III

ERGODIC GEOMETRY

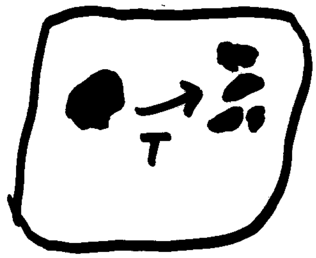
and the role of

NILPOTENT GROUPS

&

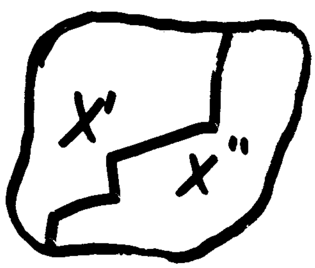
NILMANIFOLDS

ERGODIC (MEASURE PRESERVING) SYSTEM



X

Space X endowed with measure μ, $\mu(X)=1$ and transformation $T: X \to X$ preserving the measure of (measurable) sets: $\mu(T^{-1}A) = \mu(A)$



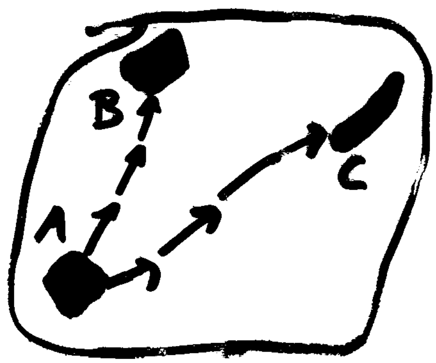
and such that one cannot split $X = X' \cup X''$ with $\mu(X')>0$, $\mu(X'')>0$

$T(X') \subset X' \quad T(X'') \subset X''$

$\Updownarrow$

whenever $\mu(A), \mu(B) > 0$, $\exists\, n \geq 1$ with $A \cap T^{-n}B \neq \emptyset$ $\quad (\Rightarrow \mu(A \cap T^{-n}B) > 0)$

WEAK MIXING SYSTEM



IF $\mu(A), \mu(B), \mu(C) > 0$

$\exists\, n \geq 1$ and two points $x, y \in A$ with $T^n x \in B$, $T^n y \in C$

$\Rightarrow \mu(A \cap T^{-n}B) > 0 \quad \mu(A \cap T^{-n}C) > 0$

KRONECKER SYSTEM

$X = Z$, a compact abelian group which contains α so that $\{\alpha, 2\alpha, 3\alpha, \dots n\alpha, \dots\}$ is dense in Z

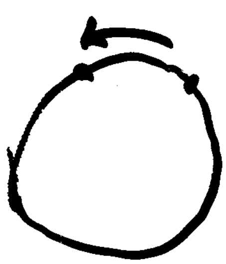
($\Leftrightarrow \{\dots, -\alpha, 0, \alpha, 2\alpha, \dots\}$ is dense)

μ = haar measure

$Tz = z + \alpha$

<u>Classical case</u>: $Z = \mathbb{R}/\mathbb{Z}$ $\quad \alpha$ irrational

$\approx [0,1) \quad Tx = x+\alpha - [x+\alpha]$



$\approx S^1 = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$, $T\zeta = e^{2\pi i \alpha}\zeta$

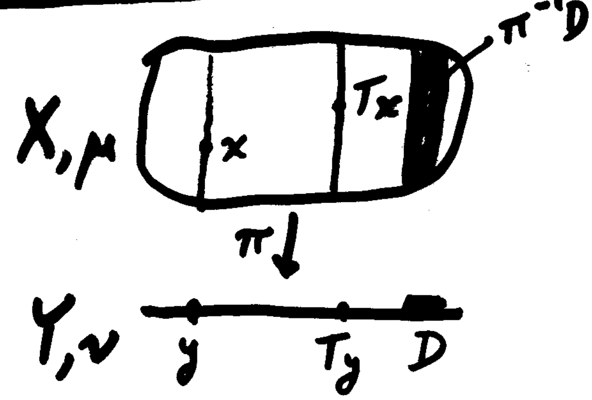
<u>Or</u>, $\quad \mathbb{Z} = \mathbb{R}^m/\mathbb{Z}^m \quad$ m-torus

$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ $\{1, \alpha_j\}$ lin. ind. over $\mathbb{Q}$

<u>Or</u>, $\quad Z$ = p-adic integers, $\alpha = 1$

<u>Or</u>, $\quad Z = \mathbb{Z}/m\mathbb{Z}$ = finite cyclic gp. $\alpha = j$, $(j,m) = 1$

