

©

General Theme: Given

an object S with
known properties and
desired properties,

- find dyn. system X

$$\ni S \in X$$

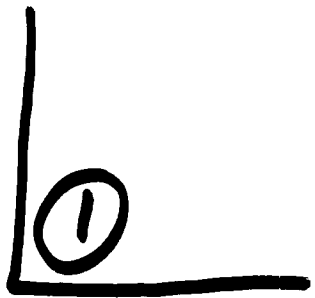
- prove

1. known $\Rightarrow X$ is "good"

2. X is "nice" \Rightarrow desired

3. good \Rightarrow nice

Thm. (Szemerédi)



✓ $S \subseteq \mathbb{Z}$ pos. upper density

✓ $l \in \mathbb{Z}$ pos. \implies

$S \ni$ an arith. seq. of length l

Fix S, l

✓ Want: $\exists j \exists b \neq 0 \ni$:

$j, j-b, j-2b, \dots, j-lb \in S$

②

S pos. upper density

Choose seqs m_k, n_k

✓ $\ni: n_k - m_k \rightarrow +\infty$

$\& \ni: \inf_k \frac{\# S_k}{n_k - m_k + 1} > 0$

✓ where $S_k := S \cap [m_k, n_k]$

✓ $\varepsilon := \inf_k \frac{\# S_k}{n_k - m_k + 1} > 0$

(e.g. $S = 2\mathbb{Z}$, $\varepsilon = 1/2$)

$$\chi := \chi_S : \mathbb{Z} \rightarrow \{0, 1\}$$

✓
$$\chi(i) = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{o.w.} \end{cases}$$

✓
$$\chi \in \{0, 1\}^{\mathbb{Z}} =: X$$

✓
$$\mathbb{Z} \curvearrowright X, (t \cdot \varphi)(i) = \varphi(i+t)$$

✓
$$Y := \{\varphi \in X \mid \exists: \varphi(0) = 1\}$$

④

$$i \in S \iff \chi(i) = 1$$

$$\iff (i, \chi)(0) = 1$$

$$\iff i, \chi \in Y$$

✓

5

$$X = \{0, 1\}^{\mathbb{Z}} \quad \text{cpt}$$

$$P := \{\text{prob. meas on } X\}$$

$$P \longleftrightarrow (C(X))^*$$

$$P \quad \text{cpt, cvx}$$

$$\forall \varphi \in X,$$

6

$$\delta_{\varphi} = \left(\begin{array}{l} \text{pt mass} \\ \text{at } \varphi \end{array} \right) \in \mathcal{P}$$

$$\delta_{\varphi}(Y) = \begin{cases} 1 & \text{if } \varphi \in Y \\ 0 & \text{o.w.} \end{cases}$$

$$i \in S \iff i.\chi \in Y$$

$$\iff \delta_{i.\chi}(Y) = 1$$

$$i \notin S \iff \delta_{i.\chi}(Y) = 0$$

7

✓ $\forall k, \gamma_k := \text{avg}_{m_k \leq i \leq n_k} \delta_{i,\gamma}$

$$= \frac{1}{n_k - m_k + 1} \left[\sum_{i=m_k}^{n_k} \delta_{i,\gamma} \right]$$

$\forall k, \gamma_k(Y) =$

$$\frac{1}{n_k - m_k + 1} \left[\#S_k \right] \geq \epsilon$$

$$\forall k, \nu_k \in \mathcal{P}$$

Subseq. $\nu_k \rightarrow \mu$

$$\mu(Y) \geq \varepsilon > 0$$

$$n_k - m_k \rightarrow +\infty$$

$\lim_{k \rightarrow \infty} \mu$ is \mathbb{Z} -invariant
↑
IOU

⑨

Want: $\exists j \exists b \neq 0 \exists:$
 $j, j-b, j-2b, \dots, j-lb \in S$

$$i \in S \iff i.\chi \in Y$$

$$j-i \in S \iff (j-i).\chi \in Y$$

$$\iff j.\chi \in i.Y$$

$$\iff \delta_{j.\chi}(i.Y) > 0$$

Want: $\exists j, \exists b \neq 0 \exists$:

$$\delta_{j,x}(Y) > 0,$$

$$\delta_{j,x}(b.Y) > 0,$$

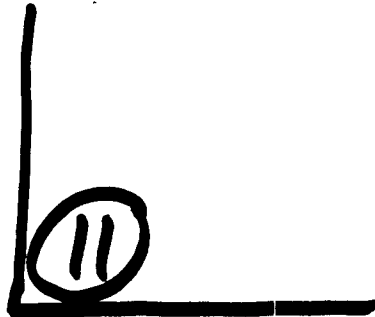
$$\delta_{j,x}(2b.Y) > 0,$$

⋮

$$\delta_{j,x}(qb.Y) > 0$$

$$\forall b, Y_b :=$$

$$Y \cap (b.Y) \cap \dots \cap (qb.Y)$$



Want: $\exists j \exists b \neq 0$

$$\Rightarrow: \delta_{j \cdot \gamma} (Y_b) > 0$$

$\forall k,$
 $v_k = \text{avg}_{m_k \leq j \leq n_k} \delta_{j \cdot \gamma}$

Want: $\exists k \exists b \neq 0$

$$\Rightarrow: v_k (Y_b) > 0$$

$$\nu_k \longrightarrow \mu$$

Want: $\exists b \neq 0 \exists$:

$$\mu(Y_b) > 0$$

$$T_1, \dots, T_q : X \longrightarrow X$$

$$T_1(x) = l \cdot x, \dots, T_q(x) = l \cdot x$$

T_1, \dots, T_q commute,
preserve μ

$$\forall b, Y_b =$$

$$Y \cap (b.Y) \cap \dots \cap ((2b).Y)$$

$$\forall b, Y_b =$$

$$Y \cap (T_1^b Y) \cap \dots \cap (T_2^b Y)$$

Want: $\exists b \neq 0 \ni$:

$$Y \cap (T_1^b Y) \cap \dots \cap (T_2^b Y)$$

has pos. μ -msn.

Multiple Recurrence

14

X std prob space

$(T_1, \dots, T_\ell: X \rightarrow X)$
commute, msr. pres

$Y \subseteq X$ has pos. msr

$\Rightarrow \exists b \neq 0 \quad \exists:$

$Y \cap (T_1^b Y) \cap \dots \cap (T_\ell^b Y)$

has pos. msr

IOU: $\mathbb{Z} \curvearrowright X$ cpt

$$\gamma \in X$$

$$n_k - m_k \rightarrow +\infty$$

$$\left(\forall k, \nu_k := \text{avg}_{m_k \leq i \leq n_k} \delta_{i, \gamma} \right)$$

$$\text{subseq } \nu_k \rightarrow \mu$$

$\Rightarrow \mu$ is \mathbb{Z} -invariant

16

Pf: $\mathcal{P} := \{\text{prob msrs on } X\}$

$$\mathcal{P} \longleftrightarrow (C(X))^*$$

\mathcal{P} cpt

$$\mathbb{Z} \ni j, C(X), (C(X))^*, \mathcal{P}$$

Fix $j \in \mathbb{Z}$

Want: $(j \cdot \nu_k) - \nu_k \rightarrow 0$

Say $j > 0$

$\forall k$

(17)

$$t_k := n_k - m_k + 1 \rightarrow +\infty$$

$$v_k = \frac{1}{t_k} \sum_{i=m_k}^{n_k} \delta_{i,x}$$

$$j.v_k = \frac{1}{t_k} \sum_{i=m_k+j}^{n_k+j} \delta_{i,x}$$

$$t_k [(j.v_k) - v_k] =$$

$$\sum_{i=n_k+1}^{n_k+j} \dots - \sum_{i=m_k}^{m_k+j-1} \dots$$

$$\mathcal{Q} := \underbrace{\mathcal{P} + \dots + \mathcal{P}}_{j \text{ times}} \subseteq C(X)^*$$

