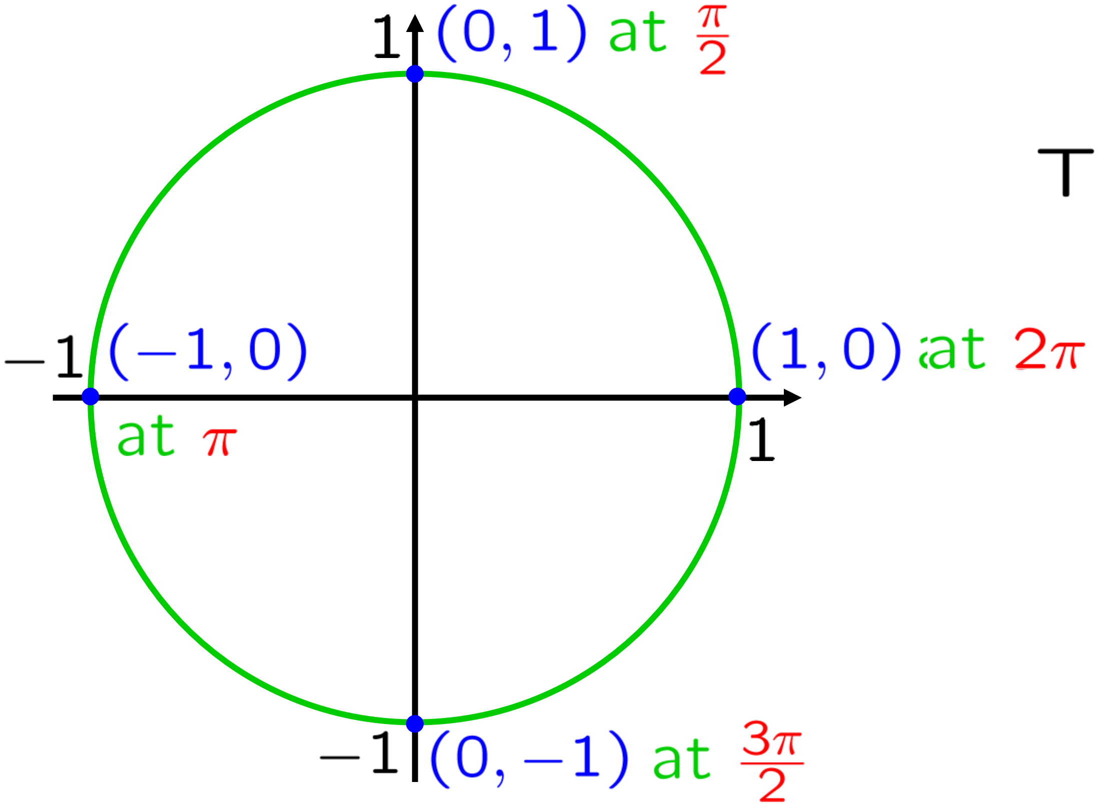
CALCULUS

Sum of angles formulas in trigonometry



The standard orbiter
is a particle
that travels
counterclockwise
at speed 1
starting at $(1, 0)$

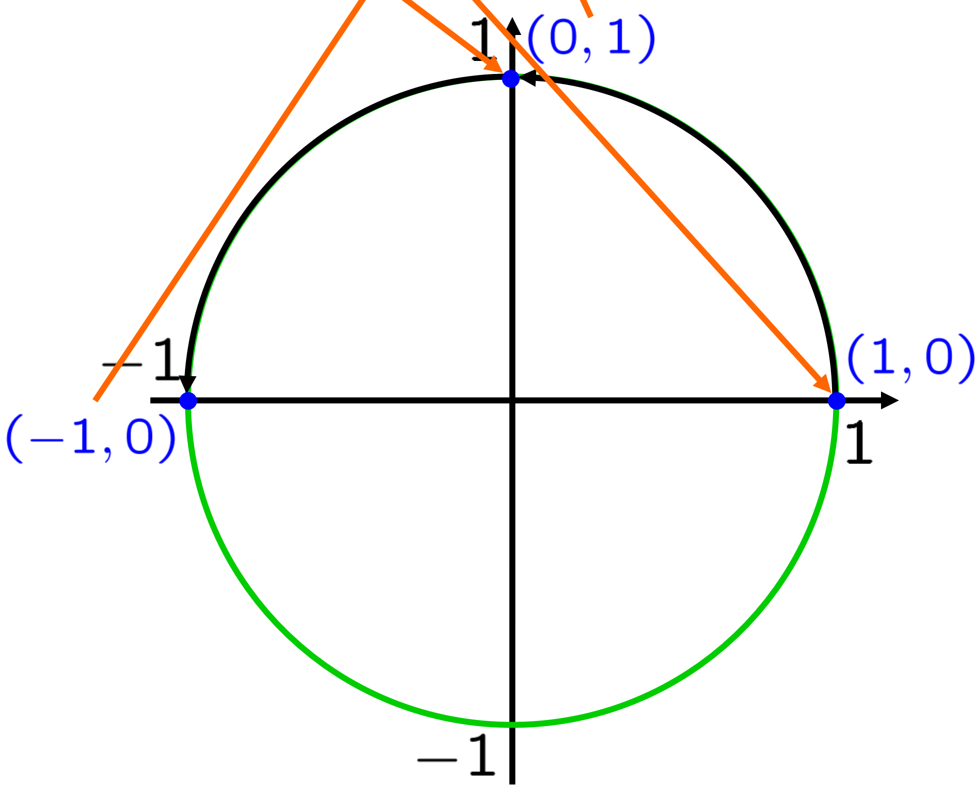
Let $R_t(x, y)$ denote the result of rotating the point (x, y) counterclockwise about the origin by t radians.

e.g.: position of std orbiter at time $\pi/2$

$R_{\pi/2}(1, 0) = (0, 1)$

$R_{\pi/2}(0, 1) = (-1, 0)$

Let's study $R_{\pi/4}$ next.



