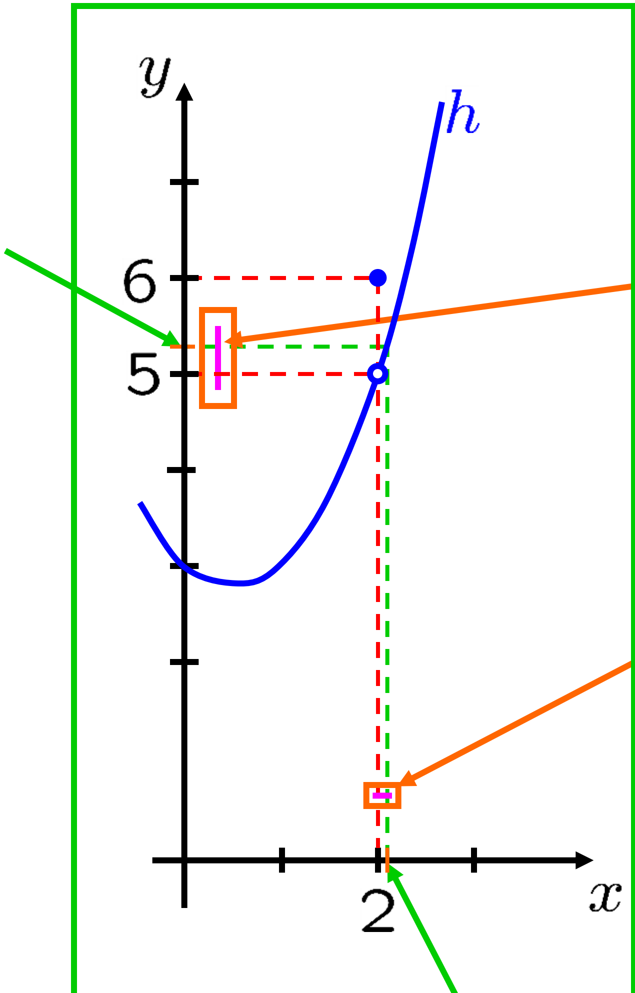
CALCULUS

The limit game and

the exact definition of a limit

THE LIMIT GAME!



No answer needed! How close is close?

1. You pick a vertical distance $\varepsilon > 0$.

   e.g.: represents tolerance for output error

2. I pick a horizontal distance $\delta > 0$.

   e.g.: represents tolerance for input error

3. You pick a point on the x-axis.

   Must meet input tolerance.

   Must be within δ of 2, but not equal to 2.

4. Apply f to the point.

   I win if output meets output tolerance.

If answer is within ε of 5, then I win; otherwise you win.

Let's play! e.g.

§2.3, p. 29, Def'n 2.3: $\lim_{x \to a} f(x) = L$ is read

the limit of $f(x)$ as x approaches a is equal to L

and means, intuitively:

if x is close to a, but not actually equal to a,

then $f(x)$ is close to L

or:

if dist(x, a) is small, but not actually 0,

then dist$(f(x), L)$ is small

and means, rigorously:

$\forall$tolerance to output error $\varepsilon > 0$,

$\exists$tolerance to input error $\delta > 0$

such that

if dist(x, a) is small enough

to meet the input tolerance of δ,

but is not actually 0,

then dist$(f(x), L)$ is small enough

to meet the output tolerance of ε

§2.3, p. 29, Def'n 2.3: $\lim_{x \to a} f(x) = L$ is read

the limit of $f(x)$ as x approaches a is equal to L

and means, rigorously:

∀tolerance to output error $\varepsilon > 0$,

∃tolerance to input error $\delta > 0$

such that

if dist(x, a) is small enough

to meet the input tolerance of δ,

but is not actually 0,

then dist$(f(x), L)$ is small enough

to meet the output tolerance of ε

and means, rigorously:

∀tolerance to output error $\varepsilon > 0$,

∃tolerance to input error $\delta > 0$

or: such that

$\forall \varepsilon > 0$, dist(x, a) is small enough

to meet the input tolerance of δ,

but is not actually 0,

then dist$(f(x), L)$ is small enough

to meet the output tolerance of ε

§2.3, p. 29, Def'n 2.3: $\lim_{x\to a} f(x) = L$ is read

the limit of $f(x)$ as x approaches a is equal to L

and means, rigorously:

