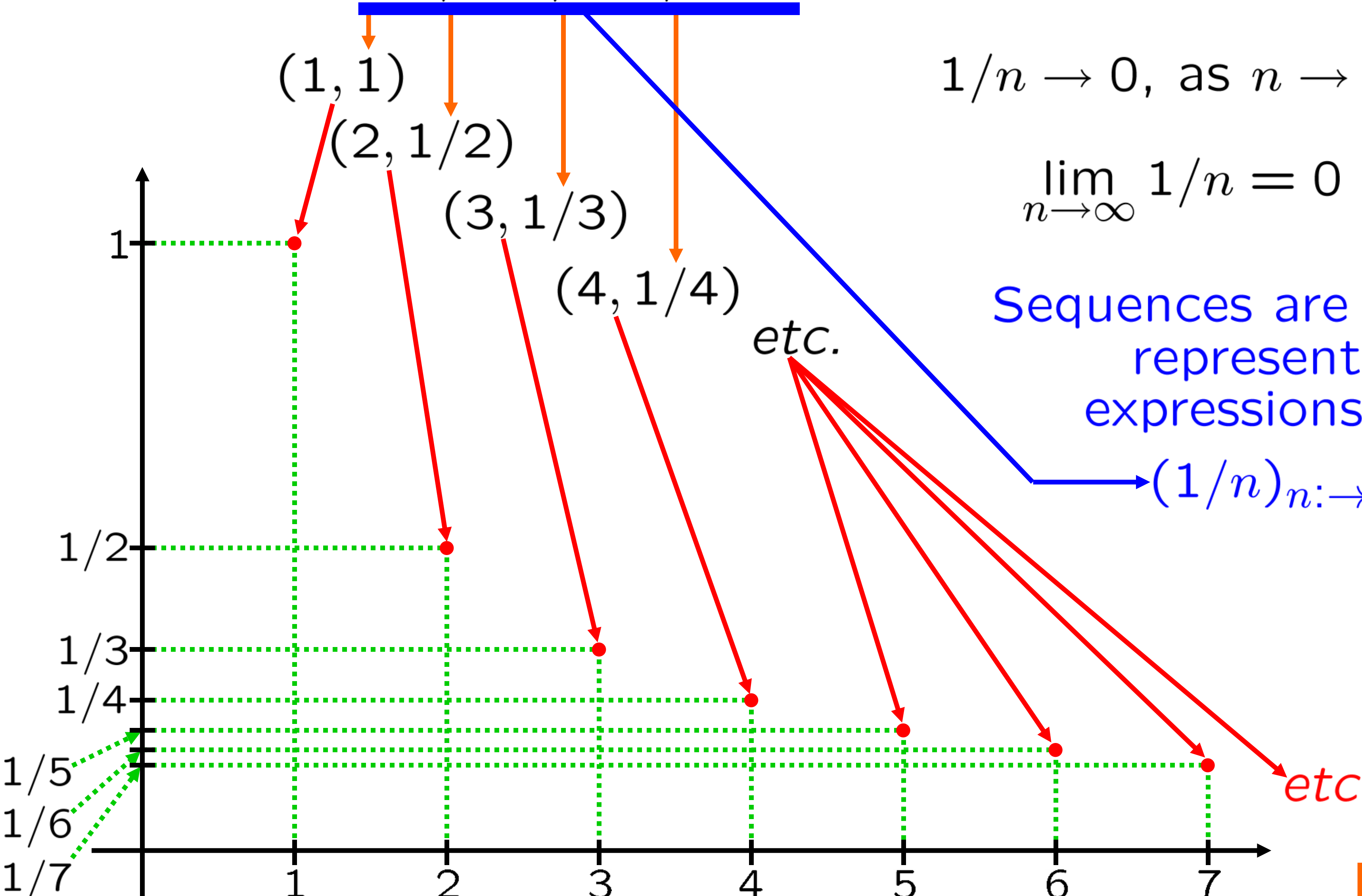
CALCULUS
Sequences

Def'n: A **sequence** is an ordered list of scalars.

visualization?

e.g.: $1, 1/2, 1/3, 1/4, \ldots \to 0$



$1/n \to 0$, as $n \to \infty$

$$\lim_{n\to\infty} 1/n = 0$$

Sequences are often represented by expressions of n.

