\mu(A_1)\mu(A_2)\cdots\mu(A_k)$$

If $f_n \in L^2(X,\mathcal{B},\mu)$, $f, g \in L^2(X,\mathcal{B},\mu)$ and $f_n \xrightarrow{L^2} f$

then $\int g(x) f_n(x)\,d\mu(x) \longrightarrow \int g(x) f(x)\,d\mu(x)$

$$\therefore \frac{1}{N}\sum \int 1_{A_0}(x)\, 1_{T^{-n}A_1}(x) \cdots 1_{T^{-kn}A_k}(x)\,d\mu(x) \longrightarrow$$

$$\mu(A_0)\mu(A_1)\mu(A_2)\cdots\mu(A_k)$$

or

$$(*)\ \frac{1}{N}\sum \mu(A_0 \cap T^{-n}A_1 \cap \cdots \cap T^{-kn}A_k) \to \mu(A_0)\mu(A_1)\cdots\mu(A_k)$$

FACT: $(X, \mathcal{B}, \mu, T)$ WM $\Longrightarrow$

$$(X \times X, \mathcal{B} \times \mathcal{B}, \mu \times \mu, T \times T) \text{ WM}$$

$$\mu \times \mu (A_1 \times A_2) = \mu(A_1)\mu(A_2)$$

$$T \times T(x_1, x_2) = (Tx_1, Tx_2)$$

Apply foregoing to $A_0 \times A_0, A_1 \times A_1, \cdots A_k \times A_k$ in $X \times X$

$$\mu \times \mu (A_0 \times A_0 \cap (T \times T)^{-n} A_1 \times A_1 \cap \cdots (T \times T)^{-kn} A_k \times A_k) =$$

$$\mu \times \mu (A_0 \cap T^{-n} A_1 \cap \cdots \cap T^{-kn} A_k \times A_0 \cap T^{-n} A_1 \cap \cdots \cap T^{-kn} A_k)$$

$$\mu(A_0 \cap T^{-n} A_1 \cap \cdots \cap T^{-kn} A_k)^2$$

$$(**) \quad \frac{1}{N} \sum \mu(A_0 \cap T^{-n} A_1 \cap \cdots \cap T^{-kn} A_k)^2 \longrightarrow \mu(A_0)^2 \mu(A_1)^2 \cdots \mu(A_k)^2$$

But $(*) + (**) \Longleftrightarrow$

$$\mu(A_0 \cap T^{-n} A_1 \cap \cdots \cap T^{-kn} A_k) \xrightarrow{\mathcal{D}} \mu(A_0)\mu(A_1) \cdot \mu$$

Proof of WM ergodic thm and all subsequent non-conventional L^2 - theorems goes by way of following <u>Hilbert</u> <u>space</u> <u>lemma</u>

$\mathcal{H}$ is Hilbert space

$\langle u, v\rangle$ denotes inner product

$\|u\| = \sqrt{\langle u,u\rangle}$ is norm in $\mathcal{H}$

Lemma: (van der Corput) If $\{u_n\}$ is bounded sequence in $\mathcal{H}$ and for each $h \geq 0$

$$\gamma_h = \lim \tfrac{1}{N}\sum \langle u_n, u_{n+h}\rangle$$

and $$\tfrac{1}{H}\sum \gamma_h \longrightarrow 0,$$

then $$\Big\| \tfrac{1}{N}\sum_1^N u_n \Big\| \longrightarrow 0 \qquad \tfrac{1}{N}\sum_1^N u_n \to 0 \text{ in } \mathcal{H}$$

More explicitly

$$\limsup_N \Big\|\tfrac{1}{N}\sum u_n\Big\|^2 \leq \liminf_H \tfrac{1}{H}\sum \limsup_N \Big|\tfrac{1}{N}\sum \langle u_n, u_{n+h}\rangle\Big|$$

$$\boxed{\text{Proof of } * \ \frac{1}{N}\sum f(T^n x)\, g(T^{2n}x) \xrightarrow{L^2} \int f d\mu \ldots \text{ in WM case}}:$$

Write $T^n f$ for the function $T^n f(x) = f(T^n x)$

$T^{2n} g$ ———————— $T^{2n} g(x) = g(T^{2n} x)$

Lemma: WM $\Rightarrow$ $f, g \in L^2$. $\int f\, T^n g\, d\mu \xrightarrow{\mathcal{D}} \int f d\mu \int g\, d\mu$ ✓

$f = 1_A$ $g = 1_B$ this is characterization of WM

Note: It suffices to prove $*$ when $\int g\, d\mu = 0$

(when g = constant this is ergodic thm)

