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NEW
0520-1. A right triangle is growing. At time $t$, its legs have lengths $3x$ and $6x$, and its area is $A$, so $x$ and $A$ are expressions of $t$. Find a formula for $dA/dt$ in terms of $x$ and $dx/dt$.

**ANSWER:**

$$A = \frac{(3x)(6x)}{2} = 9x^2$$

$$\frac{dA}{dt} = 18x \left[ \frac{dx}{dt} \right]$$
A regular decagon is growing. At time $t$, its area is $A$ and its side length is $s$, so $A$ and $s$ are expressions of $t$. Find a formula for $dA/dt$ in terms of $s$ and $ds/dt$.

**ANSWER:**

Each triangle has area $\left[\frac{\cot(\pi/10)}{4}\right] s^2$.

$$A = 5 \left[\frac{\cot(\pi/10)}{2}\right] s^2$$

$$\frac{dA}{dt} = 5 \left[\cot(\pi/10)\right] s \left[\frac{ds}{dt}\right]$$
A **silo** is a cylinder capped with a hemisphere. A certain silo is growing. At time $t$, it has base radius $x$, height $3x$ and enclosed volume $V$, so $x$ and $V$ are expressions of $t$. (The height of its hemisphere is $x$, and the height of its cylinder is $2x$.) **Find** a formula for $dV/dt$ in terms of $x$ and $dx/dt$.

\[
V = \frac{1}{2} \left( \frac{4}{3} \pi x^3 \right) + (\pi x^2)(2x) \\
= \frac{2}{3} \pi x^3 + \frac{6}{3} \pi x^3 = \frac{8}{3} \pi x^3
\]

\[
\frac{dV}{dt} = \left[ 8\pi x^2 \right] \left[ \frac{dx}{dt} \right]
\]
0520-4. Suppose \(4x^2 + y + 27 = z^4 + 6z^3\) and \(dx/dt = 6\) and \(dy/dt = 8\). Compute \(dz/dt\) at a certain moment when

\[x = 4, \ y = 3\] \ and \ \[z = 2.\]

**ANSWER:** \(4x^2 + y + 27 = z^4 + 6z^3\)

\[8x \frac{dx}{dt} + \frac{dy}{dt} = 4z^3 \frac{dz}{dt} + 18z^2 \frac{dz}{dt}\]

\[8[4][6] + [8] = 4[2^3][?] + 9 \cdot 2[2^2][?]\]

\[8 \cdot 24 + 8 \cdot 1 = 4 \cdot 8[?] + 9 \cdot 8[?]\]

\[24 + 1 = [4 + 9][?]\]

\[? = \frac{24 + 1}{4 + 9} = \frac{25}{13}\]
A streetlight is at the top of a 26 foot pole. A 6 foot tall man walks directly away from the light at a speed of 2 feet per second. How fast is his shadow growing?

**ANSWER:**

\[
\frac{dx}{dt} = 2
\]

\[
\frac{ds}{dt} = ?
\]
0520-5. A streetlight is at the top of a 26 foot pole. A 6 foot tall man walks directly away from the light at a speed of 2 feet per second. How fast is his shadow growing?

**ANSWER:**

\[
\frac{dx}{dt} = 2
\]

\[
\frac{ds}{dt} = ?
\]

By similarity of the green triangles,

\[
\frac{s}{6} = \frac{s + x}{26}.
\]
A streetlight is at the top of a 26 foot pole. A 6 foot tall man walks directly away from the light at a speed of 2 feet per second. How fast is his shadow growing?