$(1/n)_{n:\to 1,2,\ldots}$

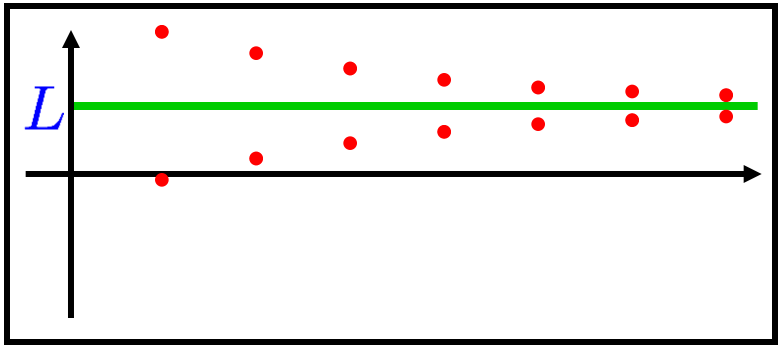
Def'n: $\lim_{n\to\infty} a_n = L$ is read

the limit of a_n as n approaches infinity is equal to L

and means, intuitively:

if n is a very large positive integer,

then a_n is close to L



and means, rigorously:

$\forall \varepsilon > 0$, $\exists N \geq 1$ such that

if $n \geq N$, then $|a_n - L| < \varepsilon$.

Alternative notation: $a_n \to L$ as $n \to \infty$, $a_1, a_2, a_3, \ldots \to L$

could replace "$\geq$" with "$>$"

traditional to use "$\geq$" for integers

Def'n: $\lim_{n\to\infty} a_n = \infty$ is read

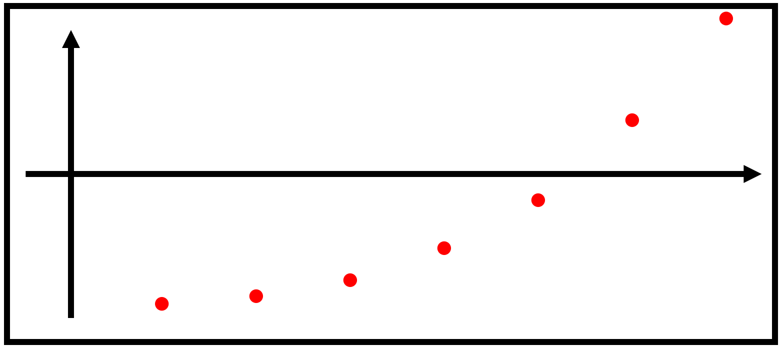
the limit of a_n as n approaches infinity is equal to infinity

$-\infty$

and means, intuitively:

if n is a very large positive integer,

then a_n is very positive



and means, rigorously:

$\forall R > 0$, $\exists N \geq 1$ such that

if $n \geq N$, then $a_n > R$.

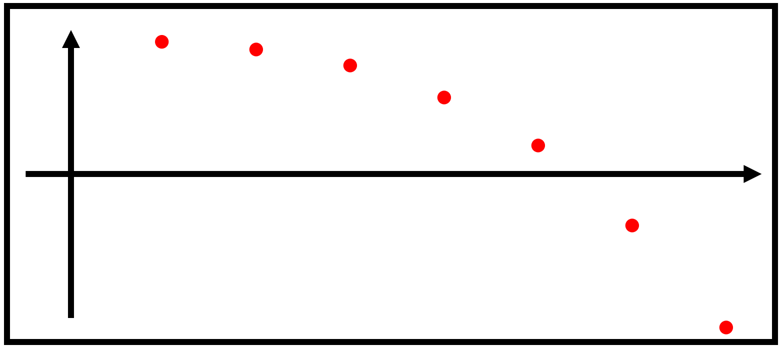
Alternative notation: $a_n \to \infty$ as $n \to \infty$, $a_1, a_2, a_3, \ldots \to \infty$

Def'n: $\lim_{n\to\infty} a_n = -\infty$ is read

the limit of a_n as n approaches infinity is equal to negative infinity

and means, intuitively:

if n is a very large positive integer,
then a_n is very negative



and means, rigorously:

$\forall R < 0$, $\exists N \geq 1$ such that
if $n \geq N$, then $a_n < R$.