Take $u_n = T^n f\, T^{2n} g$

$$\langle u_n, u_{n+h} \rangle = \int T^n f\, T^{2n} g\, T^{n+h} f\, T^{2n+2h} g\, d\mu$$

$$= \int f\, T^n g\, T^h f\, T^{n+2h} g\, d\mu$$

$$= \int (f T^h f)\, T^n (g\, T^{2h} g)\, d\mu$$

$$\frac{1}{N}\sum \langle u_n, u_{n+h} \rangle = \int (f T^h f)\, \frac{1}{N} \sum T^n (g\, T^{2h} g)\, d\mu$$

$$\longrightarrow \int f\, T^h f\, d\mu \int g\, T^{2h} g\, d\mu$$

so

$$\gamma_h = \int f T^h f \, d\mu \int g T^{2h} g \, d\mu$$

But $\int g T^{2h} g \, d\mu \xrightarrow{\mathcal{D}} \int g \, d\mu \cdot \int g \, d\mu = 0$

$\therefore \quad \frac{1}{H} \sum \gamma_h \longrightarrow 0$

so by vdC $\quad \frac{1}{N} \sum u_n = \frac{1}{N} \sum T^n f \, T^{2n} g \longrightarrow 0$

Next consider $\frac{1}{N} \sum T^n f \, T^{2n} g \, T^{3n} k$

Can assume $\int k \, d\mu = 0$

$u_n = T^n f \, T^{2n} g \, T^{3n} k$

$\langle u_n, u_{n+h} \rangle = \int (f T^h f) T^n (g T^{2h} g) \, T^{2n} (k T^{3h} k) \, d\mu$

$\gamma_h = \lim \frac{1}{N} \sum \langle u_n, u_{n+h} \rangle = \int f T^h f \, d\mu \cdot \int g T^{2h} g \, d\mu \cdot \int k T^{3h} k \, d\mu$

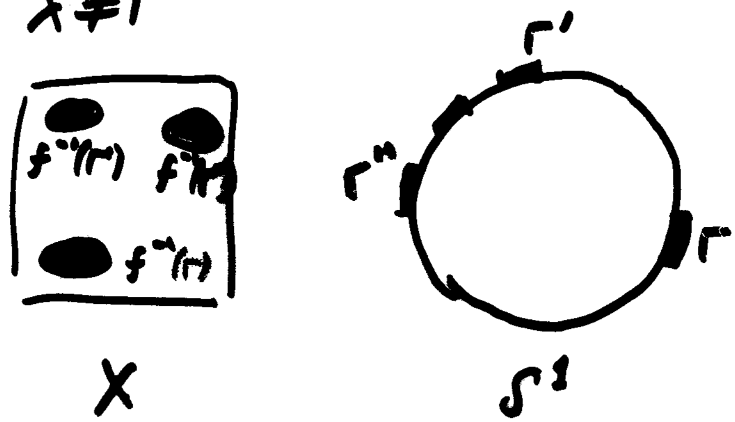
But $\int k T^{3h} k \, d\mu \xrightarrow{\mathcal{D}} 0$

$\therefore \quad \frac{1}{N} \sum u_n \xrightarrow{L^2} 0$

AND SO ON ...

<u>THIRD CHARACTERIZATION</u> of <u>WEAK MIXING</u>

<u>Theorem</u>: An ergodic system $(X, \mathcal{B}, \mu, T)$ is weakly mixing unless $\exists$ measurable $f(x)$ with $|f(x)| = 1$ and $f(Tx) = \lambda f(x)$ with $\lambda \neq 1$



$\Gamma' = \lambda^{-k}\Gamma \quad \Gamma'' = \lambda^{-k-1}\Gamma$

if $\mu(f^{-1}(\Gamma)) > 0$, so is $\mu(f^{-1}(\Gamma'))$, $\mu(f^{-1}(\Gamma''))$

But if $x_1 \in \Gamma \quad T^n x_1 \in \Gamma' \Rightarrow \Gamma' \cap \lambda^n \Gamma \neq \emptyset$

$x_2 \in \Gamma \quad T^n x_2 \in \Gamma'' \Rightarrow \Gamma'' \cap \lambda^n \Gamma \neq \emptyset$