∀tolerance to output error $\varepsilon > 0$,

∃tolerance to input error $\delta > 0$

such that

if dist(x, a) is small enough
to meet the input tolerance of δ,
but is not actually 0,

then dist$(f(x), L)$ is small enough
to meet the output tolerance of ε

or:

$\forall\varepsilon > 0$, $\exists\delta > 0$ such that

if dist$(x, a) < \delta$, but dist$(x, a) \neq 0$,

then dist$(f(x), L) < \varepsilon$

§2.3, p. 29, Def'n 2.3: $\lim\limits_{x\to a} f(x) = L$ is read

the limit of $f(x)$ as x approaches a is equal to L

and means, rigorously:

dist$(x,a) > 0$ $\Updownarrow$

$\forall\varepsilon > 0$, $\exists\delta > 0$ such that

if dist$(x,a) < \delta$, but dist$(x,a) \neq 0$,

then dist$(f(x), L) < \varepsilon$

PERFECTLY RIGOROUS,

BUT ATYPICALLY PHRASED.

MORE TYPICAL. . .

$\forall\varepsilon > 0$, $\exists\delta > 0$ such that

if dist$(x,a) < \delta$, but dist$(x,a) \neq 0$,

then dist$(f(x), L) < \varepsilon$

§2.3, p. 29, Def'n 2.3: $\lim_{x \longrightarrow a} f(x) = L$ is read

the limit of $f(x)$ as x approaches a is equal to L

and means, rigorously:

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that

if $\mathrm{dist}(x, a) < \delta$, but $\mathrm{dist}(x, a) \neq 0$,

then $\mathrm{dist}(f(x), L) < \varepsilon$

$\mathrm{dist}(x, a) > 0$ $\Updownarrow$

or:

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that

if $0 < \mathrm{dist}(x, a) < \delta$,

then $\mathrm{dist}(f(x), L) < \varepsilon$

$\mathrm{dist}(s, t) = |s - t|$

or:

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that

if $0 < |x - a| < \delta$,

then $|(f(x)) - L| < \varepsilon$

§2.3, p. 29, Def'n 2.3: $\lim_{x \to a} f(x) = L$ is read

the limit of $f(x)$ as x approaches a is equal to L

and means, rigorously:

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that

if $0 < |x - a| < \delta$, then $|(f(x)) - L| < \varepsilon$.

The typical form of the rigorous definition

FYI: Intuitive def'ns will be tested,

but not the rigorous ones.

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that

if $0 < |x - a| < \delta$,

then $|(f(x)) - L| < \varepsilon$

Def'n: $\lim_{x \to a} f(x) = L$ is read

the limit of $f(x)$ as x approaches a is equal to L

and means, rigorously:

and means, intuitively:

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that

if $0 < |x - a| < \delta$, then $|(f(x)) - L| < \varepsilon$.

then $f(x)$ is close to L

and means, rigorously:

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that

if $0 < |x - a| < \delta$, then $|(f(x)) - L| < \varepsilon$.

Def'n: $\lim_{x \to a} f(x) = L$ is read

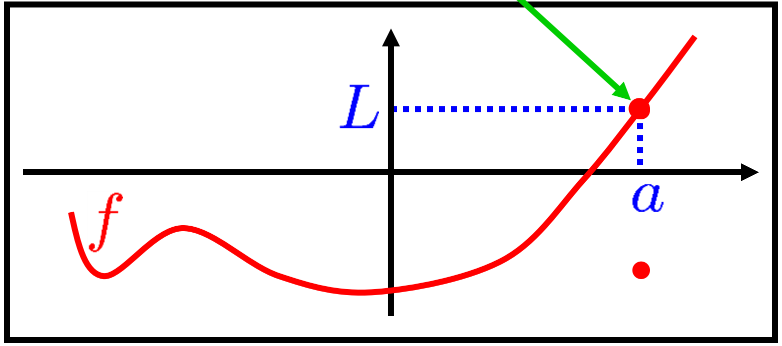
the limit of $f(x)$ as x approaches a is equal to L

a^-

and means, intuitively:

if x is close to a, but not equal to a,

then $f(x)$ is close to L



and means, rigorously:

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that

if $0 < |x - a| < \delta$, then $|(f(x)) - L| < \varepsilon$.

