CALCULUS
Continuity

Def'n 2.18, p. 42:
f is **continuous at** a
if $\lim_{x \to a} f(x) = f(a)$.

Requires: $\lim_{x \to a} f(x)$ exists

$f(a)$ exists

some (punctured nbd of a) $\subseteq$ dom$[f]$

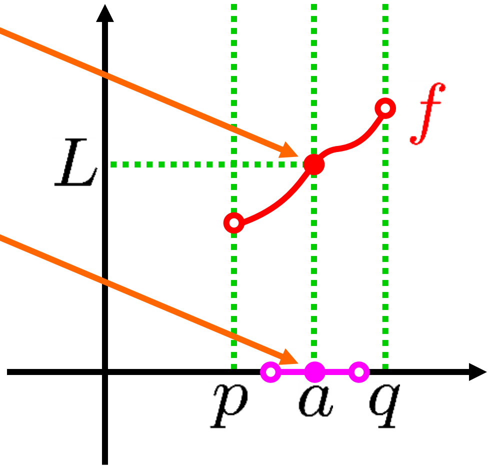
$a \in$ dom$[f]$

some (nbd of a) $\subseteq$ dom$[f]$

$\lim_{x \to p} f(x)$ DNE

$\lim_{x \to a} f(x) = L$

$\lim_{x \to q} f(x)$ DNE



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Otherwise NOT contin. at a.

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Spp

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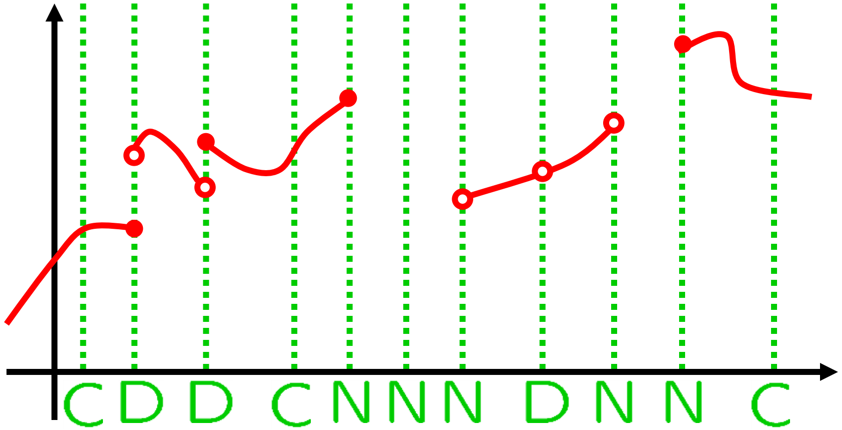
Next: Visualizing continuity and discontinuity



C=continuous
D=discontinuous
N= neither C nor D

discontinuous at a means:
not continuous at a and
some (punctured nbd of a) $\subseteq$ dom$[f]$,

Spp

Def'n 2.18, p. 42:
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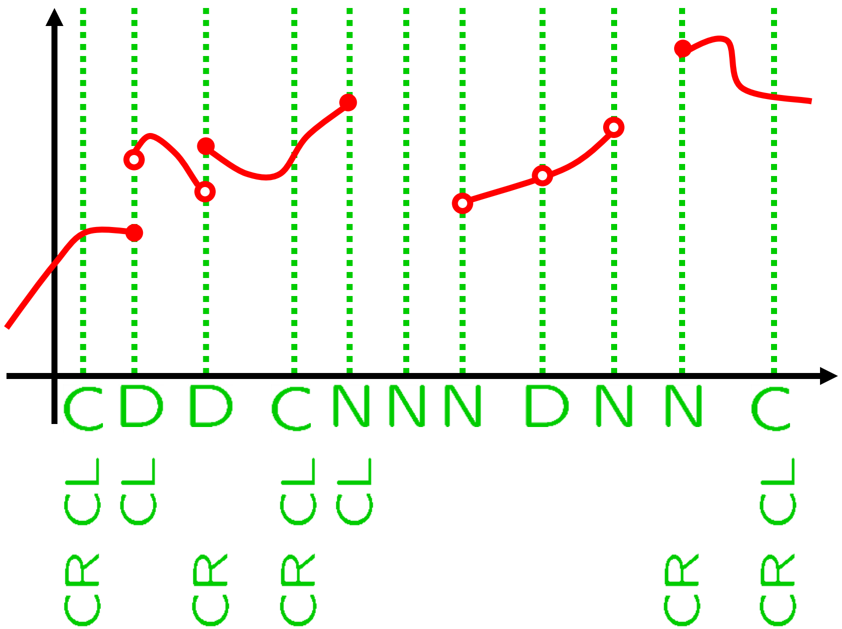
Def'n:
f is **contin. from the left (resp. right) at** a
if $\lim_{x \to a^-} f(x) = f(a)$ (resp. $\lim_{x \to a^+} f(x) = f(a)$).