The standard orbiter is a particle that travels counterclockwise at speed 1 starting at $(1, 0)$

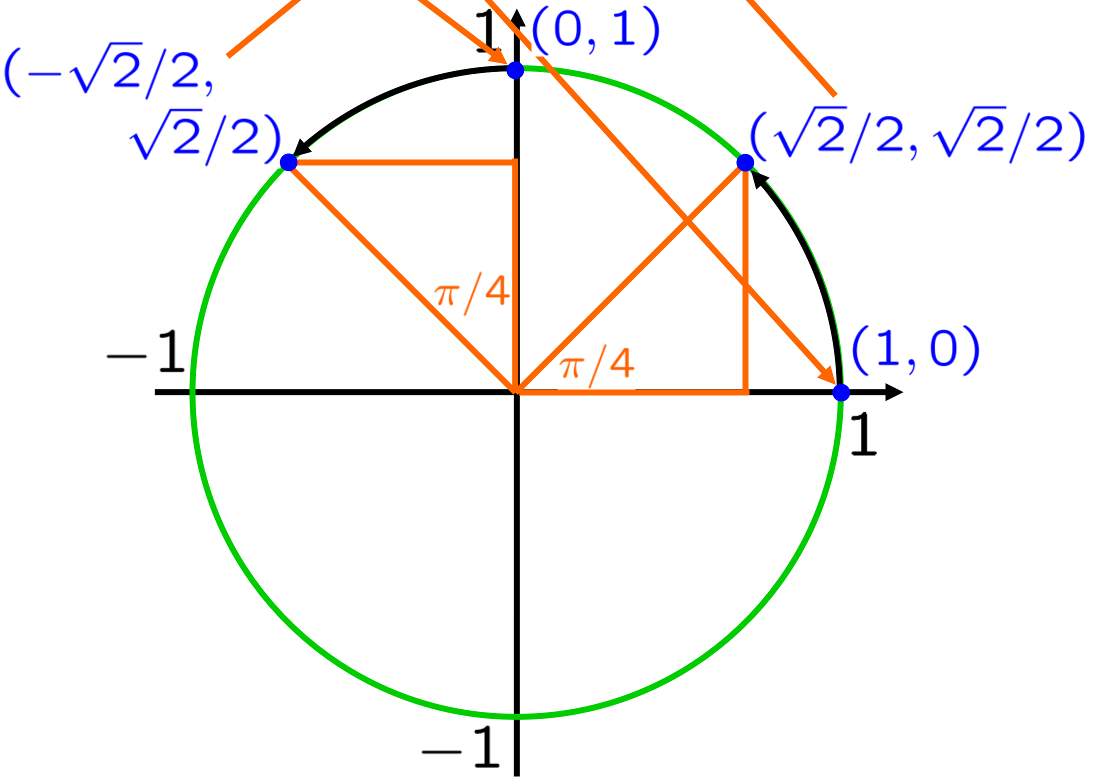
Let $R_t(x,y)$ denote the result of rotating the point (x,y) counterclockwise about the origin by t radians.

e.g.: position of std orbiter at time $\pi/4$

$R_{\pi/4}(1,0) = (\sqrt{2}/2, \sqrt{2}/2)$

$R_{\pi/4}(0,1) = (-\sqrt{2}/2, \sqrt{2}/2)$

Let's study general R_t next.



The standard orbiter is a particle that travels counterclockwise at speed 1 starting at $(1,0)$