Alternative notation: $a_n \to -\infty$ as $n \to \infty$, $a_1, a_2, a_3, \ldots \to -\infty$

SKILL
intuitive def'ns of lim

Next subtopic: Define the real number e

Spp

$1 → bank
6% nominal annual interest,
compounded annually

Goal: Describe e.

one year mark, in bank: $1.06

$1 → bank
6% nominal annual interest,
compounded semi-annually

$\frac{6\%}{2} = 3\% = 0.03$

6 months mark, in bank: $(\$1) + (0.03)(\$1)$

$= \$1.03$

Next subtopic: Define the real number e

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6 months mark, in bank: $1.03

$1 → bank
mark, in bank:
6% nominal annual interest,
compounded semi-annually

$1.03

one year mark, in bank:

Spp

Goal: Describe e.

$1 → bank
6% nominal annual interest,
compounded annually

one year mark, in bank: $1.06

$1 → bank
6% nominal annual interest,
compounded semi-annually

$\frac{6\%}{2} = 3\% = 0.03$

6 months mark, in bank: $1.03

$1 → bank
6% nominal annual interest,
compounded semi-annually

one year mark, in bank: $(1)(\$1.03)+(0.03)(\$1.03)$

$=(1.03)(\$1.03) = \$(1.03)^2$

Adding 3%
is the same as
multiplying by 1.03.

Spp

Goal: Describe e.

$\$1 \longrightarrow$ bank

6% nominal annual interest,
compounded annually

one year mark, in bank: $\$1.06$

$\$1 \longrightarrow$ bank

6% nominal annual interest,
compounded semi-annually

6 months mark, in bank: $\$1.03$

$\$1 \longrightarrow$ bank

6% nominal annual interest,
compounded semi-annually

one year mark, in bank: $\$(1.03)^2$

$\$(1.03)^2$

$\$1 \longrightarrow$ bank

6% nominal annual interest,
compounded bimonthly

one year mark, in bank:

Spp

$\$1 \longrightarrow$ bank

Goal: Describe e.

6% nominal annual interest,
compounded annually

one year mark, in bank: $1.06

$\$1 \longrightarrow$ bank

6% nominal annual interest,
compounded semi-annually

6 months mark, in bank: $1.03

$\$1 \longrightarrow$ bank

6% nominal annual interest,
compounded semi-annually

one year mark, in bank: $\$(1.03)^2$

$\$1 \longrightarrow$ bank

$$\frac{6\%}{6} = \frac{0.06}{6} = 0.01$$

6% nominal annual interest,
compounded bimonthly

one year mark, in bank: $\$(1.01)^6$

Spp

Goal: Describe e.

$\$1 \to$ bank

6% nominal annual interest,
compounded semi-annually

one year mark, in bank: $\$(1.03)^2$

$\$1 \to$ bank

6% nominal annual interest,
compounded bimonthly

one year mark, in bank: $\$(1.01)^6$

$\$1 \to$ bank

6% nominal annual interest,
compounded semi-annually

one year mark, in bank: $\$(1.03)^2$

$\$1 \to$ bank

100% nominal annual interest,
compounded annually

$\$1 \to$ bank

6% nominal annual interest,
compounded bimonthly

one year mark, in bank: $\$(1.01)^6$

Goal: Describe e.