C=continuous
D=discontinuous
N= neither C nor D
CL=continuous from the left
CR=continuous from the right

SKILL
recognize continuity visually

Next: Three types of discontinuities

Fact: continuous $\Leftrightarrow$ continuous from the left and continuous from the right

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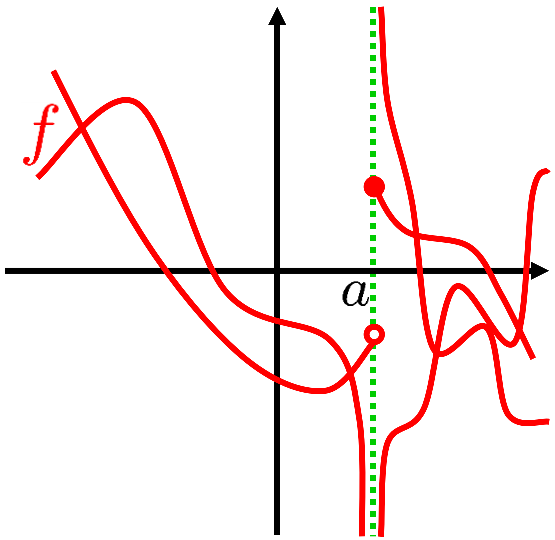
Types of Discontinuities

infinite discontinuity:

at least one one-sided limit is $\pm\infty$



f has an infinite discontinuity at a

$f(x)$ has an infinite discontinuity at $x = a$

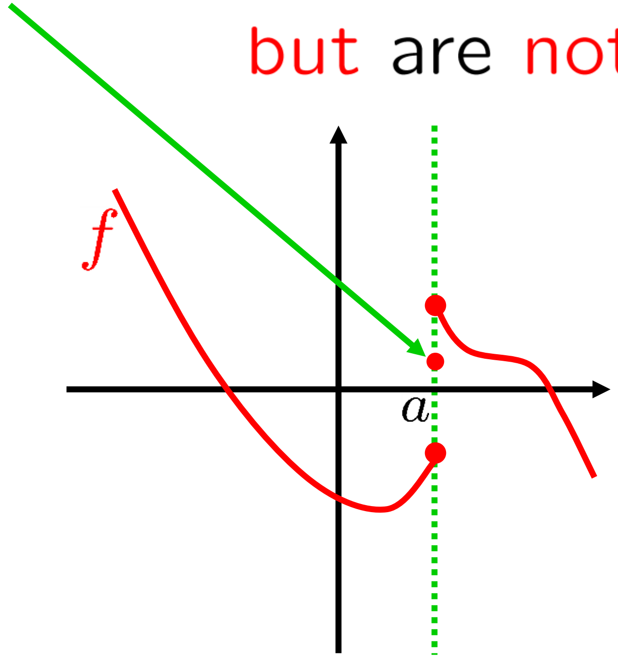
$f(t)$ has an infinite discontinuity at $t = a$

etc.

