<u>General Theme</u>: Given an object $S$ with known properties and desired properties,

— find dyn. system $X$

$$\ni: \ S \in X$$

— prove

1. known $\Rightarrow$ $X$ is "good"
2. $X$ is "nice" $\Rightarrow$ desired
3. good $\Longrightarrow$ nice

<u>Thm.</u> (Szemerédi)

✓ $S \subseteq \mathbb{Z}$ pos. upper density

✓ $\ell \in \mathbb{Z}$ pos. $\Longrightarrow$

$S \supseteq$ an arith. seq. of length $\ell$

---

Fix $S, \ell$

✓ <u>Want</u>: $\exists j \ \ \exists b \neq 0 \ni$:

$j, j-b, j-2b, \ldots, j-\ell b \in S$

②

$S$ pos. upper density

Choose seqs $m_k, n_k$

✓ $\ni$: $n_k - m_k \longrightarrow +\infty$

& $\ni$: $\inf_k \dfrac{\# S_k}{n_k - m_k + 1} > 0$

✓ where $S_k := S \cap [m_k, n_k]$

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✓ $\varepsilon := \inf_k \dfrac{\# S_k}{n_k - m_k + 1} > 0$

---

(e.g. $S = 2\mathbb{Z}$, $\varepsilon = 1/2$)

$$\chi := \chi_S : \mathbb{Z} \longrightarrow \{0,1\}$$

✓ $$\chi(i) = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{o.w.} \end{cases}$$

✓ $$\chi \in \{0,1\}^{\mathbb{Z}} =: X$$

✓ $$\mathbb{Z} \curvearrowright X, \quad (t.\phi)(i) = \phi(i+t)$$

✓ $$Y := \{\phi \in X \ni : \phi(0) = 1\}$$

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$$i \in S \iff \chi(i) = 1$$

$$\iff (i.\chi)(0) = 1$$

$$\iff i.\chi \in Y$$

✓

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$X = \{0,1\}^{\mathbb{Z}}$ cpt

$\mathcal{P} := \{$prob. msrs on $X\}$

$\mathcal{P} \hookrightarrow (C(X))^*$

$\mathcal{P}$ cpt, cvx

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$\forall \phi \in X,$

$$\delta_\phi = \begin{pmatrix}\text{pt mass} \\ \text{at } \phi\end{pmatrix} \in P$$

---

$$\delta_\phi(Y) = \begin{cases} 1 & \text{if } \phi \in Y \\ 0 & \text{o.w.} \end{cases}$$

---

$$i \in S \iff i.\chi \in Y$$

✓ $$\iff \delta_{i.\chi}(Y) = 1$$

---

✓ $$i \notin S \iff \delta_{i.\chi}(Y) = 0$$

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✓ $\forall k,\ \nu_k := \operatorname*{avg}_{m_k \le i \le n_k} \delta_{i,\gamma}$

$$= \frac{1}{n_k - m_k + 1}\left[\sum_{i=m_k}^{n_k} \delta_{i,\gamma}\right]$$

---

$\forall k,\ \nu_k(Y) =$

$$\frac{1}{n_k - m_k + 1}\left[\#S_k\right] \geq \varepsilon$$

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$$\forall k, \quad \nu_k \in \mathcal{P}$$

---

Subseq. $\nu_k \longrightarrow \mu$

$$\mu(Y) \geqslant \varepsilon > 0$$

---

$$n_k - m_k \longrightarrow +\infty$$

$\therefore$ $\mu$ is $\mathbb{Z}$-invariant

IOU

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<u>Want</u>: $\exists j \ \exists b \neq 0 \ni$:
$j, j-b, j-2b, \ldots, j-\ell b \in S$

---

$i \in S \Longleftrightarrow i . X \in Y$

---

$j-i \in S \Longleftrightarrow (j-i) . X \in Y$

$\Longleftrightarrow j . X \in i . Y$

$\Longleftrightarrow \delta_{j.X}(i.Y) > 0$

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**Want**: $\exists j, \exists b \neq 0 \ni$:

$$\delta_{j \cdot x}(Y) > 0,$$

$$\delta_{j \cdot x}(b \cdot Y) > 0,$$

$$\delta_{j \cdot x}((2b) \cdot Y) > 0,$$

$$\vdots$$

$$\delta_{j \cdot x}((\ell b) \cdot Y) > 0$$

---

$\forall b, \quad Y_b :=$

$$Y \cap (b \cdot Y) \cap \cdots \cap ((\ell b) \cdot Y)$$

<u>Want</u>: $\exists j \ \exists b \neq 0$

$\ni: \delta_{j \cdot x}(Y_b) > 0$

---

$\forall k,$

$$\nu_k = \operatorname*{avg}_{m_k \leq j \leq n_k} \delta_{j \cdot x}$$

---

<u>Want</u>: $\exists k \ \exists b \neq 0$

$\ni: \nu_k(Y_b) > 0$

$\nu_k \longrightarrow \mu$

---

<u>Want</u>: $\exists b \neq 0 \ni$:

$$\mu(Y_b) > 0$$

---

$T_1, \ldots, T_\ell : X \longrightarrow X$

$T_1(x) = 1.x, \ldots, T_\ell(x) = \ell.x$

---

$T_1, \ldots, T_\ell$ commute,

preserve $\mu$

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$\forall b,\quad Y_b =$

$Y \cap (b.Y) \cap \cdots \cap ((\ell b).Y)$

---

$\forall b,\quad Y_b =$

$Y \cap (T_1^b Y) \cap \cdots \cap (T_\ell^b Y)$

---

<u>Want</u>: $\exists b \neq 0 \ni$:

$Y \cap (T_1^b Y) \cap \cdots \cap (T_\ell^b Y)$

has pos. $\mu$-msr.