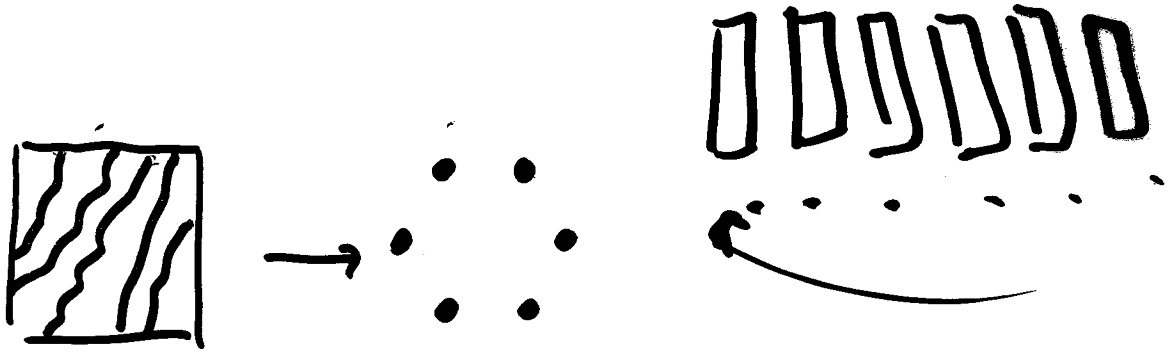
Two Cases of <u>non WM</u>: a) λ is root of unity

b) $\lambda = e^{2\pi i \alpha}$ α irrational

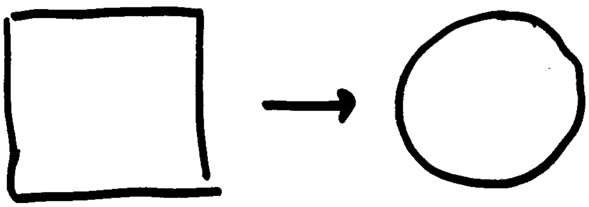
a) If $\lambda^m = 1$ $[f(Tx)]^m = [f(x)]^m \therefore f(x)^m = \text{const.}$

replace $f(x)$ by $c f(x)$;

Can assume $f(x)^m \equiv 1$

a) 

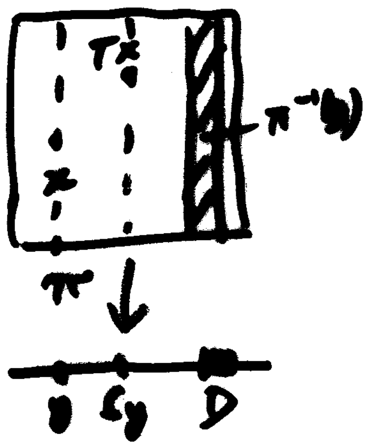
$X \to$ cyclic permutation system

b) 

$X \to$ circle with irrational rotation

Notion of FACTOR

X



Y

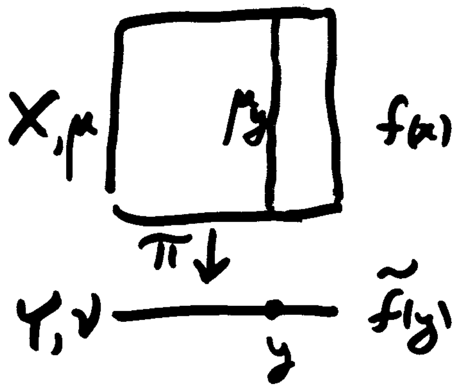
We say $(Y, \mathcal{D}, \nu, S)$ is <u>factor</u> of $(X, \mathcal{B}, \mu, T)$ if $\exists\ \pi : X \to Y$ so that for $D \in \mathcal{D}$

$\pi^{-1}(D) \in \mathcal{B}$, and $\mu(\pi^{-1}(D)) = \nu(D)$

& for $x \in X$

$$\boxed{\pi(Tx) = S\pi(x)}$$

Some Measure Theory



We have Fubini theorem:

$$\int f(x)\, d\mu(x) =$$

$$\int \left\{ \int f(x)\, d\mu_y(x) \right\} d\nu(x)$$

In other words: can decompose/disintegrate

$$\mu = \int \mu_y \, d\nu(y)$$

This gives a map: $L^p(X,\mu) \longrightarrow L^p(Y,\nu)$

$$f \longrightarrow \tilde{f}$$

where $\tilde{f}(y) = \int f(x)\, d\mu_y(x)$

In L^2 this map is characterized by the fact that $f \rightarrow \tilde{f} \circ \pi$ is the orthogonal projection of $f \in L^2(X)$ to the subspace $L^2(Y) \circ \pi \subset L^2(X)$

KRONECKER SYSTEMS

Let Z be a compact abelian (additive) group

$\alpha \in Z$ with $\mathbb{Z}\alpha$ dense in Z

dz = haar measure

(Z, α) denotes the meas. pres. system with space $= Z$, σ-alg = borel sets

measure $= dz$

$Tz = z + \alpha$

A non-trivial Kronecker system is never WM:

Take character $\chi : Z \to S^1$ $\quad \chi \not\equiv 1$

$\lambda = \chi(\alpha) \neq 1$. then $\boxed{\chi(Tz) = \chi(z+\alpha) = \lambda\,\chi(z)}$

If $(X, \mathcal{B}, \mu, T)$ has a non-trivial Kronecker factor, then $(X, \mathcal{B}, \mu, T)$ is not WM

Conversely ...