$\$1 \to$ bank

6% nominal annual interest,

compounded semi-annually

one year mark, in bank: $\$(1.03)^2$

$\$1 \to$ bank

6% nominal annual interest,

compounded bimonthly

one year mark, in bank: $\$(1.01)^6$

$\$1 \to$ bank

100% nominal annual interest,

compounded annually

$\frac{100\%}{1} = \frac{1}{1} = 1$

one year mark, in bank: $\$(1 + \boxed{1})$

$\$1 \to$ bank

100% nominal annual interest,

compounded semi-annually

$\frac{100\%}{2} = \frac{1}{2}$

one year mark, in bank: $\$\left(1 + \boxed{\frac{1}{2}}\right)^2$

Goal: Describe e.

\$1 $\rightarrow$ bank

100% nominal annual interest,
compounded annually

one year mark, in bank: $\$(1+1)$

\$1 $\rightarrow$ bank

100% nominal annual interest,
compounded semi-annually

one year mark, in bank: $\$\left(1+\frac{1}{2}\right)^2$

\$1 $\rightarrow$ bank

\$1 $\rightarrow$ bank nominal annual interest,

100% nominal annual interest,

one year m compounded quarterly $+1)$

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one year mark, in bank:

100% nominal annual interest,
compounded semi-annually

one year mark, in bank: $\$\left(1+\frac{1}{2}\right)^2$

Goal: Describe e.

$\$1 \to$ bank

100% nominal annual interest,
compounded annually

one year mark, in bank: $\$(1+1)$

$\$1 \to$ bank

100% nominal annual interest,
compounded semi-annually

one year mark, in bank: $\$\left(1+\frac{1}{2}\right)^2$

$\$1 \to$ bank

100% nominal annual interest,
compounded quarterly

one year mark, in bank: $\$\left(1+\frac{1}{4}\right)^4$

$\$1 \to$ bank

100% nominal annual interest,
compounded monthly

Spp

Goal: Describe e.

$1 → bank
100% nominal annual interest,
compounded quarterly

one year mark, in bank: $$\$\left(1+\frac{1}{4}\right)^4$$

$1 → bank
100% nominal annual interest,
compounded monthly

one year mark, in bank:

$1 → bank
100% nominal annual interest,
compounded quarterly

one year mark, in bank: $$\$\left(1+\frac{1}{4}\right)^4$$

$1 → bank
100% nominal annual interest,
compounded monthly

Goal: Describe e.

\$1 $\longrightarrow$ bank

100% nominal annual interest,
compounded quarterly

one year mark, in bank: $\$\left(1+\frac{1}{4}\right)^{4}$

\$1 $\longrightarrow$ bank

100% nominal annual interest,
compounded monthly

one year mark, in bank: $\$\left(1+\frac{1}{12}\right)^{12}$

\$1 $\longrightarrow$ bank

100% nominal annual interest,
compounded daily

one year mark, in bank: $\$\left(1+\frac{1}{365}\right)^{365}$

Goal: Describe e.

$1 $\longrightarrow$ bank

100% nominal annual interest,
compounded daily

one year mark, in bank: $$\$\left(1+\frac{1}{365}\right)^{365}$$

$1 $\longrightarrow$ bank

100% nominal annual interest,
compounded n times per year

one year mark, in bank:

$1 $\longrightarrow$ bank

100% nominal annual interest,
compounded daily

one year mark, in bank: $$\$\left(1+\frac{1}{365}\right)^{365}$$

Goal: Describe e.

$\$1 \rightarrow$ bank

100% nominal annual interest,
compounded daily

one year mark, in bank: $$\$\left(1+\frac{1}{365}\right)^{365}$$

$\$1 \rightarrow$ bank

100% nominal annual interest,
compounded n times per year

one year mark, in bank: $$\$\left(1+\frac{1}{n}\right)^{n}$$

$\$1 \rightarrow$ bank

100% nominal annual interest,
compounded continuously

one year mark, in bank: $$\$\lim_{n\rightarrow\infty}\left(1+\frac{1}{n}\right)^{n}$$

Spp

Goal: Describe e.

$1 $\longrightarrow$ bank

100% nominal annual interest,

compounded continuously

one year mark, in bank: $$\$ \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$$

$$e := \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$$

$1 $\longrightarrow$ bank

100% nominal annual interest,

compounded continuously

one year mark, in bank: $$\$ \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$$

Goal: Describe e.