Spp

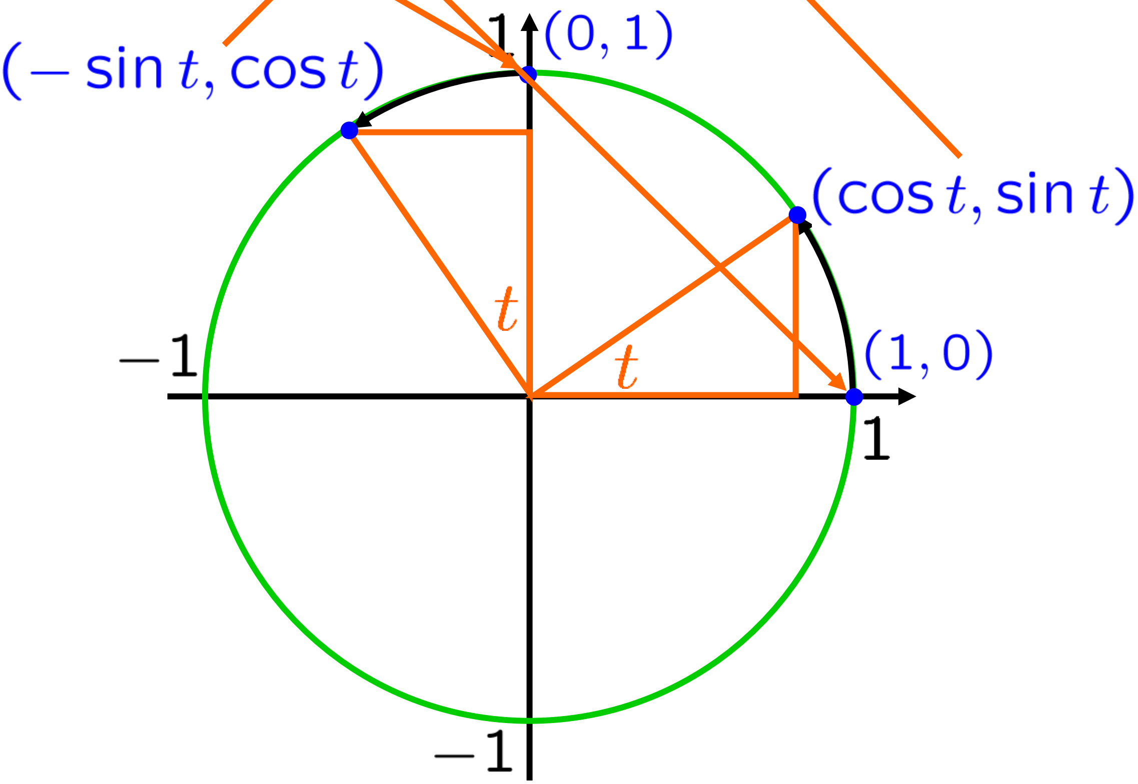
Let $R_t(x, y)$ denote the result of rotating the point (x, y) counterclockwise about the origin by t radians.

position of std orbiter at time t

$$R_t(1, 0) = (\cos t, \sin t)$$

$$R_t(0, 1) = (-\sin t, \cos t)$$

Let's study general R_t next.



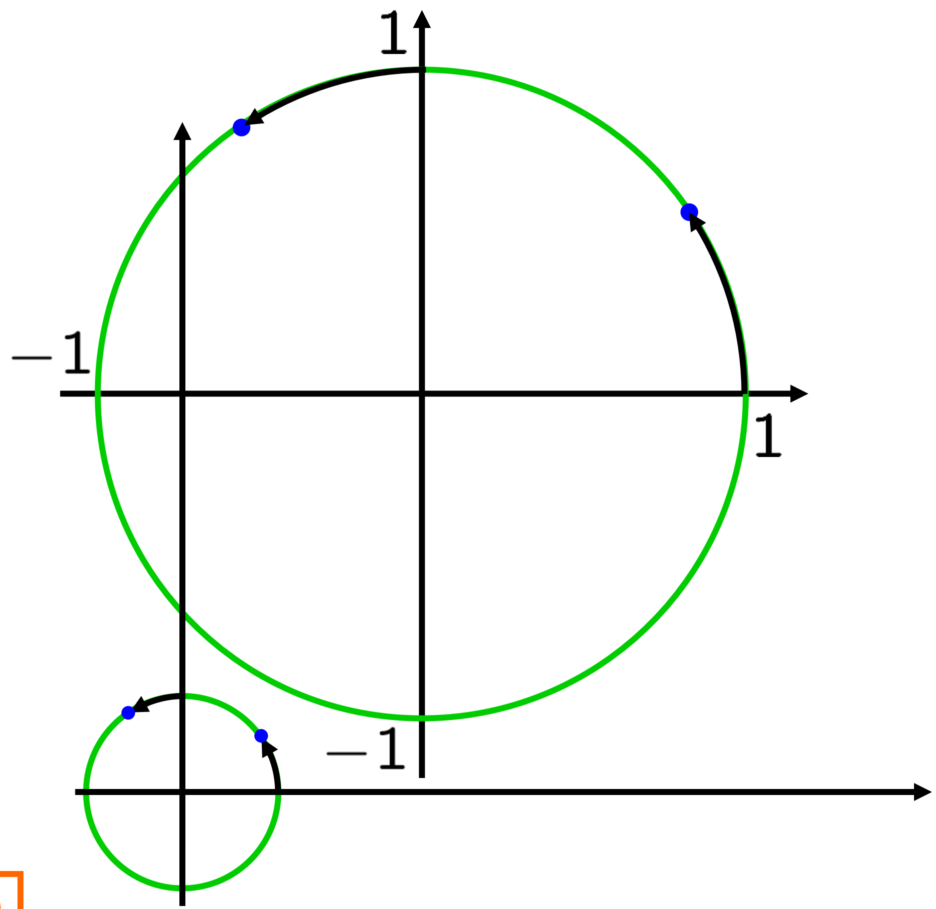
The standard orbiter is a particle that travels counterclockwise at speed 1 starting at $(1, 0)$

Let $R_t(x, y)$ denote the result of rotating the point (x, y) counterclockwise about the origin by t radians.