<u>Theorem</u>: If $(X,\mathcal{B},\mu,T)$ is ergodic but not WM

then $\exists$ Kronecker system (Z,α)

so that $X \xrightarrow{\pi} Z$ & (Z,α) is factor

of $(X,\mathcal{B},\mu,T)$ and such that every

eigenfunction $f : X \to S^1$, $f(Tx) = \lambda f(x)$

has form

$$f = c\chi(\pi(x))$$

with $\chi \in \hat{Z}$, the group of characters

on Z.

(Z,α) is <u>the</u> Kronecker factor

of $(X,\mathcal{B},\mu,T)$

Theorem: Let $(X,\mathcal{B},\mu,T)$ be an ergodic mps;

let (Z,α) be its Kronecker factor.

For f, g bdd functions on X, let $\tilde{f}, \tilde{g}$

be projections on Z. Then in $L^2(X)$

$$\lim \frac{1}{N}\sum f(T^n x)\, g(T^{2n} x) = \lim \frac{1}{N}\sum \tilde{f}(T^n z)\, \tilde{g}(T^{2n} z) = \int \tilde{f}(z+\omega)\, \tilde{g}(z+2\omega)\, d\omega$$

where $\pi(x) = z$

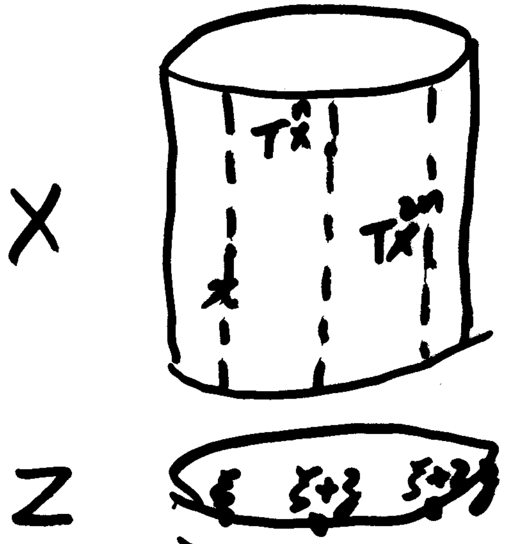
<u>Theorem:</u> Let $(X, \mathcal{B}, \mu, T)$ be an ergodic meas. preserving system; let (Z, α) be its Kronecker factor $(\pi : X \to Z)$. Let f, g be bounded meas. functions on X, and let $\tilde{f}, \tilde{g}$ be their "projections" on Z. Then in $L^2(X, \mu)$

$$\boxed{\lim_{N\to\infty} \frac{1}{N} \sum f(T^n x)\, g(T^{2n} x) = \lim_{N\to\infty} \frac{\sum \tilde{f}(T^n \xi)\, \tilde{g}(T^{2n} \xi)}{N} = \int_Z \tilde{f}(\xi + z)\, \tilde{g}(\xi + 2z)\, dz}$$

where $\xi = \pi(x)$.

Pf: Almost same as in WM case; instead of $\int g = 0$ assume $\tilde{g} = 0$

<u>Interpretation</u>



For given x, $T^n x$ & $T^{2n} x$ are constrained by the fact that the projections form arith. prog.

$\xi,\ \xi + z,\ \xi + 2z$

And – There are no further constraints

In general we look for a "small" factor $\underset{x \to \xi}{X \xrightarrow{\pi} Y}$ so that given $x \in X$ the only constraints on $T^n x, T^{2n} x, \dots T^{kn} x$ are determined by constraints on

$$T^n \xi, T^{2n} \xi, \dots T^{kn} \xi$$

inside Y. This is called a/the characteristic factor (for $T^n, \dots T^{kn}$)

If $(Y, \dots)$ is a char. factor then

$$\frac{1}{N} \sum T^n f_1 \, T^{2n} f_2 \cdots T^{kn} f_k - \frac{1}{N} \sum T^n \tilde{f}_1 \, T^{2n} \tilde{f}_2 \cdots T^{kn} \tilde{f}_k \circ \pi$$

$$\longrightarrow 0$$

in $L^2(X, \mu)$

Remark: $\exists$ a "smallest" char. factor which is the char. factor