\$1 $\rightarrow$ bank

100% nominal annual interest,
compounded continuously

one year mark, in bank: $\$ \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

$$e := \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

$\lim_{n\to\infty} n^7 = \infty$ $\qquad x :\rightarrow n^7$

(or any expression of n
that $\rightarrow \infty$, as $n \rightarrow \infty$)

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n^7}\right)^{n^7}$$

Spp

Goal: Describe e.

\$1 → bank

unrealistic

100% nominal annual interest, compounded continuously

one year mark, in bank: $\$ \underbrace{\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n}_{e}$

n	$\left(1+\frac{1}{n}\right)^n$
1	2
5	2.48832
10	2.5937425
20	2.6532977
100	2.7048138
500	2.7155685
1,000	2.7619239
10,000	2.7181459
100,000	2.7182682

$$\boxed{e} := \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$$

$\lim_{n\to\infty} n = \infty$ $\quad x :\to n$

(or any expression of n that $\to \infty$, as $n \to \infty$)

$$e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \doteq 2.72$$

Spp

Goal: Describe e.

$1 $\to$ bank

unrealistic → 100% nominal annual interest,

compounded continuously

one year mark, in bank: $$\$ \underbrace{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}_{e}$$

$1 $\to$ bank

4% nominal annual interest,

compounded n times per year

one year mark, in bank: $$\$ \left(1+\frac{0.04}{n}\right)^n$$

$1 $\to$ bank

4% nominal annual interest,

compounded continuously

one year mark, in bank: $$\$ \lim_{n\to\infty}\left(1+\frac{0.04}{n}\right)^n$$

Spp

Goal: Describe e.

$1 → bank

100% nominal annual interest, compounded continuously

unrealistic

one year mark, in bank: $\$ \underbrace{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}_{e}$

$1 → bank

4% nominal annual interest, compounded continuously

one year mark, in bank: $\$ \lim_{n\to\infty}\left(1+\frac{0.04}{n}\right)^n$

$1 → bank

4% nominal annual interest, compounded continuously

$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

one year mark, in bank: $\$ \lim_{n\to\infty}\left(1+\frac{0.04}{n}\right)^n$

Goal: Describe e.

$1 → bank
100% nominal annual interest, compounded continuously

unrealistic

one year mark, in bank: $\$ \underbrace{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}_{e}$

$1 → bank
4% nominal annual interest, compounded continuously

one year mark, in bank: $\$ \lim_{n\to\infty}\left(1+\frac{0.04}{n}\right)^n$

$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

$x :\to n/0.04$

$$e^{0.04} = \left[\lim_{n\to\infty}\left[\left(1+\boxed{\frac{1}{n/0.04}}\right)^{n/0.04}\right]^{0.04}\right]$$

$\frac{1}{n/0.04} = \frac{0.04}{n}$

$$\lim_{n\to\infty} n/0.04 = \infty$$

Spp

Goal: Describe e.

$1 → bank

100% nominal annual interest, compounded continuously

unrealistic

one year mark, in bank: $\$ \underbrace{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}_{e}$

$1 → bank

4% nominal annual interest, compounded continuously

one year mark, in bank: $\$ \underbrace{\lim_{n\to\infty}\left(1+\frac{0.04}{n}\right)^n}_{e^{0.04}}$

equal

$$e^{0.04} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n/0.04}\right)^{n/0.04}\right]^{0.04} = \lim_{n\to\infty}\left(1+\frac{0.04}{n}\right)^n$$

$\frac{1}{n/0.04} = \frac{0.04}{n}$

Spp

Goal: Describe e.

\$1 → bank

100% nominal annual interest, compounded continuously

unrealistic

one year mark, in bank: $\$ \underbrace{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}_{e}$

\$1 → bank

4% nominal annual interest, compounded continuously

one year mark, in bank: $\$ \underbrace{\lim_{n\to\infty}\left(1+\frac{0.04}{n}\right)^n}_{e^{0.04}}$

The moral:

Most continuous compounding problems (realistic or not) lead to e.

STOP

Spp