FACTORS



$\mu(\pi^{-1}D) = \nu(D)$

$T\pi(x) = \pi(Tx)$

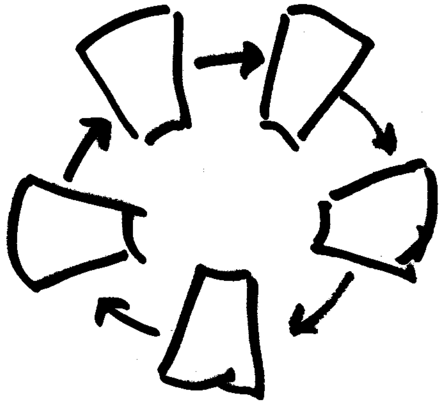
In a Kronecker system, each character χ is eigenfunction: $T\chi(z) \equiv \chi(Tz) = \chi(z+\alpha)$

$$= \chi(\alpha)\cdot\chi(z)$$

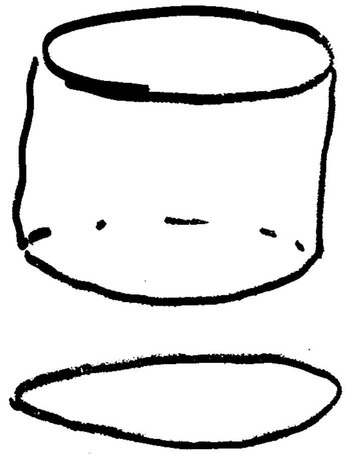
$\lambda = \chi(\alpha)$: $T\chi = \lambda\chi$

THM: An ergodic system is weak mixing unless: it has a nontrivial eigenfn

$\Leftrightarrow$ it has a nontrivial Kronecker factor



$Z = \{0,1,2,3,4\}$



$Z = S^1$

CLASSICAL (MEAN) ERGODIC THEOREM (von Neumann)

If $(X, \mathcal{B}, \mu, T)$ is any measure preserving system, $f \in L^2(X, \mathcal{B}, \mu)$, then in L^2 the limit

$$\bar{f}(x) = \lim_{N\to\infty} \frac{1}{N}\{f(x) + f(Tx) + \cdots + f(T^{N-1}x)\}$$

exists.

If $(X, \mathcal{B}, \mu, T)$ is <u>ergodic</u>, then

$$\frac{1}{N}\{f(x) + f(Tx) + \cdots + f(T^{N-1}x)\} \xrightarrow{L^2} \int f\, d\mu$$

$$\left(\text{i.e. } \int \left[\frac{1}{N}\sum_{0}^{N-1} f(T^n x) - \int f\, d\mu\right]^2 d\mu(x) \to 0\right)$$

WEAK MIXING (MEAN) ERGODIC THEOREM

If $(X, \mathcal{B}, \mu, T)$ is WM system and $f_1, \ldots, f_k$ are bounded measurable fn's, then in L^2

$$\frac{1}{N}\sum_{0}^{N-1} f_1(T^n x) f_2(T^{2n} x) \cdots f_k(T^{kn} x) \to$$

$$\int f_1\, d\mu \int f_2\, d\mu \cdots \int f_k\, d\mu$$

This cannot hold for non WM systems!

Let $\varphi(x)$ be non-trivial eigenfunction

$$\varphi(Tx) = \lambda\varphi(x)$$

φ is non-constant $(\lambda \neq 1)$ $\quad |\varphi| = 1$

Set $\quad f(x) = \varphi(x)^2 \quad g(x) = \varphi(x)^{-1}$

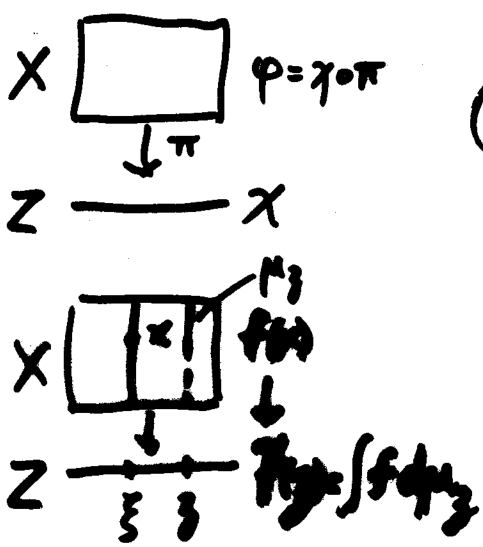
$$f(T^n x) = \lambda^{2n}\varphi(x)^2$$

$$g(T^{2n} x) = \lambda^{-2n}\varphi(x)^{-1}$$

$$f(T^n x)\, g(T^{2n} x) = \varphi(x)$$

$\therefore \lim \frac{1}{N} \sum f(T^n x)\, g(T^{2n} x) = \varphi(x)$ is not constant!