Alternative notation: $f(x) \to L$ as $x \Rightarrow a$

$\neq$ NONSTANDARD

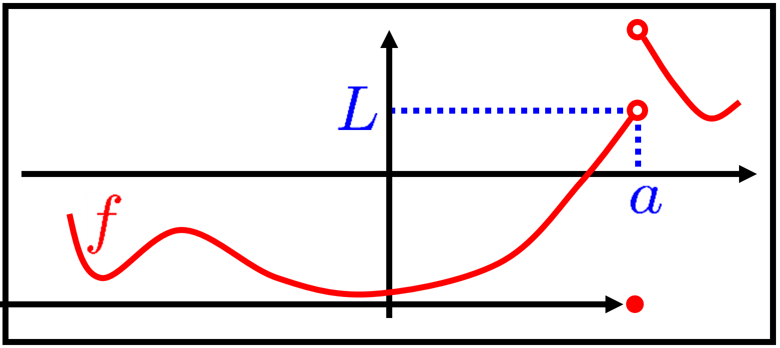
Def'n: $\lim_{x \to a^-} f(x) = L$ is read

the limit of $f(x)$ as x approaches a from the left is equal to L

a^+

and means, intuitively:

if x is close to a, but less than a,
then $f(x)$ is close to L



and means, rigorously:

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that

if $a - \delta < x < a$, then $|(f(x)) - L| < \varepsilon$.

Alternative notation: $f(x) \to L$ as $x \to a^-$, $\lim_{x \uparrow a} f(x) = L$

$x \uparrow a$

§2.4

§2.4, p. 34, Def 2.11: $\lim\limits_{x\to a^+} f(x) = L$ is read

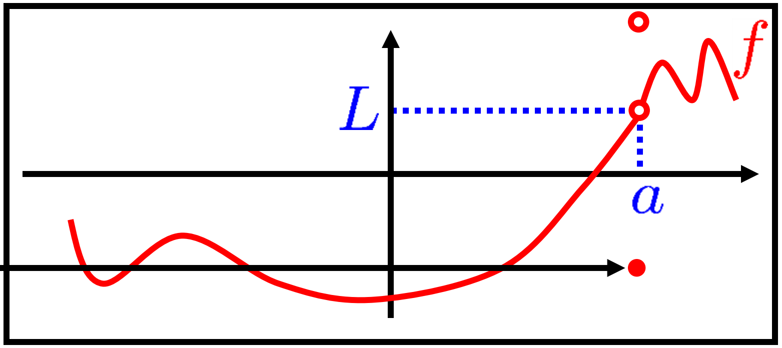
the limit of $f(x)$ as x approaches a from the right is equal to L

∞

and means, intuitively:

if x is close to a, but greater than a,

then $f(x)$ is close to L



and means, rigorously:

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that

if $a < x < a + \delta$, then $|(f(x)) - L| < \varepsilon$.

Alternative notation: $f(x) \to L$ as $x \to a^+$, $\lim\limits_{x \downarrow a} f(x) = L$

$x \downarrow a$

Fact: $\lim\limits_{x \to a} f(x) = L$ iff

both $\lim\limits_{x\to a^-} f(x) = L$ and $\lim\limits_{x\to a^+} f(x) = L$.