$R_t(1, 0) = (\cos t, \sin t)$

$R_t(5, 0)$

$R_t(0, 1) = (-\sin t, \cos t)$

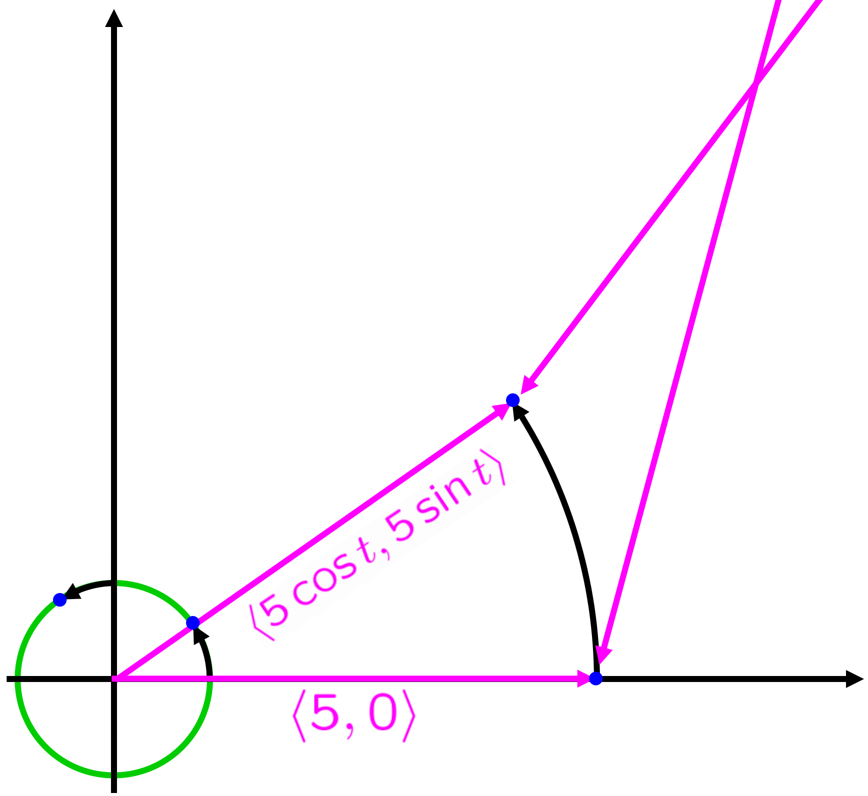


Spp

Let $R_t(x, y)$ denote the result of rotating the point (x, y) counterclockwise about the origin by t radians.

$R_t(1, 0) = (\cos t, \sin t)$ $\qquad$ $R_t(5, 0) = (\quad 5\cos t\,, \quad 5\sin t \quad)$

$R_t(0, 1) = (-\sin t, \cos t)$ $\qquad$ $R_t(0, 3)$



Let $R_t(x, y)$ denote the result of rotating the point (x, y) counterclockwise about the origin by t radians.

