SOLUTIONS TO PRACTICE TEST 4

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52. Consider the following system of linear equations over the real numbers, where x, y and z are variables and b is a real constant.

$$\begin{array}{ccccccc} x & + & y & + & z & = & 0 \\ x & + & 2y & + & 3z & = & 0 \\ x & + & 3y & + & bz & = & 0 \end{array}$$

Which of the following statements are true?

I. There exists a value of b for which the system has no solution.
II. There exists a value of b for which the system has exactly one solution.
III. There exists a value of b for which the system has more than one solution.

(A) II only
(B) I and II only
(C) I and III only
(D) II and III only
(E) I, II and III

Solution: Let

$$M := \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & b \end{bmatrix} \quad \text{and} \quad v := \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad \mathbf{0} := \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Then the given system can be written, in matrix form, as $Mv = \mathbf{0}$.

Since, for every $b \in \mathbb{R}$, $x = y = z = 0$ is a solution to $Mv = \mathbf{0}$, it follows that I is false. For every $b \in \mathbb{R}$,

$$\text{the solution } x = y = z = 0 \text{ is the } \textbf{only} \text{ solution of } Mv = \mathbf{0}$$

iff

$$\det M = 0.$$

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For every $b \in \mathbb{R}$, by expanding $\det M$ along the third row of M, we get

$$\begin{aligned} \det M &= 1 \cdot \det \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} - 3 \cdot \det \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} + b \cdot \det \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \\ &= 1 \cdot 1 - 3 \cdot 2 + b \cdot 1 \;=\; b - 5. \end{aligned}$$

Therefore, if $b = 5$, then $Mv = \mathbf{0}$ has more than one solution. Moreover, if $b \neq 5$, then $Mv = \mathbf{0}$ has exactly one solution. Thus II and III are both true. Answer: (D) $\square$

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53. In the complex plane, let C be the circle $|z| = 2$ with positive (counterclockwise) orientation. Then $\displaystyle\int_C \frac{dz}{(z-1)(z+3)^2} =$

(A) 0
(B) $2\pi i$
(C) $\dfrac{\pi i}{2}$
(D) $\dfrac{\pi i}{8}$
(E) $\dfrac{\pi i}{16}$

Solution: Define $g : \mathbb{C}\backslash\{-3\} \to \mathbb{C}$ by $g(z) = \dfrac{1}{(z+3)^2}$. We wish to compute $\displaystyle\int_C \frac{g(z)}{z-1}\, dz$.

Let $c_0 := g(1)$, $c_1 := g'(1)$, $c_2 := \dfrac{g''(1)}{2!}$, …be the Taylor coefficients of g at 1. As g is holomorphic on $\mathbb{C}\backslash\{-3\}$, it follows that g is holomorphic at 1. So choose a neighborhood N in $\mathbb{C}$ of 1 such that, for all $z \in N$, $g(z) = \displaystyle\sum_{k=0}^{\infty} c_k (z-1)^k$. Let C' be a circle in the complex plane, centered at 1, contained in N, with positive orientation. By Cauchy's Theorem, $\displaystyle\int_C \frac{g(z)}{z-1}\, dz = \int_{C'} \frac{g(z)}{z-1}\, dz$. Then $\displaystyle\int_C \frac{g(z)}{z-1}\, dz = \int_{C'} \left[\sum_{k=0}^{\infty} c_k (z-1)^{k-1}\right] dz$. Splitting off the first term,

$$\int_C \frac{g(z)}{z-1}\, dz = \left(\int_{C'} \frac{c_0}{z-1}\, dz\right) + \left(\int_{C'} \left[\sum_{k=1}^{\infty} c_k (z-1)^{k-1}\right] dz\right).$$

By Cauchy's Theorem, $\int_{C'} \left[\sum_{k=1}^{\infty} c_k (z-1)^{k-1} \right] dz = 0$; and note that this sum starts with $k = 1$, not $k = 0$. Then $\int_C \frac{g(z)}{z-1}\, dz = \int_{C'} \frac{c_0}{z-1}\, dz$. Parametrization of C' and computation gives $\int_{C'} \frac{1}{z-1}\, dz = 2\pi i$. We have $c_0 = g(1) = \frac{1}{(1+3)^2} = \frac{1}{16}$. Then $\int_{C'} \frac{c_0}{z-1}\, dz = 2\pi i c_0 = \frac{\pi i}{8}$.