Note: RHS is function lifted from Kronecker factor of $(X, \mathcal{B}, \mu, T)$

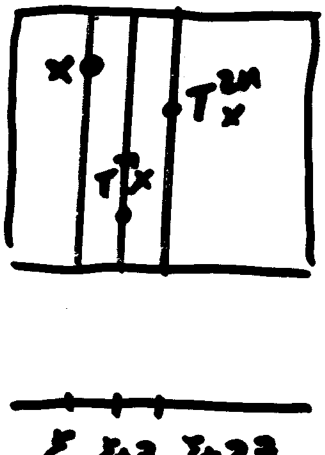


(FIRST) <u>Non-Conventional ERGODIC THEOREM</u>:

For any ergodic system $(X, \mathcal{B}, \mu, T)$ and bdd meas f, g

$$\frac{1}{N} \sum f(T^n x)\, g(T^{2n} x) \longrightarrow \int \tilde{f}(\xi + z)\, \tilde{g}(\xi + 2z)\, dz$$

in $L^2(X)$ where $\xi = \pi(x)$



Meaning of this theorem:

The only constraint on $T^n x$, $T^{2n} x$ given x derives from Kronecker factor and algebraic relation between $(\xi, \xi+n\alpha, \xi+2n\alpha)$

$$\in \{(\xi, \xi+z, \xi+2z)\}$$

$$\left(\in \{(\xi, \xi', \xi''): \xi - 2\xi' + \xi'' = 0\right)$$

Now suppose we have ψ on X, $|\psi| = 1$ and

$$\psi(Tx) = \varphi(x)\psi(x) \quad \text{where } \varphi(Tx) = \lambda\varphi(x)$$

$$\psi(T^n x) = \lambda^{\frac{n(n-1)}{2}} \varphi(x)^n \psi(x)$$

Abramov

Set $f = \psi^3$, $g = \psi^{-3}$, $h = \psi$

$$T^n f = \lambda^{\frac{3n(n-1)}{2}} \varphi^{3n} \psi^3$$

$$T^{2n} g = \lambda^{-\frac{6n(2n-1)}{2}} \varphi^{-6n} \psi^{-3}$$

$$T^{3n} h = \lambda^{\frac{3n(3n-1)}{2}} \varphi^{3n} \psi$$

$$T^n f \, T^{2n} g \, T^{3n} h = \lambda^{\frac{3n^2 - 12n^2 + 9n^2 - 3n + 6n - 3n}{2}} \varphi^{3n - 6n + 3n} \psi =$$

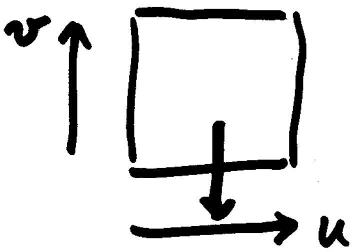
$$= \psi$$

Example of second order eigenfunction:

$$X = \mathbb{R}^2/\mathbb{Z}^2 \qquad T(u,v) = (u+\alpha,\ v+u)$$

$$\varphi(u,v) = e^{2\pi i u} \qquad \psi(u,v) = e^{2\pi i v}$$

$$T\varphi = e^{2\pi i \alpha}\varphi \qquad T\psi = \varphi\cdot\psi$$



Kronecker factor is $Z = \mathbb{R}/\mathbb{Z}$

$$(u,v) \longrightarrow u$$

If $f(x) = f(u,v)$ then

$$\tilde{f}(u) = \int f(u,v)\,dv \ : \ \int e^{2\pi i v}\,dv = 0$$

$$\therefore\ \tilde{\psi} = 0$$

This system is also an example of <u>Nilsystem</u>

$$G = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : x \in \mathbb{Z},\ y,z \in \mathbb{R} \right\}$$

$$\Gamma = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : x,y,z \in \mathbb{Z} \right\} \qquad G/\Gamma \approx \mathbb{R}^2/\mathbb{Z}^2$$

as top. space

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}\Gamma = \begin{pmatrix} 1 & x+1 & y+z \\ 0 & 1 & z+\alpha \\ 0 & 0 & 1 \end{pmatrix}\Gamma = \begin{pmatrix} 1 & x & y+z \\ 0 & 1 & z+\alpha \\ 0 & 0 & 1 \end{pmatrix}$$

(SECOND) <u>Non-conv. Ergodic thm:</u> (Weiss, F.)