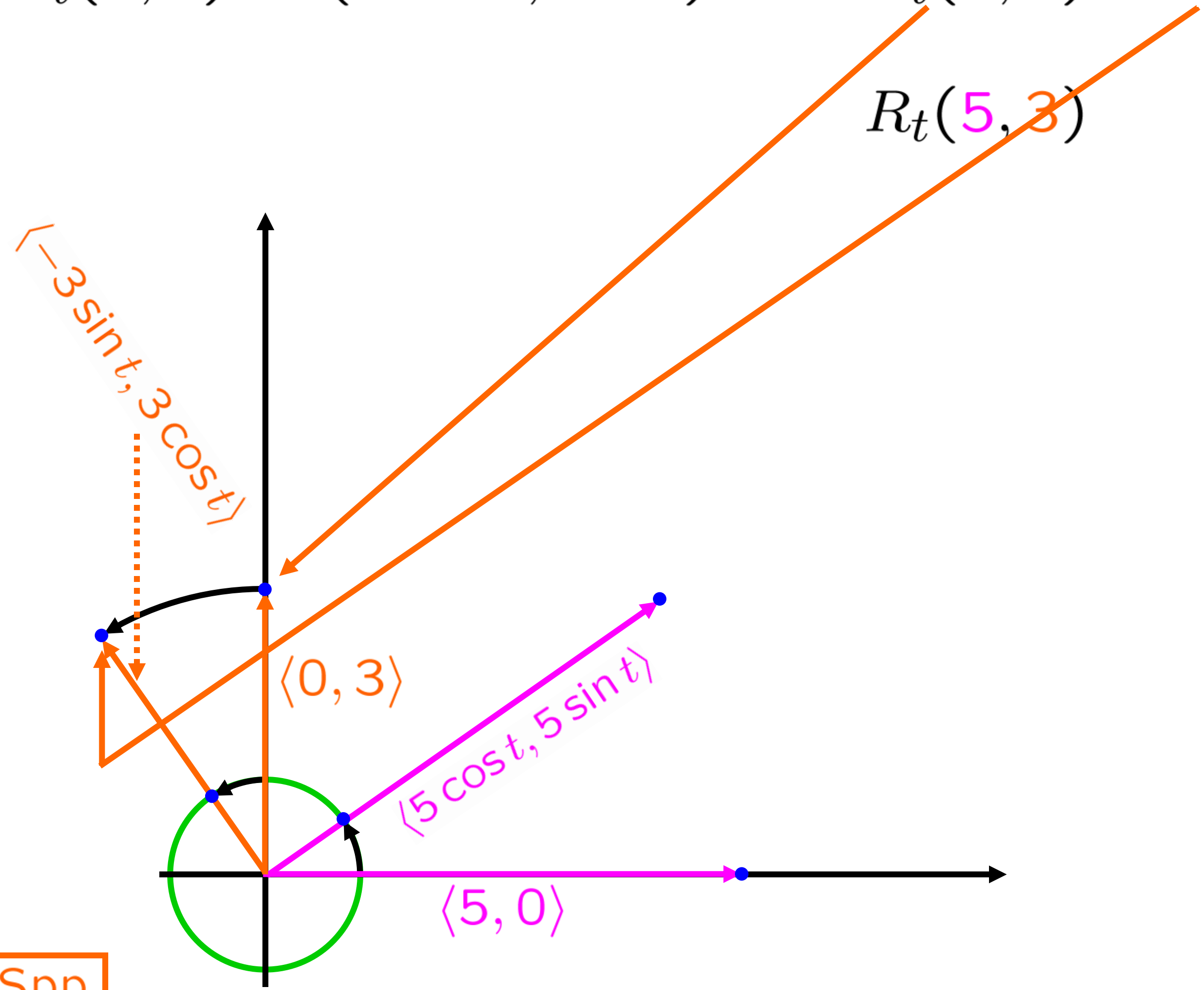
$R_t(1,0) = (\cos t, \sin t)$

$R_t(0,1) = (-\sin t, \cos t)$

$R_t(5,0) = (\quad 5\cos t \ , \quad 5\sin t \quad)$

$R_t(0,3) = (\ -3\sin t \ , \quad 3\cos t \quad)$

$R_t(5,3)$



Spp

Let $R_t(x,y)$ denote the result of rotating the point (x,y) counterclockwise about the origin by t radians.

$R_t(1,0) = (\cos t, \sin t)$ $\qquad$ $R_t(5,0) = (\; 5\cos t \; , \; 5\sin t \;)$

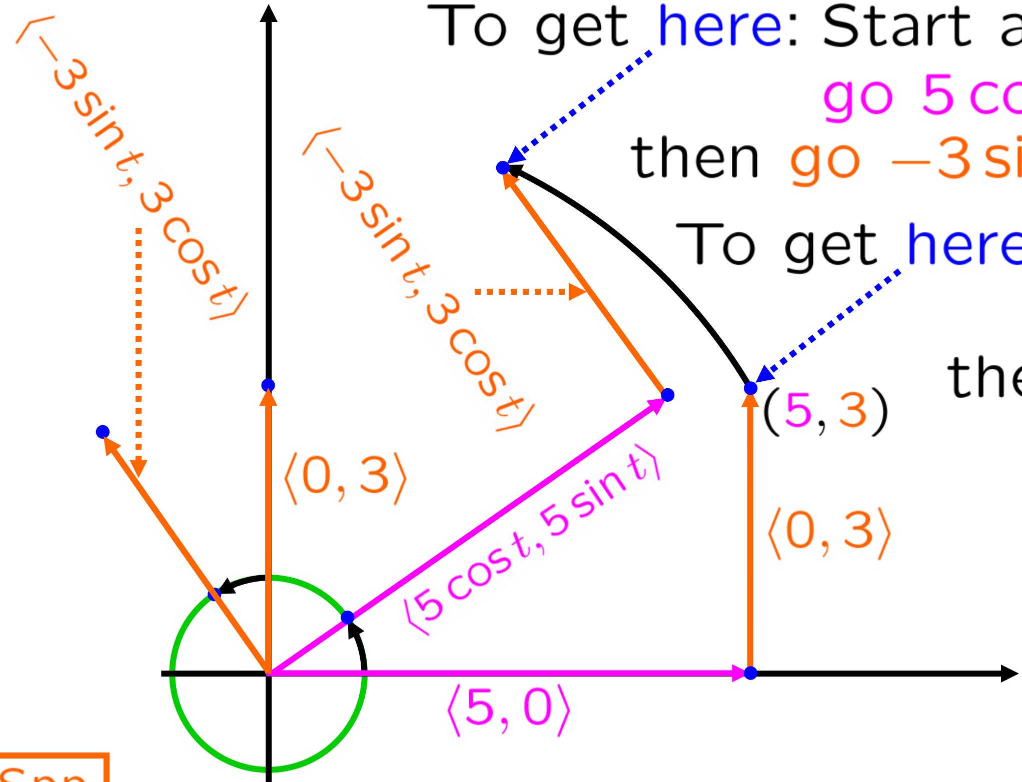
$R_t(0,1) = (-\sin t, \cos t)$ $\qquad$ $R_t(0,3) = (\; -3\sin t \; , \; 3\cos t \;)$

NOTHING SPECIAL ABOUT 5 AND 3...

$$R_t(5,3) = \left(\begin{array}{c} 5\cos t \\ -3\sin t \end{array} , \begin{array}{c} 5\sin t \\ +3\cos t \end{array} \right)$$

To get here: Start at $(0,0)$,
go $5\cos t$ horizontal, $5\sin t$ vertical
then go $-3\sin t$ horizontal, $3\cos t$ vertical

To get here: Start at $(0,0)$,
go 5 horizontal, 0 vertical
then go 0 horizontal, 3 vertical



Spp

Let $R_t(x,y)$ denote the result of rotating the point (x,y) counterclockwise about the origin by t radians.