Then $\int_C \frac{g(z)}{z-1}\, dz = \int_{C'} \frac{c_0}{z-1}\, dz = \frac{\pi i}{8}$. Answer: (D) $\square$

Alternate solution: The integrand is holomorphic inside C, except at $z = 1$. The winding number of C around $z = 1$ is equal to 1. The residue of $\frac{1}{(z-1)(z+3)^2}$ at $z = 1$ is $\frac{1}{(1+3)^2} = \frac{1}{16}$. So, by the Residue Theorem, the integral is $[1] \left[\frac{1}{16} \right] [2\pi i] = \frac{\pi i}{8}$. Answer: (D) $\square$

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54. The inside of a certain water tank is a cube measuring 10 feet on each edge and having vertical sides and no top. Let $h(t)$ denote the water level, in feet, above the floor of the tank at time t seconds. Starting at time $t = 0$, water pours into the tank at a constant rate of 1 cubic foot per second, and, simultaneously, water is removed from the tank at $0.25\,[h(t)]$ cubic feet per second. As $t \to \infty$, what is the limit of the volume of the water in the tank?

(A) 400 cubic feet
(B) 600 cubic feet
(C) $1,000$ cubic feet
(D) The limit does not exist.
(E) The limit exists, but it cannot be determined without knowing $h(0)$.

Solution: From the information in the problem, on $t > 0$, we have: $h'(t) = 1 - [1/4][h(t)]$. At time t, the water is in the shape of a 10 by 10 by $h(t)$ box, so its volume is $[10][10][h(t)]$. We therefore wish to compute $\lim_{t\to\infty} ([10][10][h(t)])$, which is equal to $100 \cdot \left[\lim_{t\to\infty} (h(t)) \right]$.

The "multiplier" for the ODE $h'(t) = 1 - [1/4][h(t)]$ is $e^{t/4}$. So, on $t \geq 0$, let's define $f(t) = [e^{t/4}][h(t)]$.

On $t > 0$, using the product rule, and $h'(t) = 1 - [1/4][h(t)]$,

$$\begin{aligned} f'(t) &= [e^{t/4}][1/4][h(t)] + [e^{t/4}][h'(t)] \\ &= [e^{t/4}][1/4][h(t)] + [e^{t/4}][1 - [1/4][h(t)]] \\ &= [e^{t/4}][1/4][h(t)] + [e^{t/4}] - [e^{t/4}][1/4][h(t)] \;=\; e^{t/4}. \end{aligned}$$