For any ergodic system $(X, \mathcal{B}, \mu, T)$
$\exists$ a nilsystem factor* $(G/\Gamma, a)$
so that if f, g, h are bdd meas fn's on X and $\tilde{f}, \tilde{g}, \tilde{h}$ are proj. on G/Γ then

$$\lim \frac{1}{N} \sum f(T^n x)\, g(T^{2n} x)\, h(T^{3n} x) =$$

$$\lim \frac{1}{N} \sum \tilde{f}(a^n \xi)\, \tilde{g}(a^{2n} \xi)\, \tilde{h}(a^{3n} \xi)$$

where $\xi = \pi(x) \in G/\Gamma$

Here G is nilpotent of <u>level 2</u>:

$[[G,G],G] = 1$ ($\Leftrightarrow$ every $g_1 g_2 g_1^{-1} g_2^{-1}$ commutes with every g_3)

A third analogous result was proved by T. Ziegler with G nilpotent of level 3.

*Nilsystem: G nilpotent gp, Γ subgroup with $\mu(G/\Gamma) < \infty$, μ G-inv.

a is fixed element in G

$T(g\Gamma) = ag\Gamma$

$T^n(g\Gamma) = a^n g\Gamma$

Algebraic Explanation of Relevance of Nilpotent Groups

What is geometric progression in a non-commutative gp?

We would like product of geom. prog. to be geometric prog.

$$(u_1, u_1 v_1, u_1 v_1^2, u_1 v_1^3) \times (u_2, u_2 v_2, u_2 v_2^2, u_2 v_2^3) =$$
$$(u_1 u_2, u_1 u_2 v_1 v_2 w, u_1 u_2 (v_1 v_2 w)^2 w', \dots)$$

Theorem of Hall-Petresco

Let G be any group, $G^{(1)} = [G,G]$, $G^{(2)} = [G^{(1)}, G]$, etc. $G^{(i+1)} = [G^{(i)}, G]$; for any $x, y \in G$ $\exists z \in G$ and $\underline{w_i \in G^{(i)}}$

$$(x, x^2, x^3, \dots) \times (y, y^2, y^3, \dots) =$$
$$(z, z^2 w_1, z^3 w_1^3 w_2, z^4 w_1^6 w_2^6 w_3, \dots z^{\binom{n}{1}} w_1^{\binom{n}{2}} w_2^{\binom{n}{3}} \dots w_{n-1}^{\binom{n}{n}}$$

It turns out such expressions do form a group! (Lazard, Liebman)

$$\boxed{G \text{ is nilpotent of level } l \text{ iff } G^{(l)} = \{1\}}$$

__Definition__ In any group G, a __geometric__ [redacted].
is a sequence of the form

$$\left(g,\ gz,\ gz^2 w_1,\ gz^3 w_1^3 w_2,\ \cdots\ gz^{\binom{n}{1}} w_1^{\binom{n}{2}} \cdots w_{n-1}^{\binom{n}{n}} \right)$$

where g, z are any elt's of G and
$w_i \in G^{(i)}$, $i = 1, 2, \dots n-1$

> Note: In a nilpotent group of level ℓ, $\ell+1$ [redacted] determine all the rest

__Theorem__: In any group the term by term product of geometric progressions is a g.p. and the term by term inverse of a g.p is a g.p.

Now let Γ be any subgroup of G.

__Def^n__: A sequence $x_1, x_2, \dots x_n$ in G/Γ is a geometric prog. if it can be expressed as $g_1\Gamma, g_2\Gamma, \dots g_n\Gamma$ with $g_1 g_2 \dots g_n$ a geometric prog.

<u>Fundamental Thm</u>: If G is a nilpotent group of level l, $\Gamma < G$, and

$$x_1, \ldots x_l, x_{l+1}, \ldots x_n$$

a geometric progression in G/Γ, then the first $l+1$ terms determine the rest.