$R_t(1,0) = (\cos t, \sin t)$

$R_t(0,1) = (-\sin t, \cos t)$

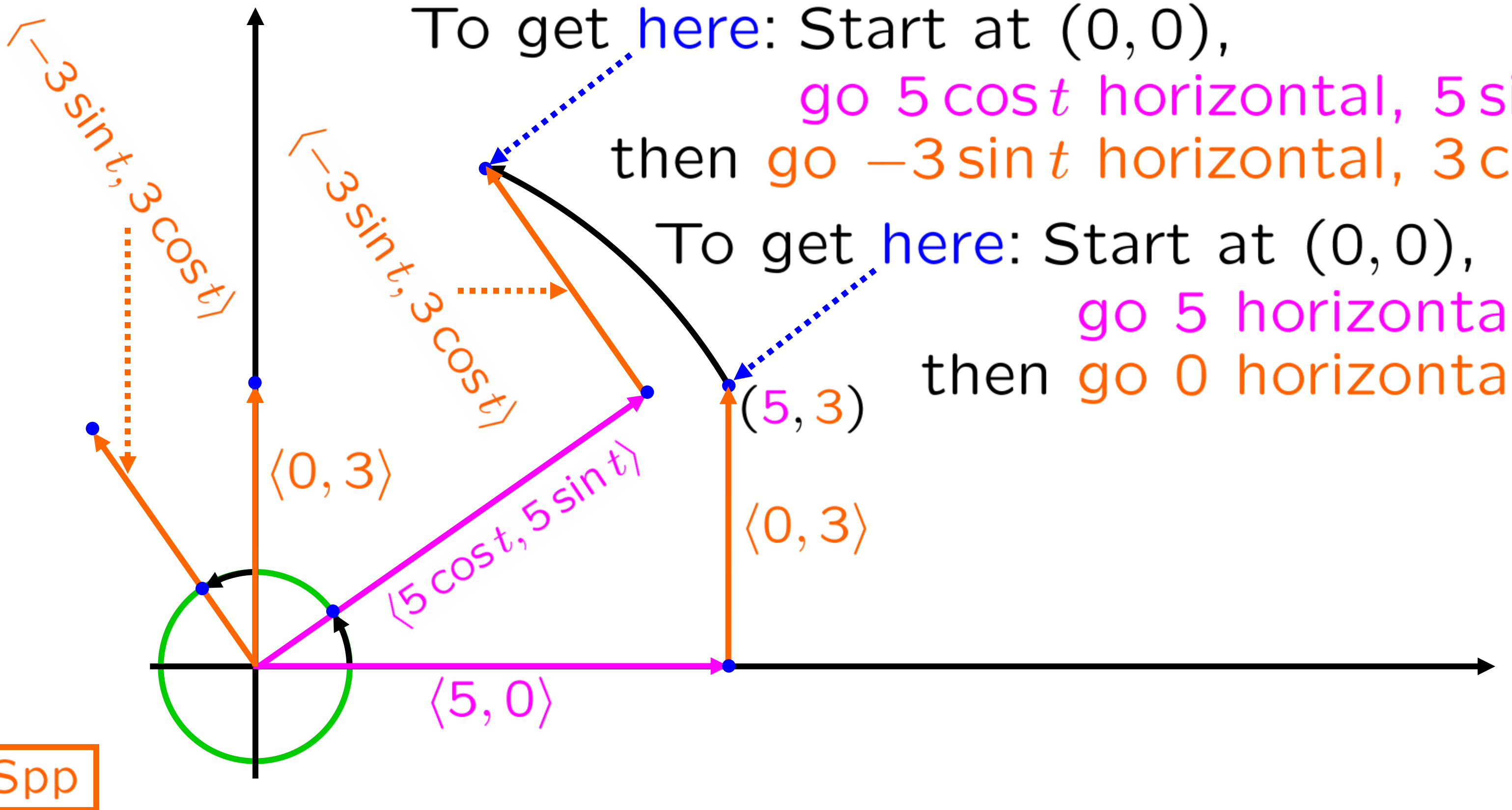
$$R_t(x,y) = \begin{pmatrix} x\cos t & , & x\sin t \\ -y\sin t & & +y\cos t \end{pmatrix}$$

NOTHING SPECIAL ABOUT 5 AND 3...

$$R_t(5,3) = \begin{pmatrix} 5\cos t & , & 5\sin t \\ -3\sin t & & +3\cos t \end{pmatrix}$$

To get here: Start at $(0,0)$,
go $5\cos t$ horizontal, $5\sin t$ vertical
then go $-3\sin t$ horizontal, $3\cos t$ vertical

To get here: Start at $(0,0)$,
go 5 horizontal, 0 vertical
then go 0 horizontal, 3 vertical



Spp

Let $R_t(x,y)$ denote the result of rotating the point (x,y) counterclockwise about the origin by t radians.

$$R_t(1,0) = (\cos t, \sin t)$$

$$R_t(0,1) = (-\sin t, \cos t)$$

$$R_t(x,y) = \begin{pmatrix} x\cos t & x\sin t \\ -y\sin t & +y\cos t \end{pmatrix}$$

Key point: Rotation is linear!

$$R_t(1,0) = (\cos t, \sin t)$$

$$R_t(0,1) = (-\sin t, \cos t)$$

Let $R_t(x,y)$ denote the result of rotating the point (x,y) counterclockwise about the origin by t radians.

$$R_t(x,y) = \left(\begin{matrix} x\cos t \\ -y\sin t \end{matrix} , \begin{matrix} x\sin t \\ +y\cos t \end{matrix} \right)$$

Key point: Rotation is linear!