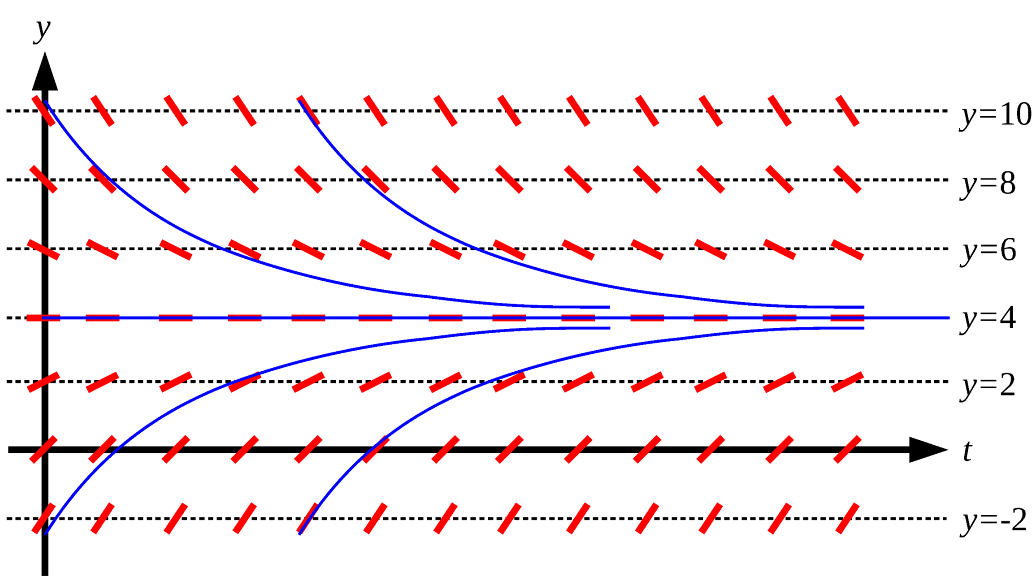
Antidifferentiating to get $f(t)$, fix a constant C such that, on $t \geq 0$, we have $f(t) = 4e^{t/4} + C$. Then $h(t) = [e^{-t/4}][f(t)] = 4 + Ce^{-t/4}$. Then $\lim_{t\to\infty} (h(t)) = 4$, so $100 \cdot \left[\lim_{t\to\infty} (h(t))\right] = 400$. Answer: (A) □

Alternate solution: From the information in the problem, on $t > 0$, we have: $h'(t) = 1 - [1/4][h(t)]$. At time t, the water is in the shape of a 10 by 10 by $h(t)$ box, so its volume is $[10][10][h(t)]$. We therefore wish to compute $\lim_{t\to\infty} ([10][10][h(t)])$, which is equal to $100 \cdot \left[\lim_{t\to\infty} (h(t))\right]$.

Define $F : \mathbb{R} \to \mathbb{R}$ by $F(y) = 1 - [1/4]y$. Then, on $t > 0$, we have $h'(t) = F(h(t))$. We now form a "line field" (also known as a "slope field" or a "direction field") by:

- forming a ty-coordinate plane, with t on the horizontal axis and y on the vertical axis,
- choosing several points $q_i := (t_i, y_i)$ on the plane and
- at each point q_i drawing a small line segment that passes through q_i and has slope $F(y_i)$.

This line field is visualized by the 91 red line segments appearing below.



For example, $F(2) = 1/2$, and, along the dashed line $y = 2$, there are 13 red line segments, each of which has slope $1/2$.

Each blue curve (including straight curve on $y = 4$) satisfies:

at each if its points, the curve is tangent to the line field.

Rephrasing this geometric property in a symbolic way, each blue curve

is a graph of a solution to $h'(t) = F(h(t))$.

The geometry drives the conclusion that each solution has a graph is asymptotic, as $t \to \infty$, to the line $y = 4$. That is, each solution h satisfies $\lim\limits_{t\to\infty} (h(t)) = 4$, so $100 \cdot \left[\lim\limits_{t\to\infty} (h(t))\right] = 400$. Answer: (A) $\square$

NOTE: For ODEs of the form $h'(t) = G(t, h(t))$, where G is a function of two variables, we may understand the behavior of solutions by visualizing line fields and then "flowing" along them. This translates an ODE problem into one where you can use your geometric acuity.

NOTE: Many ODEs have only finitely many constant (or "steady state") solutions, and, in some nice situations, every solution tends (as time $\to \infty$) toward one of those. Such steady state solutions h are constant, and so satisfy $h' = 0$. So, to find them, we set h' to zero and solve for h. In this problem, if we set h' to zero in $h'(t) = 1-[1/4][h(t)]$, we get $0 = 1 - [1/4][h(t)]$, and solving this for $h(t)$ gives $h(t) = 4$. So the only steady state solution is given by: On $t \geq 0$, $h(t) = 4$.

If we somehow know that each solution of $h'(t) = 1-[1/4][h(t)]$ tends toward steady state, then we'll solve this problem very quickly. In this context, it's worth noting that, if a and b are contants, and $b \neq 0$, then $h'(t) = a + b \cdot [h(t)]$ will have exactly one steady state solution. If, moreover, $b < 0$, then all solutions will tend toward that steady state solution as $t \to \infty$. In this problem, $a = 1$ and $b = -1/4$.