Types of Discontinuities

jump discontinuity:

the two one-sided limits exist (and are finite),
but are not equal

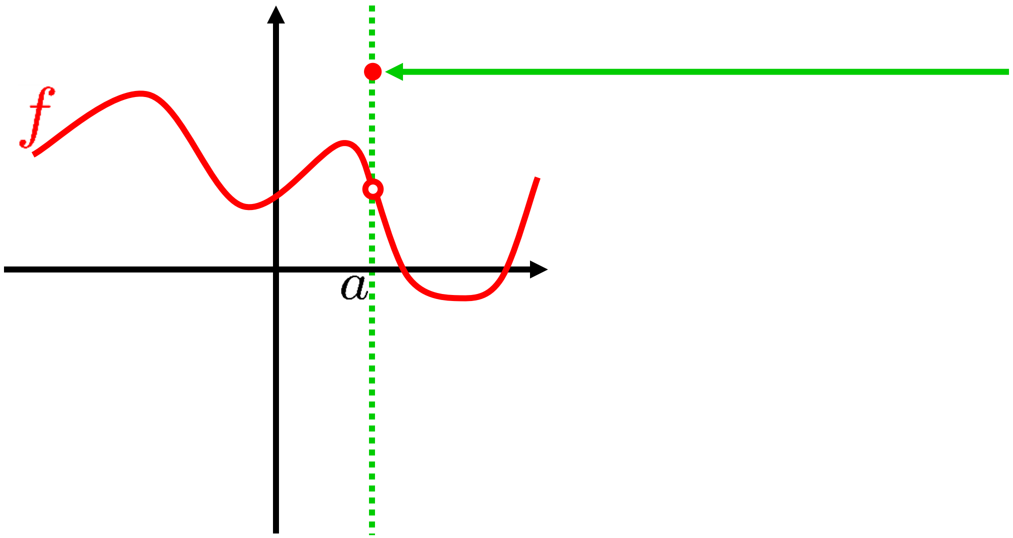


f has a jump discontinuity at a
$f(x)$ has a jump discontinuity at $x = a$
$f(t)$ has a jump discontinuity at $t = a$
etc.

Types of Discontinuities

removable discontinuity:

the two-sided limit exists (and is finite),
but either the function is not defined
or its value not equal to the limit.



f has a removable discontinuity at a
$f(x)$ has a removable discontinuity at $x = a$
$f(t)$ has a removable discontinuity at $t = a$
etc.

Next: Some theory

Def'n 2.18, p. 42:

f is **continuous at** a if $\lim_{x \to a} f(x) = f(a)$.

Def'n:

f is **contin. from the left (resp. right) at** a if $\lim_{x \to a^-} f(x) = f(a)$ (resp. $\lim_{x \to a^+} f(x) = f(a)$).

Theorem: Let $a \in \mathbb{R}$.

Assume f and g are continuous at a.

Then $\forall p, q \in \mathbb{R}$, $pf + qg$ continuous at a

and fg is continuous at a

and $(g(a) \neq 0) \Rightarrow (f/g$ is continuous at $a)$.

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Spp

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and fg is continuous $\overbrace{}^{\text{from the right}}$ at a

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Def'n: Say $-\infty \leq a < b \leq \infty$.

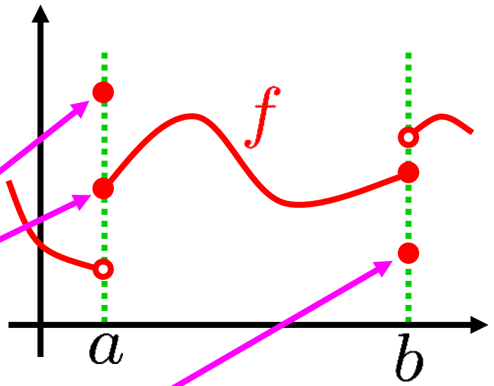
f is **continuous on** (a,b) means: $\forall x \in (a,b)$, f is continuous at x

f is **continuous on** $[a,b)$ means: f is continuous on (a,b) and f is continuous from the right at a

$a \neq -\infty$

f is **continuous on** $(a,b]$ means: f is continuous on (a,b) and f is continuous from the left at b

$b \neq \infty$



f contin on $(a,b]$

Spp

Def'n 2.18, p. 42:
f is **continuous at** a if $\lim_{x \to a} f(x) = f(a)$.