MULTIPLY by x.

$R_t(1,0) = (\cos t, \sin t)$ $\qquad$ $R_t(x,0) = (x\cos t , x\sin t)$

$R_t(0,1) = (-\sin t, \cos t)$ $\qquad$ $R_t(0,y) = (-y\sin t , y\cos t)$

MULTIPLY by y. $\qquad$ ADD

$$R_t(x,y) = \left(\begin{matrix} x\cos t \\ -y\sin t \end{matrix} , \begin{matrix} x\sin t \\ +y\cos t \end{matrix} \right)$$

Let $R_t(x,y)$ denote the result of rotating the point (x,y) counterclockwise about the origin by t radians.

$$R_t(x,y) = \left(\begin{array}{c} x\cos t \\ -y\sin t \end{array}, \begin{array}{c} x\sin t \\ +y\cos t \end{array} \right)$$

Let $R_u(x,y)$ denote the result of rotating the point (x,y) counterclockwise about the origin by u radians.

$$R_u(x,y) = \left(\begin{array}{c} x\cos u \\ -y\sin u \end{array}, \begin{array}{c} x\sin u \\ +y\cos u \end{array} \right)$$

Then $R_u(R_t(1,0))$ is the result of rotating the point $(1,0)$ counterclockwise about the origin by $t+u$ radians.

This is the same as the position of the standard orbiter at time $t+u$.

$$R_u(R_t(1,0)) = (\cos(t+u), \sin(t+u))$$

Spp

Let $R_t(x,y)$ denote the result of rotating the point (x,y) counterclockwise about the origin by t radians.

This is the same as the position of the standard orbiter at time t.

$$R_t(x,y) = \left(\begin{array}{c} x\cos t \\ -y\sin t \end{array}, \begin{array}{c} x\sin t \\ +y\cos t \end{array} \right)$$

$$R_t(1,0) = (\cos t, \sin t)$$

Let $R_u(x,y)$ denote the result of rotating the point (x,y) counterclockwise about the origin by u radians.

$$R_u(x,y) = \left(\begin{array}{c} x\cos u \\ -y\sin u \end{array}, \begin{array}{c} x\sin u \\ +y\cos u \end{array} \right)$$

$$R_u(\cos t, \sin t) = \left(\begin{array}{c} [\cos t][\cos u] \\ -[\sin t][\sin u] \end{array}, \begin{array}{c} [\cos t][\sin u] \\ +[\sin t][\cos u] \end{array} \right)$$

Then $R_u(R_t(1,0))$ is the result of rotating the point $(1,0)$ counterclockwise about the origin by $t+u$ radians.

This is the same as the position of the standard orbiter at time $t+u$.

$$R_u(R_t(1,0)) = (\cos(t+u), \sin(t+u))$$

Spp

$$R_t(1,0) = (\quad \cos t \;,\quad \sin t \quad)$$

$$R_t(1,0) = (\cos t, \sin t)$$

$$R_u(\cos t, \sin t) = \begin{pmatrix} [\cos t][\cos u] & , & [\cos t][\sin u] \\ -[\sin t][\sin u] & & +[\sin t][\cos u] \end{pmatrix}$$

$$\parallel \qquad\qquad //$$

$$R_u(R_t(1,0)) = (\cos(t+u), \sin(t+u))$$

$$R_u(R_t(1,0)) = (\cos(t+u), \sin(t+u))$$

$$(\cos(t+u), \sin(t+u)) = \left(\begin{array}{c} [\cos t][\cos u] \\ -[\sin t][\sin u] \end{array}, \begin{array}{c} [\cos t][\sin u] \\ +[\sin t][\cos u] \end{array} \right)$$

$$\cos(t+u) = \begin{array}{c} [\cos t][\cos u] \\ -[\sin t][\sin u] \end{array}$$

$$\sin(t+u) = \begin{array}{c} [\cos t][\sin u] \\ +[\sin t][\cos u] \end{array}$$

$$= \left(\begin{array}{c} [\cos t][\cos u] \\ -[\sin t][\sin u] \end{array}, \begin{array}{c} [\cos t][\sin u] \\ +[\sin t][\cos u] \end{array} \right)$$

$$(\cos(t+u), \sin(t+u))$$

$$\cos(t+u) = [\cos t][\cos u] - [\sin t][\sin u]$$

$$\sin(t+u) = [\cos t][\sin u] + [\sin t][\cos u]$$

$$\sin(t+u) = [\sin t][\cos u] + [\cos t][\sin u]$$

$$\cos(t+u) = [\cos t][\cos u] - [\sin t][\sin u]$$

$t :\to \alpha$

$u :\to \beta$

$$\sin(\alpha+\beta) = [\sin\alpha][\cos\beta] + [\cos\alpha][\sin\beta]$$

$$\cos(\alpha+\beta) = [\cos\alpha][\cos\beta] - [\sin\alpha][\sin\beta]$$

$\beta :\to \alpha$

$$\sin(2\alpha) = [\sin\alpha][\cos\alpha] + [\cos\alpha][\sin\alpha]$$

$$\cos(2\alpha) = [\cos\alpha][\cos\alpha] - [\sin\alpha][\sin\alpha]$$

$$\sin(\alpha+\beta) = [\sin\alpha][\cos\beta] + [\cos\alpha][\sin\beta]$$

$$\cos(\alpha+\beta) = [\cos\alpha][\cos\beta] - [\sin\alpha][\sin\beta]$$

$$\begin{aligned}\sin(2\alpha) &= [\sin\alpha][\cos\alpha] + [\cos\alpha][\sin\alpha] \\ &= 2[\sin\alpha][\cos\alpha]\end{aligned}$$