Def'n: $\lim_{x \to a^+} f(x) = \infty$ is read

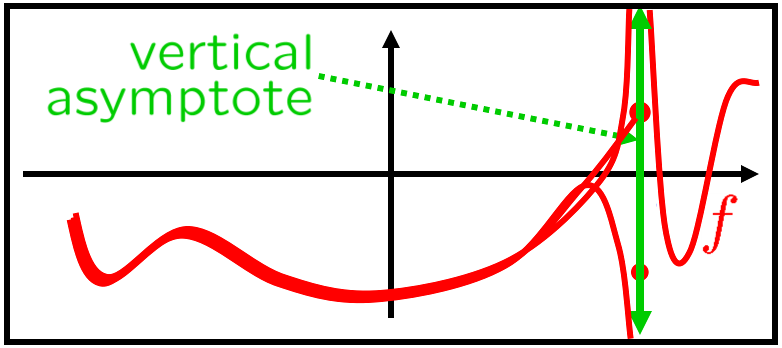
the limit of $f(x)$ as x approaches a from the right is equal to infinity

$-\infty$

and means, intuitively:

if x is close to a, but greater than a,

then $f(x)$ is very positive



and means, rigorously:

unneeded

$\forall R > 0$, $\exists \delta > 0$ such that

if $a < x < a + \delta$, then $f(x) > R$.

Alternative notation: $f(x) \to \infty$ as $x \to a^+$, $\lim_{x \downarrow a} f(x) = \infty$

$x \downarrow a$

"> 0" needed only on ε
traditional on ε and δ
on other i/p or o/p specifications,
traditional not to say "> 0" or "< 0",
but I will . . .

Spp

Def'n: $\lim_{x\to a^+} f(x) = -\infty$ is read

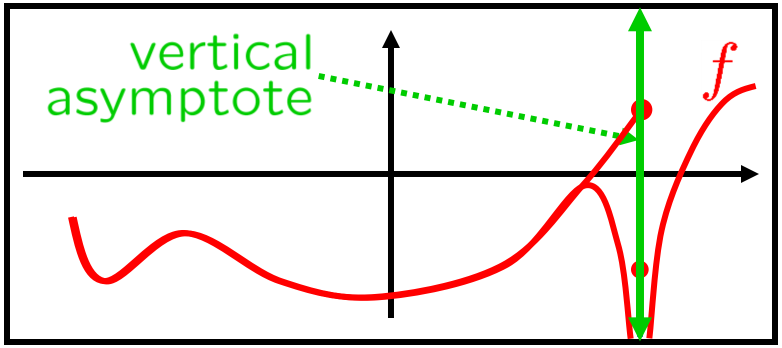
the limit of $f(x)$ as x approaches a from the right is equal to negative infinity

a

and means, intuitively:

if x is close to a, but greater than a,

then $f(x)$ is very negative



and means, rigorously:

$\forall R < 0$, $\exists \delta > 0$ such that

if $a < x < a + \delta$, then $f(x) < R$.

Alternative notation: $f(x) \to -\infty$ as $x \to a^+$, $\lim_{x \downarrow a} f(x) = -\infty$

$x \downarrow a$

Spp

Def'n: $\lim_{x \to a} f(x) = -\infty$ is read

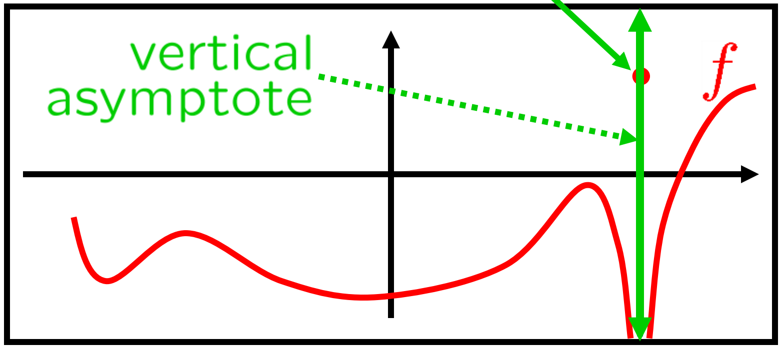
the limit of $f(x)$ as x approaches a is equal to negative infinity

∞

and means, intuitively:

if x is close to a, but not equal to a,

then $f(x)$ is very negative



and means, rigorously:

$\forall R < 0$, $\exists \delta > 0$ such that

if $0 < |x - a| < \delta$, then $f(x) < R$.