## Multiple Recurrence

$X$ std prob space

$\left( \begin{array}{l} T_1, \ldots, T_\ell : X \longrightarrow X \\ \text{commute, msr. pres} \end{array} \right)$

$Y \subseteq X$ has pos. msr.

$\Longrightarrow \exists\, b \neq 0 \ni:$

$Y \cap (T_1^{\,b} Y) \cap \cdots \cap (T_\ell^{\,b} Y)$

has pos. msr.

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IOU: $\mathbb{Z} \curvearrowright X$ cpt

$x \in X$

$n_k - m_k \to +\infty$

$$\left( \forall k, \; \nu_k := \operatorname*{avg}_{m_k \le i \le n_k} \delta_{i.x} \right)$$

subseq $\nu_k \to \mu$

$\Longrightarrow \mu$ is $\mathbb{Z}$-invariant

<u>Pf</u>: $\mathcal{P} := \{\text{prob msrs on } X\}$

$\mathcal{P} \hookrightarrow (C(X))^*$

$\mathcal{P}$ cpt

$\mathbb{Z} \circlearrowright X, C(X), (C(X))^*, \mathcal{P}$

Fix $j \in \mathbb{Z}$

<u>Want</u>: $(j \cdot \nu_k) - \nu_k \to 0$

Say $j > 0$

$\boxed{\forall k}$

$t_k := n_k - m_k + 1 \longrightarrow +\infty$

$$\nu_k = \frac{1}{t_k} \sum_{i=m_k}^{n_k} \delta_{i.x}$$

$$j.\nu_k = \frac{1}{t_k} \sum_{i=m_k+j}^{n_k+j} \delta_{i.x}$$

$$t_k \left[ (j.\nu_k) - \nu_k \right] =$$

$$\sum_{i=n_k+1}^{n_k+j} \cdots - \sum_{i=m_k}^{m_k+j-1} \cdots$$

$$Q := \underbrace{P + \cdots + P}_{j \text{ times}} \subseteq C(X)^*$$

---

$\forall k,$

$$t_k\left[(j \cdot \nu_k) - \nu_k\right] \in Q - Q$$

---

$P$ cpt $\therefore$ $Q - Q$ cpt

---

$$t_k \longrightarrow +\infty$$

---

$$(j \cdot \nu_k) - \nu_k \longrightarrow 0$$

QED

<u>Cor.</u> $\mathbb{Z}$ is amenable, i.e., $\forall \mathbb{Z} \circlearrowright X$ cpt $\exists \mathbb{Z}$-invariant prob. msr. on $X$

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<u>Rmk.</u> $n_k - m_k \longrightarrow \infty$

$\therefore \mathbb{Z} \cap [m_k, n_k]$ is a "Følner sequence"

# Multiple Recurrence

$X$ std prob space

($T_1, \dots, T_\ell : X \longrightarrow X$ ~~invertible~~ not needed

commute, msr. pres)

$Y \subseteq X$ has pos. msr.

$\Rightarrow \exists\, b \neq 0 \ni:$

$Y \cap (T_1^b Y) \cap \dots \cap (T_\ell^b Y)$

has pos. msr.

# Poincaré Recurrence

$X$ (std) prob. space

$T : X \to X$ ~~invertible~~ msr pres

not needed

$Y \subseteq X$ has pos msr

$\Rightarrow \exists b \neq 0 \ni:$

$Y \cap T^b Y$ pos msr.

---

Pf: $\exists i \neq j \ni: T^i Y \cap T^j Y$ pos msr

$Y \cap T^{j-i} Y$ pos. msr.

QED

# Sketch pf of MR

Factor of $X$:

① msr sp. $X \subseteq \mathbb{R}$

② commuting, msr. pres.

$\overline{T}_1, \dots, \overline{T}_\ell : \overline{X} \to \overline{X}$

③ msr. pres. $\overline{\Phi} : X \to \overline{X}$

$\ni: \forall i, \ \overline{\Phi} \circ T_i = \overline{T}_i \circ \overline{\Phi}$

Poset: $\check{X} \leq \bar{X}$ means

$\exists\ \Phi : \bar{X} \longrightarrow \check{X}$

$\ni \forall i,\ \Phi \circ \bar{T}_i = \check{T}_i \circ \Phi$

& $\ni$: $\Phi \circ \bar{\Phi} = \check{\Phi}$

---

$\check{X} \approx \bar{X}$ means

$\exists$ a.e. bijective $\Phi$ ...

---

$\check{X} < \bar{X}$ means

$\check{X} \leq X$ & $\check{X} \not\approx X$

$\mathfrak{M} := \{\text{MR factors of } X\}$

point $\in \mathfrak{M}$

Any chain in $\mathfrak{M}$ has an upper bound

No max elt of $\mathfrak{M}$ is $< X$, i.e.

$\check{X}$ MR, $\check{X} < X$

$\Rightarrow \exists\, \overline{X}$ MR $\ni: \check{X} < \overline{X}$

Can develop a
structure thm
for lcsc gp actions
$\approx$ composition series
in abstract gp
theory

Zimmer, Furstenberg

## Summary:

pos upper density
$\Rightarrow$ point in $\{0,1\}^{\mathbb{Z}}$
and Følner seq
$\Rightarrow$ invariant msr.

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multiple recurrence
on $\{\phi \ni : \phi(0)=1\}$
$\Rightarrow$ arb. long arith. progressions

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cf. General Theme

<u>General Theme</u>: Given an object $S$ with known properties and desired properties,

- find dyn. system $X$

  $\ni: \; S \in X$

- prove

  1. known $\Rightarrow$ $X$ is "good"
  2. $X$ is "nice" $\Rightarrow$ desired
  3. good $\Rightarrow$ nice