18

$$\forall k, \quad t_k [(j \cdot \nu_k) - \nu_k] \in \mathcal{Q} - \mathcal{Q}$$

$$\mathcal{P}_{cpt} \therefore \mathcal{Q} - \mathcal{Q} \text{ cpt}$$

$$t_k \longrightarrow +\infty$$

$$(j \cdot \nu_k) - \nu_k \longrightarrow 0$$

QED

Cor. \mathbb{Z} is amenable,
i.e., $\forall \mathbb{Z}G \curvearrowright X$ cpt
 $\exists \mathbb{Z}$ -invariant probs.
msl. on X

Rmk. $n_k - m_k \rightarrow \infty$
 $\therefore \mathbb{Z} \cap [m_k, n_k]$ is
a "Følner sequence"

Multiple Recurrence

14

X std prob space

$(T_1, \dots, T_\ell: X \rightarrow X$ invertible
commute, MSR. pres)

not needed

$Y \subseteq X$ has pos. MSR

$\Rightarrow \exists b \neq 0 \quad \exists:$

$Y \cap (T_1^b Y) \cap \dots \cap (T_\ell^b Y)$

has pos. MSR

Poincaré Recurrence

20

X ^(std) prob. space

$T: X \rightarrow X$ invertible m.s.r. pres

not needed

$Y \subseteq X$ has pos m.s.r.

$\Rightarrow \exists b \neq 0 \ni$

$Y \cap T^b Y$ pos m.s.r.

Pf: $\exists i \neq j \ni T^i Y \cap T^j Y$
pos m.s.r.

$Y \cap T^{j-i} Y$ pos. m.s.r.
QED

21

Sketch pf of MR

Factor of X :

① msr sp. $X \subseteq \mathbb{R}$

② commuting, msr. pres.

$$\overline{T}_1, \dots, \overline{T}_2: \overline{X} \rightarrow \overline{X}$$

③ msr. pres. $\overline{\Phi}: X \rightarrow \overline{X}$

$$\exists: \forall i, \overline{\Phi} \circ \overline{T}_i = \overline{T}_i \circ \overline{\Phi}$$

Poset: $\check{X} \leq \bar{X}$ means

$$\exists \Psi: \bar{X} \rightarrow \check{X}$$

$$\exists: \forall i, \Psi \circ \bar{\tau}_i = \check{\tau}_i \circ \Psi$$

$$\& \exists: \Psi \circ \bar{\Phi} = \check{\Phi}$$

$\check{X} \approx \bar{X}$ means

\exists a.e. bijective $\Psi \dots$

$\check{X} < \bar{X}$ means

$$\check{X} \leq \bar{X} \quad \& \quad \check{X} \not\approx \bar{X}$$

$\mathcal{M} := \{MR \text{ factors of } X\}$

point $\in \mathcal{M}$

Any chain in \mathcal{M}
has an upper bound

No max elt of \mathcal{M}
is $< X$, i.e.

$\forall MR, \forall X < X$
 $\Rightarrow \exists \bar{X} MR$
 $\exists: \forall X < \bar{X}$

Can develop a
structure then
for lscse gp actions
 \approx composition series
in abstract gp
theory

Zimmer, Furstenberg

25

Summary:

pos upper density
 \Rightarrow point in $\{0,1\}^{\mathbb{Z}}$
and Følner seq
 \Rightarrow invariant msr.

multiple recurrence
on $\{\phi \in \{0,1\}^{\mathbb{Z}} : \phi(0) = 1\}$
 \Rightarrow arb. long arith.
progressions

cf. General Theme

General Theme: Given
an object S with
known properties and
desired properties,

- find dyn. system X
 $\ni S \in X$

- prove

1. known $\Rightarrow X$ is "good"

2. X is "nice" \Rightarrow desired

3. good \Rightarrow nice