Alternative notation: $f(x) \to -\infty$ as $x \to a$

Spp

Def'n: $\lim\limits_{x \to a} f(x) = \infty$ is read

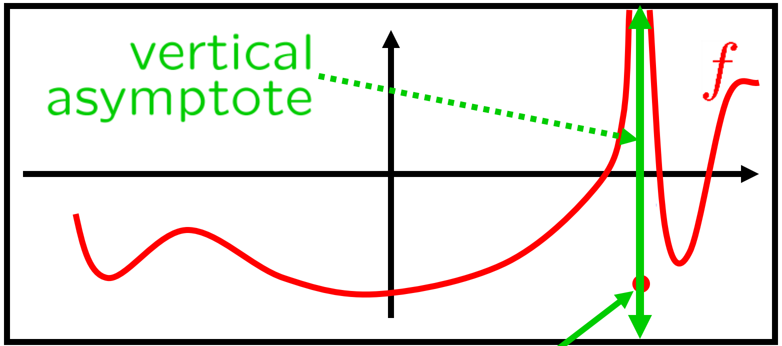
the limit of $f(x)$ as x approaches a (a^-)

is equal to infinity

and means, intuitively:

if x is close to a, but not equal to a,

then $f(x)$ is very positive



and means, rigorously:

$\forall R > 0$, $\exists \delta > 0$ such that

if $0 < |x - a| < \delta$, then $f(x) > R$.

Alternative notation: $f(x) \to \infty$ as $x \to a$

Spp

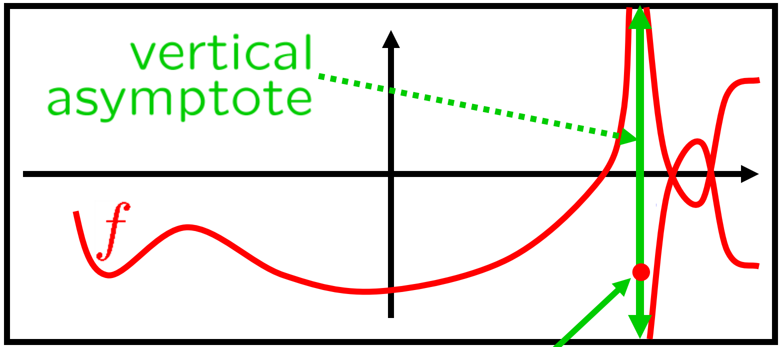
Def'n: $\lim_{x \to a^-} f(x) = \infty$ is read

the limit of $f(x)$ as x approaches a from the left
is equal to infinity

$-\infty$

and means, intuitively:

if x is close to a, but less than a,
then $f(x)$ is very positive



and means, rigorously:

$\forall R > 0$, $\exists \delta > 0$ such that

if $a - \delta < x < a$, then $f(x) > R$.

Alternative notation: $f(x) \to \infty$ as $x \to a^-$, $\lim_{x \uparrow a} f(x) = \infty$

$x \uparrow a$

Fact: $\lim_{x \to a} f(x) = \infty$ iff

both $\lim_{x \to a^-} f(x) = \infty$ and $\lim_{x \to a^+} f(x) = \infty$.

Spp

Def'n: $\lim_{x \to a^-} f(x) = -\infty$ is read

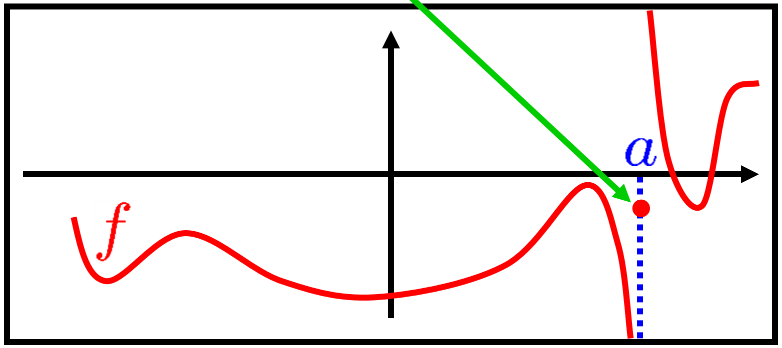
the limit of $f(x)$ as x approaches a from the left

is equal to negative infinity

and means, intuitively:

if x is close to a, but less than a,

then $f(x)$ is very negative



and means, rigorously:

$\forall R < 0$, $\exists \delta > 0$ such that

if $a - \delta < x < a$, then $f(x) < R$.