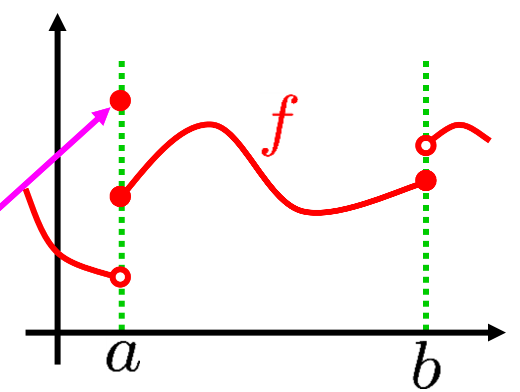
Def'n 2.19, p. 42:
f is **continuous** if, $\forall a \in \text{dom}[f]$, f is contin. at a.

Def'n:
f is **contin. from the left (resp. right) at** a if $\lim_{x \to a^-} f(x) = f(a)$ (resp. $\lim_{x \to a^+} f(x) = f(a)$).

Def'n: Say $-\infty \leq a < b \leq \infty$.

f is **continuous on** (a,b) means:
$\forall x \in (a,b)$, f is continuous at x

f is **continuous on** $[a,b]$ means:
f is continuous on (a,b) and $a \neq -\infty$
f is continuous from the right at a
and $b \neq \infty$
f is continuous from the left at b



f contin on $[a,b]$

Spp

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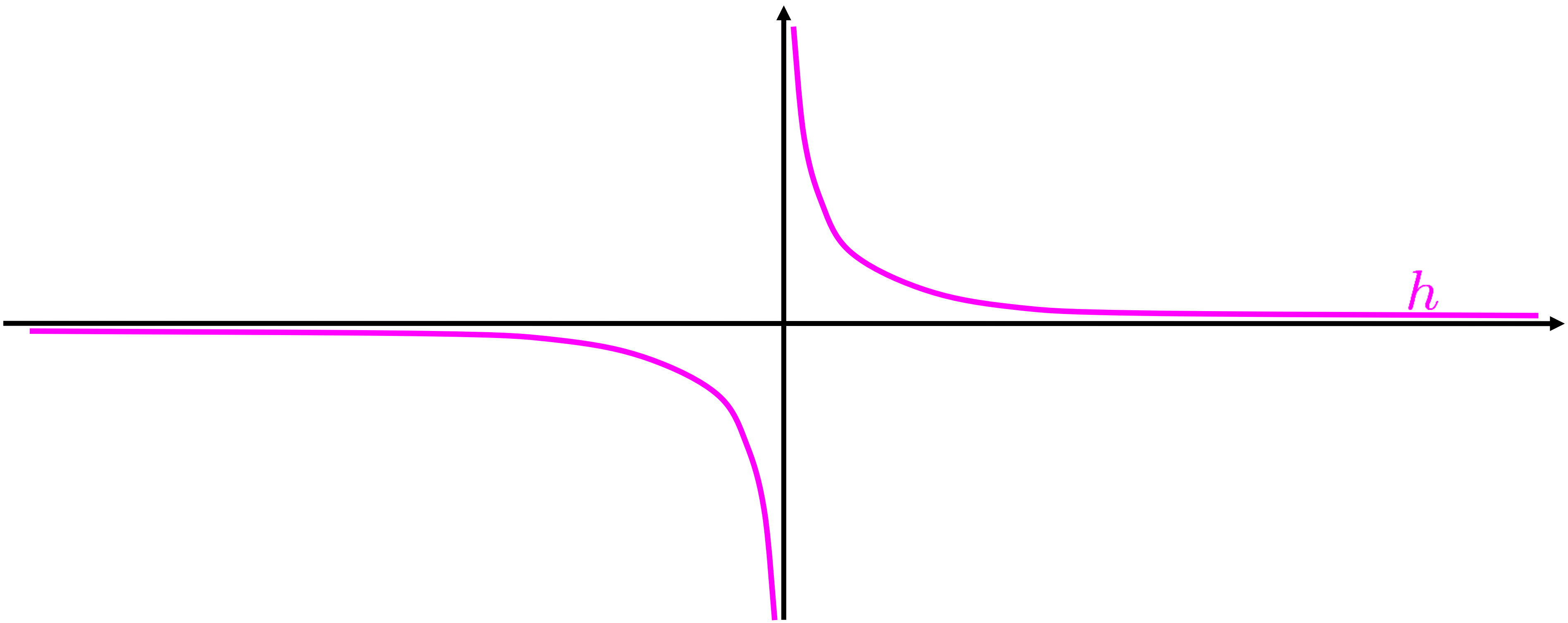