Pf: If we have two expressions for

$$(x_1, \ldots, x_{l+1}) = (g_1\Gamma, \ldots g_{l+1}\Gamma) = (g_1'\Gamma, \ldots g_{l+1}'\Gamma)$$

with $\{g_i\}$ & $\{g_i'\}$ forming g.p.'s, then

$$g_1^{-1}g_1',\ g_2^{-1}g_2' \cdots g_{l+1}^{-1}g_{l+1}' = \gamma_1, \gamma_2 \cdots \gamma_l$$ which

is a g.p. <u>in Γ</u>. the unique extension of this is <u>still in Γ</u>. Therefore for the unique continuations $g_{l+2}, g_{l+3}, \cdots g_n$

$g_{l+2}', \ldots g_n'$, $\quad g_j^{-1}g_j' \in \Gamma \Rightarrow g_j\Gamma = g_j'\Gamma$ ■

Therefore:

In a nil[redacted]system for G nilpotent of level l: $(G/\Gamma, a)$ $\quad T g\Gamma = a g\Gamma$

$l+1$ terms of $(g\Gamma, a^n g\Gamma, a^{2n} g\Gamma, \ldots a^{ln} g\Gamma \ldots)$ determine all the rest.

Affine PRLPD Geometry in $\mathbb{R}^m = V$

What is a parallelopiped (prlpd) of dimension d?

$\mathbb{R}^m$ acts on V, on $V \times V$, on $(V\times V)\times(V \times V)$, etc.

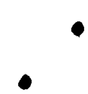
$$\tau_u(v_1, v_2, \dots v_{2^d}) = (v_1+u, v_2+u, \dots v_{2^d}+u)$$

It is transitive on V but not on the rest.

Write $(v_1, \dots) \cong (v_1', \dots)$ if $\exists\, u$

with $(v_1', \text{—}) = \tau_u(v_1 \dots)$

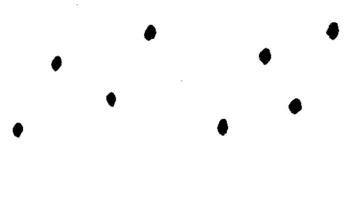
A prlpd of dim 1 is a pair (u_1, u_2) with $u_1 \cong u_2$ (so it's any pair)



A prlpd of dim 2 is a 2^2-tuple $(u_1\, u_2\, u_3\, u_4)$ with $(u_1, u_2) \cong (u_3, u_4)$ 2-prlpd

dim 3: $(u_1, u_2, u_3, u_4\; u_5\; u_6\; u_7\; u_8)$

$(u_1, u_2, u_3, u_4) \cong (u_5\, u_6\, u_7, u_8)$ 3-prlpd



ETC.

Denote by $\Pi^{[d]} \subset V^{2^d}$ the subset of d-prllpd's in V^{2^d}

(Coordinate permutation) <u>Theorem</u>:

The map $(u_1\, u_2) \to (u_2\, u_1)$

Takes $\Pi^{[1]} \to \Pi^{[1]}$ $\quad (= V \times V)$

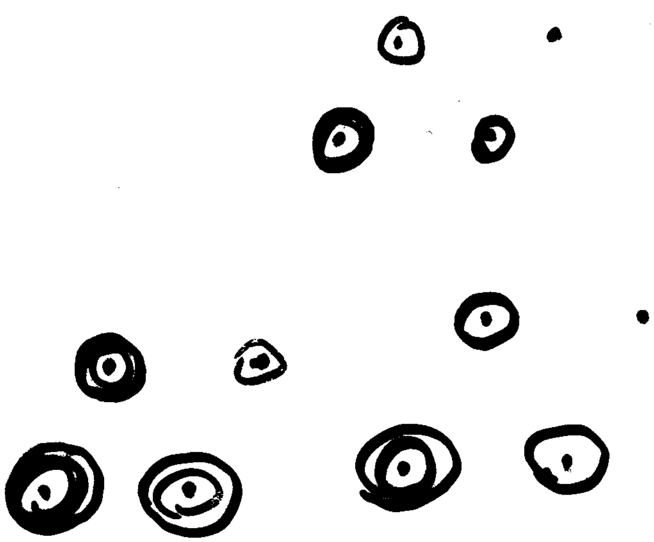
The map $(u_1\, u_2\, u_3\, u_4) \to (u_1\, u_3\, u_2\, u_4)$

Takes $\Pi^{[2]} \to \Pi^{[2]}$ $\quad (\subset V \times V \times V \times V)$

The maps

$$(u_1\, u_2\, u_3\, u_4\, u_5\, u_6\, u_7\, u_8) \begin{array}{l} \nearrow (u_1\, u_2\, u_5\, u_6\;\; u_3\, u_4\, u_7\, u_8) \\ \searrow (u_1\, u_3\, u_5\, u_7\;\; u_2\, u_4\, u_6\, u_8) \end{array}$$

take $\Pi^{[3]} \to \Pi^{[3]}$ $\quad (\subset V^8)$

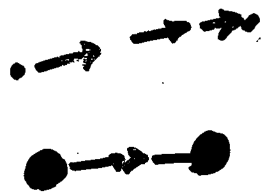


ETC.