Note that, if we were studying $h'(t) = -1 + [1/4][h(t)]$, then, the line field for **that** ODE would be similar to the figure shown above, except that each red line segment would be "flipped" vertically. Then those above $y = 4$ would have positive slope, while those below would have negative slope. In that case, solutions would flow asymptotically toward steady state in **negative** time (*i.e.*, as $t \to -\infty$). In positive time, as $t \to \infty$, they would flow **away from** steady state.

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55. Suppose that f is a twice-differentiable function on the set of real numbers and that $f(0)$, $f'(0)$ and $f''(0)$ are all negative. Suppose f'' has all three of the following properties:

I. It is increasing on the interval $[0, \infty)$.
II. It has a unique zero in the interval $[0, \infty)$.
III. It is unbounded on the interval $[0, \infty)$.

Which of the same three properties does f necessarily have?

(A) I only
(B) II only
(C) III only
(D) II and III only
(E) I, II and III

Solution: As f'' has Property I, we see that $f''(0)$ is a lower bound for f'' on $[0, \infty)$. Since f'' is bounded below on $[0, \infty)$ and since f'' has Property III, we see that f'' is not bounded above on $[0, \infty)$. Fix $a > 0$ such that $f''(a) > 0$. Let L be the tangent line to the graph of $y = f'(x)$ at $x = a$. Then L has positive slope.

Because f'' has Property I, it follows that f' is concave up on $[0, \infty)$. Then, on $x > 0$, the graph of $y = f'(x)$ lies above L. So, because L has positive slope, it follows that f' is not bounded above on $[a, \infty)$. Fix $b > a$ such that $f'(b) > 0$. Let M be the tangent line to $y = f(x)$ at $x = b$. Then M has positive slope.

Because $f''(a) > 0$ and because f'' has Property I, it follows that $f'' > 0$ on $[a, \infty)$. Then f is concave up on $[a, \infty)$. Then, on $x > a$, the graph of $y = f(x)$ lies above M. As M has positive slope, we see that f is not bounded above on $[b, \infty)$. Fix $c > b$ such that $f(c) > 0$.

Since f'' has Property II, choose $t \in (0, \infty)$ such that $f''(t) = 0$. Then, because f'' has Property I, we see that $f'' < 0$ on $(0, t)$ and that $f'' > 0$ on (t, ∞). Then f' is decreasing on $[0, t]$ and increasing on $[t, \infty)$. Because $f'(0) < 0$ and because f' is decreasing on $[0, t]$, it follows that $f' < 0$ on $[0, t]$. Then f is decreasing on $[0, t]$. So, since $f(0) < 0$, it follows that $f < 0$ on $[0, t]$. Then f has no zeroes on $[0, t]$.

Claim: The function f has at most one zero on $[0, \infty)$. *Proof of claim:* Let $u, v \in (t, \infty)$ and assume that $u < v$ and $f(u) = f(v) = 0$. We aim for a contradiction.

Since f has no zeroes on $[0,t]$, we see that $t<u$. Because $f''>0$ on (t,∞), it follows that f is concave up on $[t,\infty)$. Then, on $t<x<v$, the graph of $y=f(x)$ lies below the secant line segment S from $(t,f(t))$ to $(v,f(v))$. Let p be the intersection of S with the line $x=u$. Then, as $t<u<v$, the point $(u,f(u))$ lies below p. Because $f(t)<0=f(v)$, we see that the point p lies stricly below the x-axis. Therefore, the point $(u,f(u))$ lies strictly below the x-axis. Then $f(u)<0$. However, $f(u)=0$, contradition. *End of proof of claim.*

Because $f'(0)<0$, it follows that f is NOT increasing on $[0,\infty)$. That is, f does NOT have Property I. Since $f(0)<0$ and $f(c)>0$, we conclude, from the Claim, that f has exactly one zero in $[0,\infty)$. That is, f has Property II. Since f is not bounded above on $[b,\infty)$, it follows that f has Property III. Answer: (D) □

Alternate solution:

Lemma: Let $g:\mathbb{R}\to\mathbb{R}$ be differentiable and let $a\geq 0$. Assume that $g(0)<0$, and that $g'<0$ on $[0,a]$ and that g' is increasing and unbounded on $[a,\infty)$. Then there exists $b>0$ such that $g<0$ on $[0,b]$ and such that g is increasing and unbounded on $[b,\infty)$.