$h(x) = 1/x$

h is continuous,
but h is not continuous at 0.

Def'n 2.18, p. 42:
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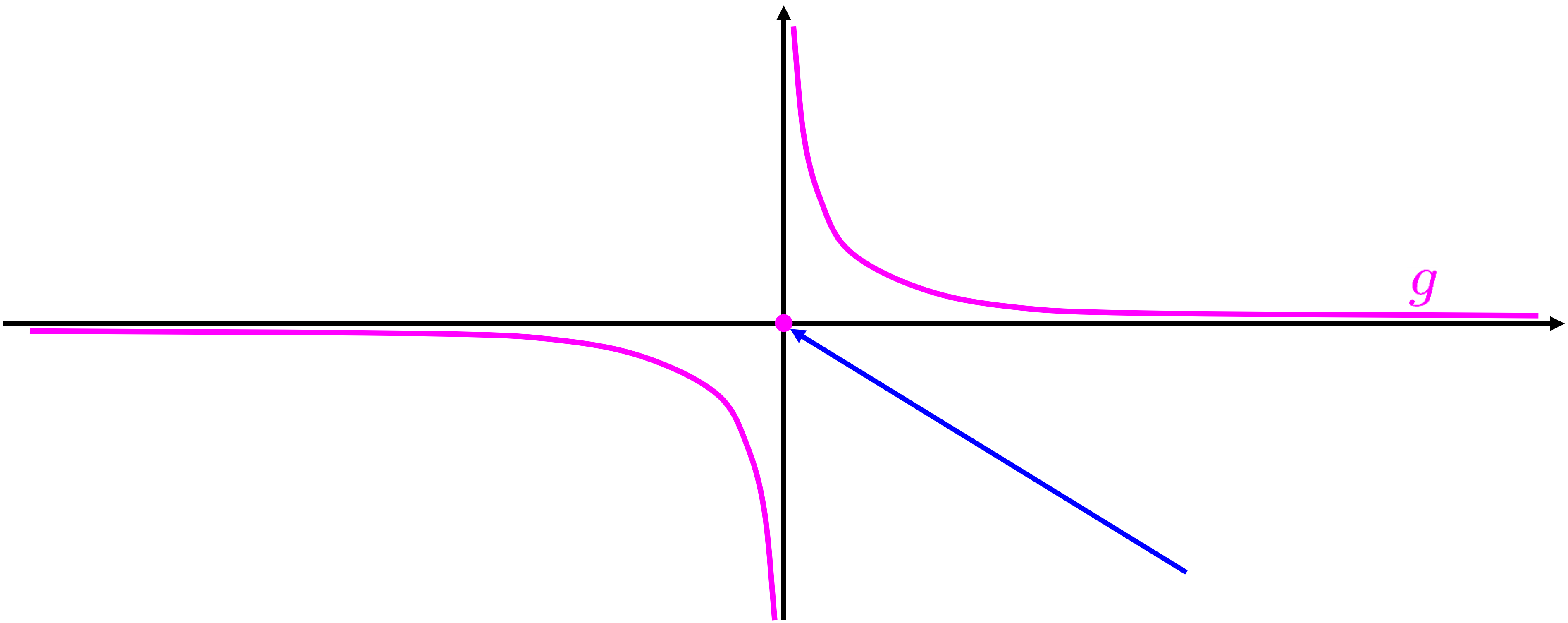

$$g(x) = \begin{cases} 1/x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

g is NOT continuous.