**ANSWER:**

\[
\frac{ds}{dt} = \frac{(ds/dt) + (dx/dt)}{6} = \frac{26}{26} = \frac{6? + 2}{6} = \frac{26?}{26}
\]

\[
\frac{dx}{dt} = 2
\]

\[
\frac{ds}{dt} = ?
\]

By similarity of the green triangles,

\[
\frac{s}{6} = \frac{s + x}{26}
\]

\[
? = \frac{12}{20} = \frac{3}{5} \text{ feet/second}
\]
A jet flies in a straight line, with constant speed and altitude. It flies directly over a radar station, and, a few minutes later, the radar's instruments show that the plane is 13 miles away, and that its distance from the radar station is increasing at 600 mph. Assuming that the altitude of the jet is 5 miles greater than that of the station, find the speed of the jet.
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**ANSWER:** \( \left[ \frac{dz}{dt} \right]_{t_0} = 13 \)

\[
\left[ \frac{dz}{dt} \right]_{t_0} = 600
\]

\[
\frac{dx}{dt} = ?
\]

\[
x^2 + 5^2 = z^2
\]
A jet flies in a straight line, with constant speed and altitude. It flies directly over a radar station, and, a few minutes later, the radar’s instruments show that the plane is 13 miles away, and that its distance from the radar station is increasing at 600 mph. 

Assuming that the altitude of the jet is 5 miles greater than that of the station, find the speed of the jet.

**ANSWER:**

\[
\frac{dz}{dt} \bigg|_{t \to t_0} = 13
\]

\[
\frac{dz}{dt} \bigg|_{t \to t_0} = 600
\]

\[
x = \sqrt{z^2 - 5^2}
\]

\[
[x]_{t \to t_0} = \sqrt{13^2 - 5^2} = 12
\]

\[
\frac{dx}{dt} = ?
\]

\[
x^2 + 5^2 = z^2
\]

\[
2x\frac{dx}{dt} + 0 = 2z\frac{dz}{dt}
\]

\[
2[12][?] + 0 = 2[13][600]
\]
A jet flies in a straight line, with constant speed and altitude. It flies directly over a radar station, and, a few minutes later, the radar’s instruments show that the plane is 13 miles away, and that its distance from the radar station is increasing at 600 mph. Assuming that the altitude of the jet is 5 miles greater than that of the station, find the speed of the jet.

**ANSWER:**

\[ 2[12][?] + 0 = 2[13][600] \]

\[ ? = \frac{2[13][600]}{2[12]} = 650 \text{ mph} \]
Water is being drained, at a rate of 6 cubic meters per minute, from a conical container of height 10 meters, whose top is a circle whose radius is 5 meters. When the water level is 4 meters, how fast is the water level decreasing?
Water is being drained, at a rate of 6 cubic meters per minute, from a conical container of height 10 meters, whose top is a circle whose radius is 5 meters. When the water level is 4 meters, how fast is that level decreasing?

**Answer:**

\[ V := \text{volume of the water} \]
\[ r := \text{radius of the top of the water} \]
\[ h := \text{height of the water} \]

\[ \frac{r}{h} = \frac{5}{10} \]

\[ \frac{dV}{dt} = -6 \]

\[ V = \frac{1}{3} \pi r^2 h \]

\[ [h]_{t \rightarrow t_0} = 4 \]

\[ \left[ \frac{dh}{dt} \right]_{t \rightarrow t_0} = -? \]
Water is being drained, at a rate of 6 cubic meters per minute, from a conical container of height 10 meters, whose top is a circle whose radius is 5 meters. When the water level is 4 meters, how fast is that level decreasing?

**ANSWER:**

\[
\frac{r}{h} = \frac{5}{10}, \text{ so } r = \frac{1}{2} h.
\]

\[
V = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi \left(\frac{1}{2} h\right)^2 h = \frac{1}{12} \pi h^3
\]

\[
[h]_{t: \to t_0} = 4
\]

\[
\left[\frac{dh}{dt}\right]_{t: \to t_0} = -?\]

\[
\frac{dV}{dt} = \frac{1}{12} \pi (3h^2) \frac{dh}{dt}
\]

\[-6 = \frac{1}{12} \pi (3(4^2))(-?)\]
Water is being drained, at a rate of 6 cubic meters per minute, from a conical container of height 10 meters, whose top is a circle whose radius is 5 meters. When the water level is 4 meters, how fast is that level decreasing?