Proof of Lemma: Let $S:=\{x\in[a,\infty)\,|\,g'(x)\leq 0\}$.

Since g' is increasing on $[a,\infty)$, it follows that $g'(a)$ is a lower bound for g' on $[a,\infty)$. So, as g' is unbounded on $[a,\infty)$, we see that g' has no upper bound on $[a,\infty)$. Choose $c\in[a,\infty)$ such that $g'(c)>0$. Then, because g' is increasing on $[a,\infty)$, it follows that $g'>0$ on $[c,\infty)$. Then c is an upper bound for S. We have $g'(a)<0$, so $a\in S$, so $S\neq\emptyset$. Let b be the supremum of S.

Then $g'<0$ on $[a,b)$ and $g'>0$ on (b,∞). Since $g'<0$ on $[0,a]$ and on $[a,b)$, we see that $g'<0$ on $[0,b)$, and so g is decreasing on $[0,b]$. So, since $g(0)<0$, we conclude that $g<0$ on $[0,b]$. Since $g'>0$ on (b,∞), it follows that g is increasing on $[b,\infty)$. It remains to show that g is unbounded on $[b,\infty)$.

Let L be the tangent line to $y=g(x)$ at $x=c$. Since $g'(c)>0$, it follows that L has positive slope. Since g' is increasing on $[a,\infty)$, we see that g is concave up on $[a,\infty)$. So, on $x>a$, the graph of $y=g(x)$ lies above L. So, since L has positive slope, g is unbounded on $[c,\infty)$, and, therefore, on $[b,\infty)$. *End of proof of lemma.*

Because $f'(0) < 0$, it follows that f is NOT increasing on $[0, \infty)$. That is, f does NOT have Property I. By the Lemma (using f' for g, and 0 for a, and r for b), choose $r > 0$ such that $f' < 0$ on $[0, r]$ and such that f' is increasing and unbounded on $[r, \infty)$. By the Lemma (using f for g, and r for a, and s for b), choose $s > 0$ such that $f < 0$ on $[0, s]$ and such that f is increasing and unbounded on $[s, \infty)$. As f is increasing and unbounded on $[s, \infty)$, it follows that f has a unique zero on $[s, \infty)$. So, since $f < 0$ on $[0, s]$, we see that f has Property II. Finally, since f is unbounded on on $[s, \infty)$, and, therefore, on $[0, \infty)$, we see that f has Property III. Answer: (D) $\square$

NOTE: The fastest way to solve this problem is to draw the graph of a possible f'', taking care to incorporate Properties I, II and III, into the graph, and taking care, also, to incorporate $f''(0) < 0$. Then sketch f', taking care to incorporate $f'(0) < 0$. Then sketch f taking care to incorporate $f(0) < 0$. It should quickly become clear that, for f, any attempt to make I hold is impossible, and that any attempt to make II or III fail is impossible. The arguments given above simply amount to saying why these things are impossible.

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56. For every nonempty set S and every metric d on S, which of the following is a metric on S?

(A) $4 + d$
(B) $e^d - 1$
(C) $d - |d|$
(D) d^2
(E) $\sqrt{d}$

Solution: For any $x \in S$, $(4 + d)(x, x) = 4 \neq 0$. Then $4 + d$ cannot be a metric on S. Thus (A) is not correct.