Note: All This makes sense for <u>any</u> group action!

MACKEY'S PHILOSOPHY & ERGODIC PRLPD GEOMETRY

Ergodicity was once called *metric transitivity*



MACKEY: Think of ergodicity as actual transitivity

(So $X = G/L$ L a "virtual" subgroup)

ergodic action $\longleftrightarrow$ transitive

ergodic component $\longleftrightarrow$ orbit

when action not ergodic — when action not transitive

Note: ergodic comp's have disjoint support — orbits either = or disjoint

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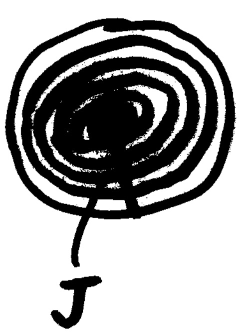
Recall: Any meas. pres. action decomposes to ergodic "components"



These are parametrized by some measure space

$$\mu = \int_J \mu_y \, d\theta(y)$$

(This is well defined in "almost everywhere" sense)

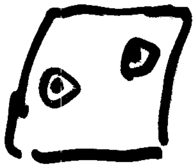
HOST–KRA THEORY

Assume $(X, \mathcal{B}, \mu, T)$ is ergodic

$X \times X$ (shorthand for $(X \times X, \mathcal{B} \times \mathcal{B}, \mu \times \mu, T \times T)$

may or may not be ergodic

1-prlpd : all pairs (x_1, x_2) $\quad \Pi^{[1]} = X \times X$



this carries measure $\mu^{[1]} = \mu \times \mu$

2-prlpd : all $(x_1\, x_2\, x_3\, x_4)$ with $(x_1\, x_2)$ & $(x_3\, x_4)$

belonging to same ergodic component of $\Pi^{[1]}$

$\Pi^{[2]} \subset X^4$ $\quad \cancel{\mu^{[2]} = \int \mu_y \times \mu_y}$

$$\mu^{[2]} = \int_{J_1} (\mu \times \mu)_y \times (\mu \times \mu)_y \, d\theta'(y)$$

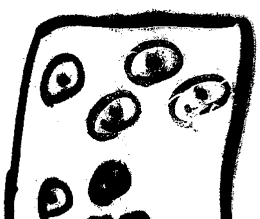
where $\quad \mu \times \mu = \int_J (\mu \times \mu)_y \, d\theta'(y)$

is ergodic decomp.

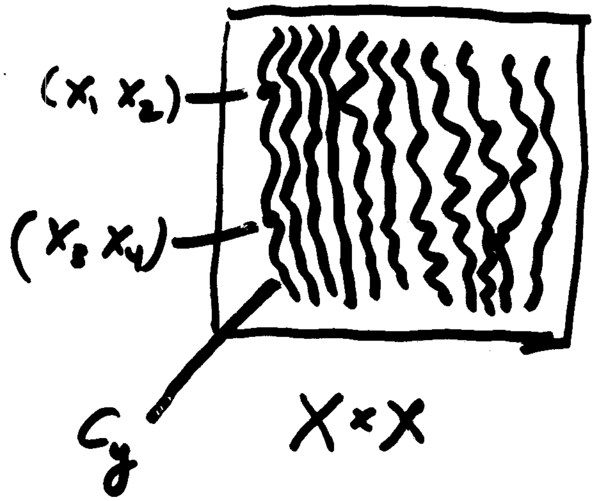
3-prlpd : all $(x_1\, x_2 \cdots x_8)$ with $(x_1 \cdots x_4)$ & $(x_5 \cdots x_8)$

in same ergodic component of

$(\Pi^{[2]}, \mu^{[2]})$



$$\mu^{[3]} = \int_{J_2} \mu_y^{[2]} \times \mu_y^{[2]} \, d\theta''(y)$$



C_y a particular ergodic comp. with measure $(\mu\times\mu)_y$

$C_y \times C_y \subset \Pi^{[2]}$

Space $\Pi^{[2]}$ of 2-prllpd's (parallelograms) is union of $C_y \times C_y$

Measure $\mu^{[2]}$ on $\Pi^{[2]}$ is integral of $(\mu\times\mu)_y \times (\mu\times\mu)_y$

Theorem If $(X, \mathcal{B}, \mu, T)$ is WM then

$$\mu^{[d]} = \underbrace{((\mu\times\mu)\times(\mu\times\mu))\times}_{2^d}$$

on $\Pi^{[d]} = X^{2^d}$

So there is no interesting geometry

(Coordinate permutation) <u>Theorem</u>:

The map $(u_1, u_2) \to (u_2, u_1)$ is measure preserving on $(\Pi^{[1]} = X \times X, $ [illegible]

The map $(u_1 u_2 u_3 u_4) \to (u_1 u_3 u_2 u_4)$ is meas. pres. on $(\Pi^{[2]}, \mu^{[2]})$

the maps

$$(u_1 u_2 u_3 u_4 u_5 u_6 u_7 u_8) \begin{array}{l} \nearrow (u_1 u_2 u_5 u_6 u_3 u_4 u_7 u_8) \\ \searrow (u_1 u_3 u_5 u_7 u_2 u_4 u_6 u_8) \end{array}$$

are meas. preserving on $(\Pi^{[3]}, \mu^{[3]})$

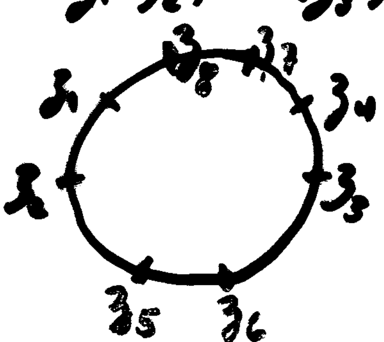
ETC.

<u>Kronecker systems</u>

(Z, α) is ergodic

$(Z \times Z, (\alpha, \alpha))$ is not ergodic; its decomposi[tion] is by cosets modulo diagonal.

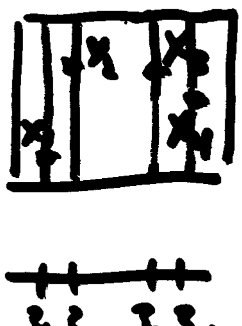
So $(z_1 z_2) \cong (z_3 z_4) \Leftrightarrow z_2 - z_1 = z_4 - z_3$



L. In passage to factors prlpd's $\rightarrow$ prlpd's

<u>Theorem</u> For any ergodic system $(X, \mathcal{B}, \mu, T)$ the 2-prlpd's are determined by projection to its Kronecker, and $\Pi_X^{[2]}$ is lift of $\Pi_Z^{[2]}$ and $\mu_X^{[2]}$ is lift of $\mu_Z^{[2]}$

X



Z

$$\mu = \int_Z \mu_z \, dz$$

$$\mu_X^{[2]} = \int_{\Pi_Z^{[2]}} \mu_{z_1} \times \mu_{z_2} \times \mu_{z_3} \times \mu_{z_4} \, d\mu_Z^{[2]}(z_1, z_2, z_3, z_4)$$

something similar happens in each dimension

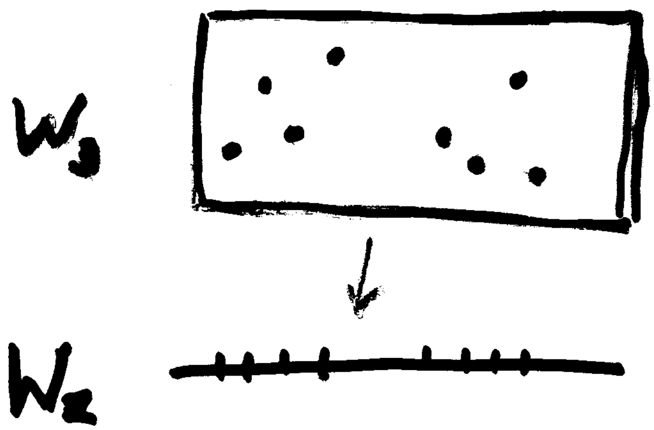
<u>Theorem</u> For any ergodic system $(X, \mathcal{B}, \mu, T)$ and for each d, the d-prlpd's are determined by projection to a "characteristic factor" W_d, so that

$$\Pi_X^{[d]} = \pi_d^{-1}\left(\Pi_{W_d}^{[d]}\right)$$ and

$$\mu_X^{[d]} = \int_{\Pi_{W_d}^{[d]}} \mu_{z_1} \times \mu_{z_2} \times \cdots \times \mu_{z_{2^d}} \, d\mu_{W_d}^{[d]}(z_1, z_2, \ldots, z_{2^d})$$

Ergodic prlpd Geometry produces a tower of characteristic factors

$$
\begin{array}{l}
X \\
\downarrow \\
\vdots \\
\downarrow \\
W_d \\
\downarrow \\
W_{d-1} \\
\downarrow \\
\vdots \\
W_3 \\
\downarrow \\
W_2 = Z = \underline{\text{Kronecker}}
\end{array}
$$



I Each W_d is "characteristic" for

$$(x, T^n x, T^{2n} x, \ldots, T^{dn} x)$$

That means that <u>all</u> constraints on these $(d+1)$-tuples are determined by ~~their~~ images

$$(z, Tz, T^n z, \ldots T^{dn} z)$$

in W_d

OR...