$$\sin(\alpha+\beta) = [\sin\alpha][\cos\beta] + [\cos\alpha][\sin\beta]$$

$$\cos(\alpha+\beta) = [\cos\alpha][\cos\beta] - [\sin\alpha][\sin\beta]$$

$$\sin(2\alpha) = [\sin\alpha][\cos\alpha] + [\cos\alpha][\sin\alpha]$$

$$\cos(2\alpha) = [\cos\alpha][\cos\alpha] - [\sin\alpha][\sin\alpha]$$

$$\sin(\alpha+\beta) = [\sin\alpha][\cos\beta] + [\cos\alpha][\sin\beta]$$

$$\cos(\alpha+\beta) = [\cos\alpha][\cos\beta] - [\sin\alpha][\sin\beta]$$

$$\sin(2\alpha) = 2[\sin\alpha][\cos\alpha]$$

$$2[\sin\alpha][\cos\alpha]$$

$$\begin{aligned}\cos(2\alpha) &= [\cos\alpha][\cos\alpha] - [\sin\alpha][\sin\alpha] \\ &= [\cos^2\alpha] - [\sin^2\alpha]\end{aligned}$$

$$\cos(2\alpha) = [\cos\alpha][\cos\alpha] - [\sin\alpha][\sin\alpha]$$

$$\sin(\alpha+\beta) = [\sin\alpha][\cos\beta] + [\cos\alpha][\sin\beta]$$

$$\cos(\alpha+\beta) = [\cos\alpha][\cos\beta] - [\sin\alpha][\sin\beta]$$

$$\sin(2\alpha) = 2[\sin\alpha][\cos\alpha]$$

$$\cos(2\alpha) = [\cos\alpha][\cos\alpha] - [\sin\alpha][\sin\alpha]$$

$$= [\cos^2\alpha] - [\sin^2\alpha]$$

$$[1-\sin^2\alpha] - [\sin^2\alpha] \qquad\qquad [\cos^2\alpha] - [1-\cos^2\alpha]$$

$$1 - 2[\sin^2\alpha] \qquad\qquad 2[\cos^2\alpha] - 1$$

Spp

SUM OF ANGLES FORMULAS

$$\sin(\alpha+\beta) = [\sin\alpha][\cos\beta] + [\cos\alpha][\sin\beta]$$

$$\cos(\alpha+\beta) = [\cos\alpha][\cos\beta] - [\sin\alpha][\sin\beta]$$

$$\sin(2\alpha) = 2[\sin\alpha][\cos\alpha]$$

$$\cos(2\alpha) = 1 - 2[\sin^2\alpha] = [\cos^2\alpha] - [\sin^2\alpha] = 2[\cos^2\alpha] - 1$$

$$\cos(2\alpha) =$$

$$[\cos^2\alpha] \quad - \quad [\sin^2\alpha]$$

$$\parallel \qquad\qquad \parallel$$

$$1 - 2[\sin^2\alpha] \qquad\qquad 2[\cos^2\alpha] - 1$$

SUM OF ANGLES FORMULAS

$$\sin(\alpha+\beta) = [\sin\alpha][\cos\beta] + [\cos\alpha][\sin\beta]$$

$$\cos(\alpha+\beta) = [\cos\alpha][\cos\beta] - [\sin\alpha][\sin\beta]$$

$$\sin(2\alpha) = 2[\sin\alpha][\cos\alpha]$$

DOUBLE ANGLE FORMULAS

$$\cos(2\alpha) = 1 - 2[\sin^2\alpha]$$

$$\cos(2\alpha) = [\cos^2\alpha] - [\sin^2\alpha]$$

$$\cos(2\alpha) = 2[\cos^2\alpha] - 1$$

$$\sin(2\alpha) = 2[\sin\alpha][\cos\alpha]$$

$$[\sin^2\alpha]$$

$$\cos(2\alpha) = [\cos^2\alpha] - [\sin^2\alpha] = 2[\cos^2\alpha] - 1$$

SUM OF ANGLES FORMULAS

$$\sin(\alpha+\beta) = [\sin\alpha][\cos\beta] + [\cos\alpha][\sin\beta]$$

$$\cos(\alpha+\beta) = [\cos\alpha][\cos\beta] - [\sin\alpha][\sin\beta]$$

DOUBLE ANGLE FORMULAS

$$\sin(2\alpha) = 2[\sin\alpha][\cos\alpha]$$

$$\cos(2\alpha) = 1 - 2[\sin^2\alpha] = [\cos^2\alpha] - [\sin^2\alpha] = 2[\cos^2\alpha] - 1$$

STOP