Assume that S has at least two points. Let $a, b \in S$ be distinct. Then $d(a, b) \geq 0$, so $|d(a, b)| = d(a, b)$. Then

$$(d - |d|)(a, b) \quad = \quad d(a, b) \; - \; |d(a, b)| \quad = \quad 0.$$

Then $d - |d|$ cannot be a metric on S. Thus (C) is not correct.

Let $S = \mathbb{R}$ and let $d : S \times S \to [0, \infty)$ be the metric defined by $d(x, y) = |x - y|$. Define $\phi : S \times S \to [0, \infty)$ by $\phi(x, y) = e^{d(x,y)} - 1$. We have $\phi(1, 2) = e - 1$, $\phi(2, 3) = e - 1$ and $\phi(1, 3) = e^2 - 1$. Since

$e^2 > 2e - 1$, we get $e^2 - 1 > (e-1) + (e-1)$, and so

$$\phi(1,3) \qquad > \qquad [\phi(1,2)] \quad + \quad [\phi(2,3)],$$

and so ϕ is not a metric on S. Thus (B) is not correct.

Let $S = \mathbb{R}$ and let $d : S \times S \to [0, \infty)$ be the metric defined by $d(x,y) = |x - y|$. Define $\psi : S \times S \to [0, \infty)$ by $\psi(x,y) = [d(x,y)]^2$. We have $\psi(1,2) = 1$, $\psi(2,3) = 1$ and $\psi(1,3) = 4$. Since $4 > 1 + 1$,

$$\psi(1,3) \qquad > \qquad [\psi(1,2)] \quad + \quad [\psi(2,3)],$$

and so ψ is not a metric on S. Thus (D) is not correct.

Lemma 1: Let $f : [0, \infty) \to [0, \infty)$ be concave down. Assume that $f(0) = 0$. Then f is subadditive.

Proof of Lemma 1: By definition of "subadditive", let $a, b \geq 0$ be given, and we wish to prove that $f(a+b) \leq [f(a)] + [f(b)]$.

This is clear if $b = 0$. We therefore assume that $b > 0$. Let L be the secant line between $(0, f(0))$ and $(b, f(b))$. Let M be the secant line between $(a, f(a))$ and $(a+b, f(a+b))$. Since f is concave down, the slope of M is less than or equal to the slope of L. That is,

$$\frac{[f(a+b)] - [f(a)]}{(a+b) - a} \leq \frac{[f(b)] - [f(0)]}{b - 0}.$$

So, since $f(0) = 0$, since $(a+b) - a = b$ and since $b - 0 = b$, we get

$$\frac{[f(a+b)] - [f(a)]}{b} \leq \frac{f(b)}{b}.$$

Multiplying by b gives $[f(a+b)] - [f(a)] \leq f(b)$. It then follows that $f(a+b) \leq [f(a)] + [f(b)]$, as desired. *End of proof of Lemma 1.*

Lemma 2: Let S be a set and let $d : S \times S \to [0, \infty)$ be a metric on S. Let $f : [0, \infty) \to [0, \infty)$ be subadditive and nondecreasing. Assume, for all $t \in [0, \infty)$, that:

$$[\ f(t) \ = \ 0\ ] \qquad \Leftrightarrow \qquad [\ t \ = \ 0\ ].$$

Then $f \circ d : S \times S \to [0, \infty)$ is a metric on S.

Proof of Lemma 2: We must prove that $f \circ d$ separates points, is symmetric and satisifies the triangle inequality. For all $x, y \in S$, we have: $f(d(x,y)) = 0$ iff $d(x,y) = 0$ iff $x = y$. Thus $f \circ d$ separates points. For all $x, y \in S$, we have: $f(d(x,y)) = f(d(y,x))$, so $f \circ d$ is symmetric. It remains to prove the triangle inequality.