Def'n: $\lim\limits_{x \to a^-} f(x) = -\infty$ is read

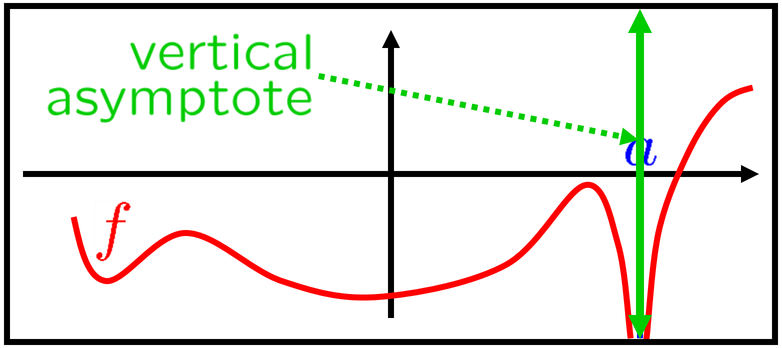
the limit of $f(x)$ as x approaches a from the left is equal to negative infinity

$-\infty$

and means, intuitively:

if x is close to a, but less than a,

then $f(x)$ is very negative



and means, rigorously:

$\forall R < 0$, $\exists \delta > 0$ such that

if $a - \delta < x < a$, then $f(x) < R$.

Alternative notation: $f(x) \to -\infty$ as $x \to a^-$, $\lim\limits_{x \uparrow a} f(x) = -\infty$

$x \uparrow a$

Fact: $\lim\limits_{x \to a} f(x) = -\infty$ iff

both $\lim\limits_{x \to a^-} f(x) = -\infty$ and $\lim\limits_{x \to a^+} f(x) = -\infty$.

Spp

Def'n: $\lim_{x \to -\infty} f(x) = -\infty$ is read

the limit of $f(x)$ as x approaches negative infinity

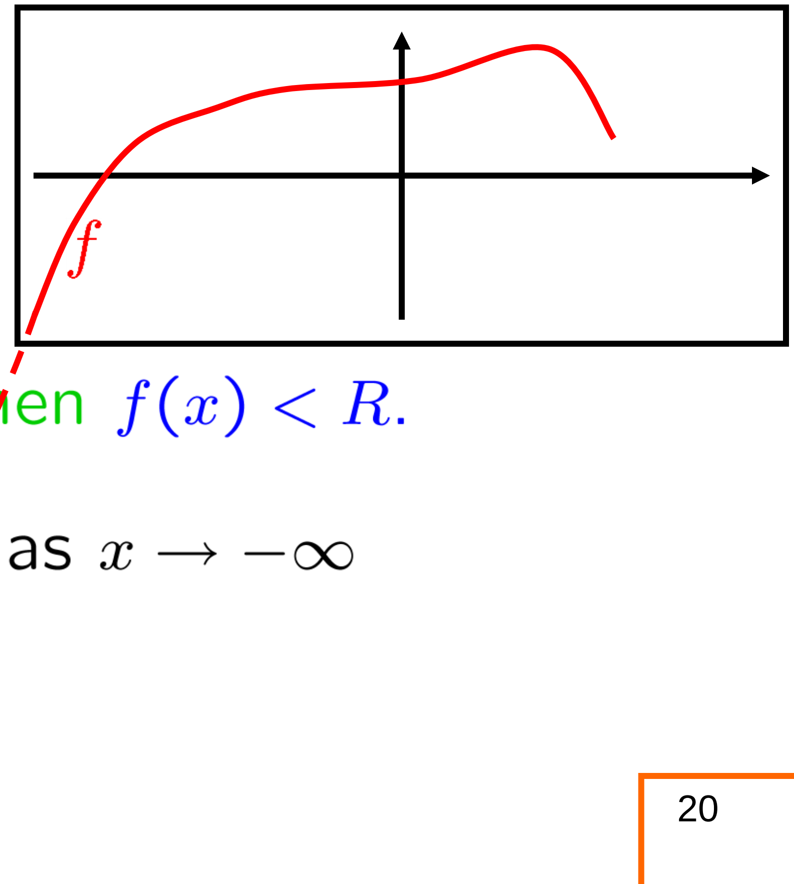
is equal to negative infinity

∞

and means, intuitively:

if x is very negative,

then $f(x)$ is very negative



and means, rigorously:

$\forall R < 0$, $\exists S < 0$ such that

if $x < S$, then $f(x) < R$.

Alternative notation: $f(x) \to -\infty$ as $x \to -\infty$

Spp

Def'n: $\lim_{x \to -\infty} f(x) = \infty$ is read

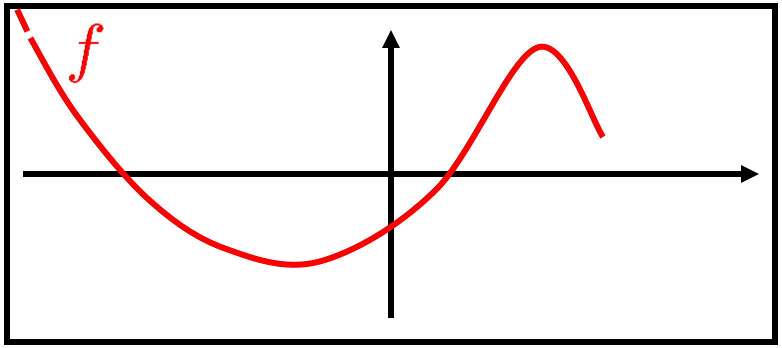
the limit of $f(x)$ as x approaches negative infinity is equal to infinity

∞

and means, intuitively:

if x is very negative,

then $f(x)$ is very positive



and means, rigorously:

$\forall R > 0$, $\exists S < 0$ such that

if $x < S$, then $f(x) > R$.

Alternative notation: $f(x) \to \infty$ as $x \to -\infty$

Spp

Def'n: $\lim_{x \to \infty} f(x) = \infty$ is read

the limit of $f(x)$ as x approaches infinity

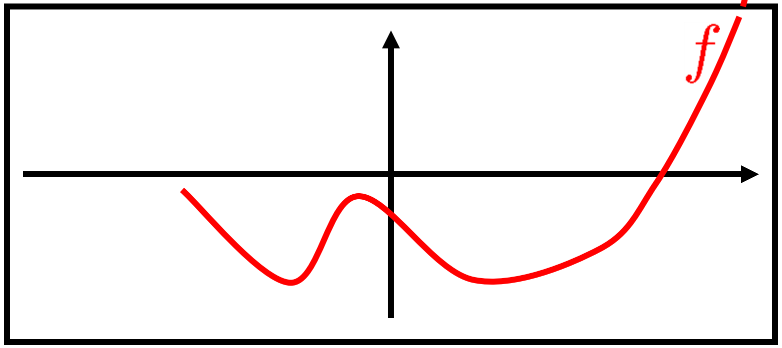
is equal to infinity

$-\infty$

and means, intuitively:

if x is very positive,

then $f(x)$ is very positive



and means, rigorously:

$\forall R > 0$, $\exists S > 0$ such that

if $x > S$, then $f(x) > R$.