Thm. If $f_1, \dots, f_d$ are bdd. meas fn's on X and $\tilde{f}_1, \dots, \tilde{f}_d$ are their projections on W_d:

$$\frac{1}{N}\sum f_1(T^n x)\cdots f_d(T^{dn}x) -$$

$$\frac{1}{N}\sum \tilde{f}_1(T^n z)\cdots \tilde{f}_d(T^{dn} z) \longrightarrow 0$$

in $L^2(X)$ where $z = \mathrm{proj}_{X \to W_d}(x)$

II $\quad W_d$ is a nilsystem of level $d-1$

Where does the nilpotent group come from?

For any ergodic system $(X, \mathcal{B}, \mu, T)$ we define the H-K group $\mathcal{G}(X)$ to be group of <u>ALL</u> measure preserving trans. of X (like T^n) <u>compatible</u> <u>with</u> <u>the</u> <u>geometry</u>:

so if $g \in \mathcal{G}(X)$, then for each d

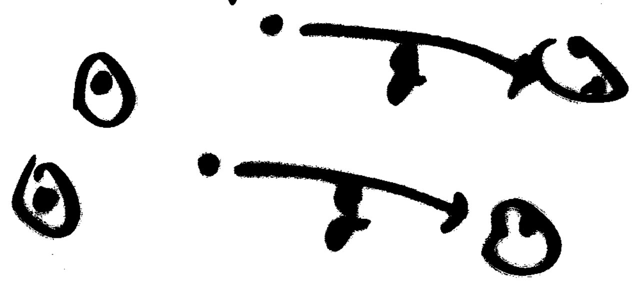
if $(x_1 \dots x_{2^d}, x_1', \dots x_{2^d}')$ in a $(d+1)$ prlpd so

is $(x_1 \dots x_{2^d}, gx_1' \dots gx_{2^d}')$

and the transf of $\Pi^{[d+1]} \circlearrowleft$ given

in this form: $1 \times 1 \times \dots \times 1 \times g \times g \times \dots \times g$

preserves $\mu^{[d+1]}$



By coordinate permutation: if

$1 \times 1 \times g \times g$ preserves $\mu^{[2]}$

so do

$1 \times g \times 1 \times g$, $g \times g \times 1 \times 1$, $g \times 1 \times g \times 1$.

Taking commutators, if $g' = [g_1, g_2]$ and $g_1, g_2 \in \mathcal{G}(X)$

$[g_1 \times g_1 \times 1 \times 1, 1 \times g_2 \times 1 \times g_2] = 1 \times g' \times 1 \times 1$ preserves 2-prlpd

similarly

$[g_1 \times g_1 \times 1 \times 1, g_2 \times 1 \times g_2 \times 1] = g' \times 1 \times 1 \times 1$

ETC

From this we deduce:

<u>Theorem</u> $\mathcal{G}(W_2)$ is commutative

Idea of proof: if $g_1, g_2, g_3, g_4 \in [\mathcal{G}, \mathcal{G}]$

and (x_1, x_2, x_3, x_4) is a 2-prlpd

so are $(g_1 x_1, x_2, x_3, x_4)$ &

$(g_1 x_1, g_2 x_2, x_3, x_4)$ &

$(g_1 x_1, g_2 x_2, g_3 x_3, g_4 x_4)$

Defining $x \sim y$ if $y = gx$ $g \in [\mathcal{G}, \mathcal{G}]$

then 2-prlpd on W_2 are determined

by $W_2/\sim$ <u>If this relation weren't trivial could</u> reduce W_2

Similarly

<u>Theorem</u> $\mathcal{G}(W_3)$ is nilpotent of level 2.

Pf: the triple commutator $[[\ ,\ ],\ ]$ of

$g_1 \times g_1 \times g_1 \times g_1 \times 1 \times 1 \times 1 \times 1$, $g_2 \times g_2 \times 1 \times 1 \times g_2 \times g_2 \times 1 \times 1$

& $g_3 \times 1 \times g_3 \times 1 \times g_3 \times 1 \times g_3 \times 1$ is $[g_1, g_2] g_3 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1$