For all $x, y, z \in S$, we have $d(x, z) \leq [d(x, y)] + [d(y, z)]$, so, as f is nondecreasing, we get $f(d(x, z)) \leq f([d(x, y)] + [d(y, z)])$, and, by subadditivity of f, we get

$$f(\ [d(x,y)]\ +\ [d(y,z)]\ )\quad \leq \quad [f(d(x,y))]\ +\ [f(d(y,z))],$$

and, together, these imply $f(d(x,z)) \leq [f(d(x,y))] + [f(d(y,z))]$. Thus $f \circ d$ satisfies the triangle inequality. *End of proof of Lemma 2.*

Let d be a metric on a set S. By Lemma 1, $\sqrt{\bullet} : [0, \infty) \to [0, \infty)$ is subadditive. So, by Lemma 2, $\sqrt{d}$ is a metric on S. Thus (E) is correct. Answer: (E) $\square$

NOTE: Basically, one is looking for a function that is concave down and nondecreasing to compose with a metric and have a serious hope of getting another metric. The functions $e^{\bullet} - 1$ and $\bullet^2$ don't have the right concavity. The functions $4 + \bullet$ and $\bullet - |\bullet|$ don't work for even simpler reasons. Only $\sqrt{\bullet}$ is a serious contender, and Lemmas 1 and 2 do show that it works.

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57. Let $\mathbb{R}$ be the field of real numbers and $\mathbb{R}[x]$ the ring of polynomials in x with coefficients in $\mathbb{R}$. Which of the following subsets of $\mathbb{R}[x]$ is a subring of $\mathbb{R}[x]$?

I. All polynomials in $\mathbb{R}[x]$ whose coefficient of x is zero
II. All polynomials in $\mathbb{R}[x]$ whose degree is an even integer, together with the zero polynomial
III. All polynomials in $\mathbb{R}[x]$ whose coefficients are rational numbers

(A) I only
(B) II only
(C) I and III only
(D) II and III only
(E) I, II and III

Solution: Let W_I, W_{II} and W_{III} be the three subsets of $\mathbb{R}[x]$ described in I, II and III, respectively. Let $U := \{1, x, x^2, x^3, x^4 \ldots\}$ be the set of monomials in x. Then $\mathbb{R}[x]$ is the $\mathbb{R}$-span of U.

Let $U_I := U \backslash \{x\}$, so $U_I = \{1, x^2, x^3, x^4, \ldots\}$. Then W_I is the $\mathbb{R}$-span of U_I. Since U_I is closed under multiplication, so is W_I. Since W_I is closed under finite $\mathbb{R}$-linear combinations, it follows that W_I is closed under subtraction. Then W_I is a subring of $\mathbb{R}[x]$.

We have: $x^2+x \in W_{II}$ and $x^2 \in W_{II}$. However, $(x^2+x)-x^2 = x$, and $x \notin W_{II}$. Thus W_{II} is not closed under subtraction, and is therefore not a subring of $\mathbb{R}[x]$.

Let $\mathbb{Q}$ be the field of rational numbers and $\mathbb{Q}[x]$ the ring of polynomials in x with coefficients in $\mathbb{Q}$. Then $\mathbb{Q}[x]$ is a subring of $\mathbb{R}[x]$. So, since $W_{III} = \mathbb{Q}[x]$, it follows that W_{III} is a subring of $\mathbb{R}[x]$.

Answer: (C) $\square$

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58. Let f be a real-valued function defined and continuous on the set $\mathbb{R}$ of real numbers. Which of the following must be true of the set $S := \{f(c) \,|\, 0 < c < 1\}$?

I. S is a connected subset of $\mathbb{R}$.
II. S is an open subset of $\mathbb{R}$.
III. S is a bounded subset of $\mathbb{R}$.

(A) I only
(B) I and II only
(C) I and III only
(D) II and III only
(E) I, II and III

Solution: The image of a connected set under a continuous function is again connected. So I is true.

If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = (x - (1/2))^2$, then $S = [0, 1/4)$, which is not open. So II is not necessarily true.

Let $T := \{f(c) \,|\, 0 \leq c \leq 1\}$. Since $[0, 1]$ is a compact subset of $\mathbb{R}$ and since the image of a compact set under a continuous function is again compact, it follows that T is compact. Then T is bounded. So, since $S \subseteq T$, it follows that S is bounded.

Answer: (C) $\square$