Alternative notation: $f(x) \to \infty$ as $x \to \infty$

Spp

Def'n: $\lim_{x \to \infty} f(x) = -\infty$ is read

the limit of $f(x)$ as x approaches infinity

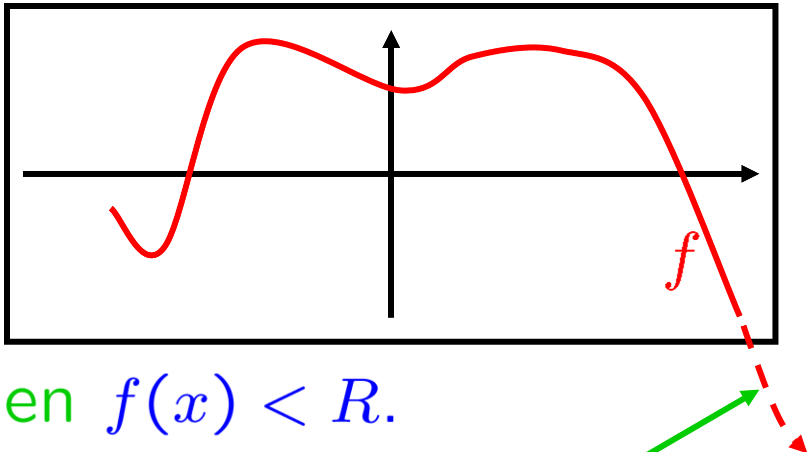
is equal to negative infinity

L

and means, intuitively:

if x is very positive,

then $f(x)$ is very negative



and means, rigorously:

$\forall R < 0$, $\exists S > 0$ such that

if $x > S$, then $f(x) < R$.

Alternative notation: $f(x) \to -\infty$ as $x \to \infty$

Spp

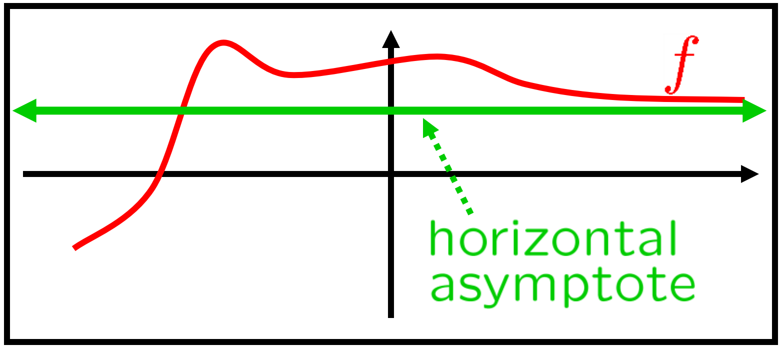
§4.8, p. 81, Def 4.10: $\lim_{x \to \infty} f(x) = L$ is read

the limit of $f(x)$ as x approaches infinity is equal to L

$-\infty$

and means, intuitively:

if x is very positive,
then $f(x)$ is close to L



and means, rigorously:

$\forall \varepsilon > 0$, $\exists S > 0$ such that

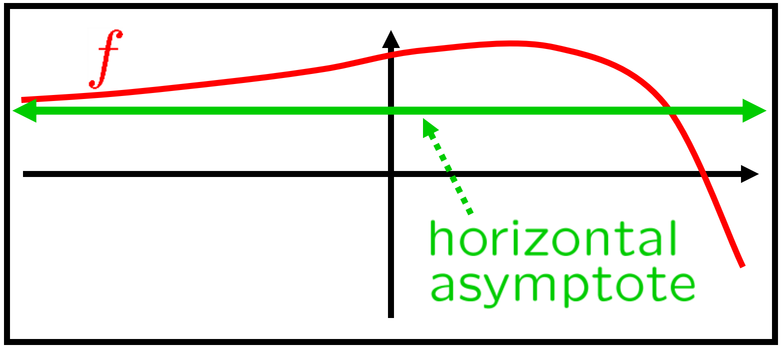
if $x > S$, then $|(f(x)) - L| < \varepsilon$.

Alternative notation: $f(x) \to L$ as $x \to \infty$

Def'n: $\lim_{x\to-\infty} f(x) = L$ is read

the limit of $f(x)$ as x approaches negative infinity
is equal to L

and means, intuitively:
if x is very negative,
then $f(x)$ is close to L



and means, rigorously:

$\forall \varepsilon > 0$, $\exists S < 0$ such that

if $x < S$, then $|(f(x)) - L| < \varepsilon$.

Alternative notation: $f(x) \to L$ as $x \to -\infty$

STOP

SKILL
intuitive def'ns of lim

Spp