**Answer:**

$$-6 = \frac{1}{12} \pi (3(4^2)) (-?)$$

$$\frac{-72}{3\pi(4^2)} = -?$$

$$? = \frac{72}{3\pi(4^2)} \approx 0.4775 \text{ meters per second}$$
A camera at (4, 0) is following a UFO that strafes in from above, following the curve $y = x^2 + 2$ from left to right. At the moment when the UFO is at the point (1, 3), retreating back into outer space, the angle between the camera and the horizontal is increasing at 2 radians per second.

a. What is the rate of change in the $x$-coordinate of the UFO at that moment?

b. What is the rate of change in the $y$-coordinate of the UFO at that moment?
a. What is the rate of change in the \( x \)-coordinate of the the UFO at that moment?

b. What is the rate of change in the \( y \)-coordinate of the the UFO at that moment?

At time \( t_0 \), UFO at \((1, 3)\)

\[
[x]_{t: \rightarrow t_0} = 1
\]
\[
[y]_{t: \rightarrow t_0} = 3
\]

At time \( t_0 \), this angle is increasing at 2 radians per second.

\[
[d\theta/dt]_{t: \rightarrow t_0} = 2
\]

Goal: Compute \( a \) and \( b \).
NEW ANSWER:

\[ \tan \theta = \frac{y}{4-x} \]

\[ [\tan \theta]_{t_0} = \frac{3}{4-1} = 1 \]

\[ [\sec^2 \theta]_{t_0} = [1 + \tan^2 \theta]_{t_0} = 1 + 1^2 = 2 \]

At time \( t_0 \),
- UFO at \((1,3)\)
- Camera at \((4,0)\)

\[ [x]_{t_0} = 1 \]
\[ [y]_{t_0} = 3 \]

\[ a := [dx/dt]_{t_0} \]
\[ b := [dy/dt]_{t_0} \]

Goal: Compute \( a \) and \( b \).
\[
\tan \theta = \frac{y}{4 - x}
\]

\[
\left[ \sec^2 \theta \right]_{t \to t_0} = 2
\]

\[
(\sec^2 \theta) \left( \frac{d\theta}{dt} \right) = \frac{(4 - x)(dy/dt) - (y)(-dx/dt)}{(4 - x)^2}
\]

\[
2(2) = \left[ (\sec^2 \theta) \left( \frac{d\theta}{dt} \right) \right]_{t \to t_0} = \frac{(4 - 1)(b) - (3)(-a)}{(4 - 1)^2}
\]

at time \( t_0 \),

UFO at \((1, 3)\)

\[
[x]_{t \to t_0} = 1
\]

\[
[y]_{t \to t_0} = 3
\]

\[
a := [dx/dt]_{t \to t_0}
\]

\[
b := [dy/dt]_{t \to t_0}
\]

Goal: Compute \( a \) and \( b \).
(2)(2) = \left[ (\sec^2 \theta) \left( \frac{d\theta}{dt} \right) \right]_{t \to t_0} = \frac{(4 - 1)(b) - (3)(-a)}{(4 - 1)^2}

4 = \frac{(4 - 1)(b) - (3)(-a)}{9}

36 = 3b + 3a

At time $t_0$,

UFO at (1, 3)

\[ [x]_{t \to t_0} = 1 \]
\[ [y]_{t \to t_0} = 3 \]

\[ a := \left[ \frac{dx}{dt} \right]_{t \to t_0} \]
\[ b := \left[ \frac{dy}{dt} \right]_{t \to t_0} \]

Goal: Compute $a$ and $b$. 
0520-8. ANSWER:

\[ y = x^2 + 2 \]

\[
\frac{dy}{dt} = 2x \frac{dx}{dt} + 0
\]

\[ b = 2(1)a + 0 = 2a \]

at time \( t_0 \),

\[ [x]_{t:\rightarrow t_0} = 1 \]
\[ [y]_{t:\rightarrow t_0} = 3 \]

\[ a := [dx/dt]_{t:\rightarrow t_0} \]
\[ b := [dy/dt]_{t:\rightarrow t_0} \]

Goal: Compute \( a \) and \( b \).
0520-8. ANSWER:

\[
b = 2a \\
36 = 3b + 3a \\
36 = 3(2a) + 3a = 9a \\
a = 36/9 = 4 \\
b = 2a = 8
\]

\[
b = 2(1)a + 0 = 2a
\]

at time \( t_0 \),
UFO at \((1, 3)\)

\[
[x]_{t \rightarrow t_0} = 1 \\
[y]_{t \rightarrow t_0} = 3
\]

\[
a := [dx/dt]_{t \rightarrow t_0} \\
b := [dy/dt]_{t \rightarrow t_0}
\]

Goal: Compute \( a \) and \( b \).
0520-8.

a. What is the rate of change in the $x$-coordinate of the the UFO at that moment?

**ANSWER:**
$$a = 4$$

b. What is the rate of change in the $y$-coordinate of the the UFO at that moment?

**ANSWER:**
$$b = 8$$

at time $t_0$, UFO at $(1, 3)$
$$[x]_{t_0} = 1$$
$$[y]_{t_0} = 3$$

$$a := \left[\frac{dx}{dt}\right]_{t_0}$$
$$b := \left[\frac{dy}{dt}\right]_{t_0}$$

Goal: Compute $a$ and $b$. 
a. What is the rate of change in the $x$-coordinate of the UFO at that moment?

ANSWER: $a = 4$

b. What is the rate of change in the $y$-coordinate of the UFO at that moment?

ANSWER: $b = 8$

NOTES:

- The speed of the UFO at time $t_0$ is $\sqrt{4^2 + 8^2} \approx 8.9443$

- No units of distance are given in the problem, so none can be given in the answer.
Sand is being poured, at a rate of 4 cubic meters per minute, into a conical pile that is always 5/8 as high as it is wide. How fast is the width of the pile increasing when the pile 16 meters wide and 10 meters high?
Sand is being poured, at a rate of 4 cubic meters per minute, into a conical pile that is always $5/8$ as high as it is wide. How fast is the width of the pile increasing when the pile 16 meters wide and 10 meters high?

**ANS:**

$$V := \text{volume of the pile} = \frac{1}{3} \pi \left( \frac{w}{2} \right)^2 \left( \frac{5w}{8} \right)$$

$$\left. \left[ \frac{dW}{dt} \right] \right|_{t:t_0} = ?$$

$$\frac{dV}{dt} = 4$$

$$[w]_{t:t_0} = 16$$
Sand is being poured, at a rate of 4 cubic meters per minute, into a conical pile that is always 5/8 as high as it is wide. How fast is the width of the pile increasing when the pile 16 meters wide and 10 meters high?

ANSA: $V := \text{volume of the pile} = \frac{1}{3} \pi \left(\frac{w}{2}\right)^2 \left(\frac{5w}{8}\right)$

$$\left[ \frac{dw}{dt} \right]_{t \to t_0} = ?$$

$$\frac{dV}{dt} = \frac{d}{dt} \left[ \frac{5}{96} \pi w^3 \right]$$

$$\frac{dV}{dt} = 4$$

$$[w]_{t \to t_0} = 16$$

$$4 = \left[ \frac{5}{32} \pi (16^2) \right] \left[ \frac{dw}{dt} \right]$$
Sand is being poured, at a rate of 4 cubic meters per minute, into a conical pile that is always $5/8$ as high as it is wide. How fast is the width of the pile increasing when the pile is 16 meters wide and 10 meters high?

**ANS:**

$$4 = \left[ \frac{5}{32} \pi (16^2) \right]$$

$$= [40\pi]$$

$$? = \frac{4}{40\pi} = \frac{1}{10\pi} \approx 0.03183$$

meters per minute