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Def'n 2.19, p. 42:
f is **continuous** if,
$\forall a \in \text{dom}[f]$, f is contin. at a.

Theorem: The following collections of functions are continuous: polynomials, rational functions

$b^{\bullet}$ $(b > 0)$ exponential functions

$\log_b$ $(b \in (0, \infty) \backslash \{1\})$ logarithmic functions

$\sin, \cos, \tan, \cot, \sec, \csc$ trigononmetic functions

$\sqrt[n]{\bullet}$, $n = 1, 3, 5, \ldots$

$\arctan, \text{arccot}$

Theorem: The following collections of functions are continuous on their domains,
but are not continuous: $\arcsin, \arccos$

$\sqrt[n]{\bullet}$, $n = 2, 4, 6, \ldots$

$\arcsin : [-1, 1] \to [-\pi/2, \pi/2]$

Next: Intermed. Value Th'm

contin. from the right at -1, but not contin. at -1
contin. at all numbers in $(-1, 1)$
contin. from the left at 1, but not contin. at 1
contin. on $[-1, 1]$, *i.e.*, contin. on its domain, but not contin.

Spp

cf. §2.5, p. 42, THEOREM 2.22 (THE INTERMEDIATE VALUE THEOREM):

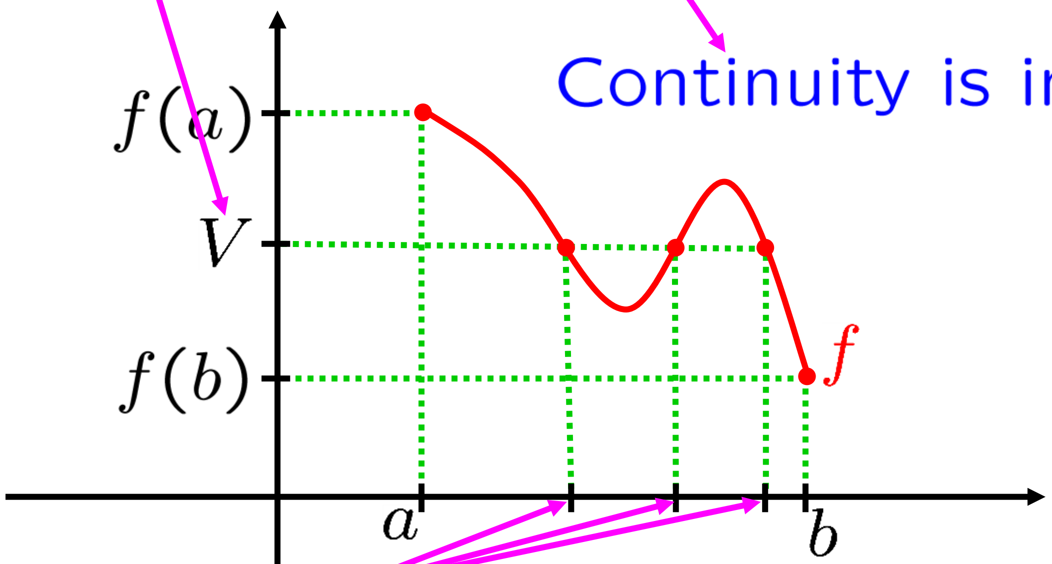
Suppose f is continuous on the compact interval $[a, b]$.

Let V be any number between $f(a)$ and $f(b)$, but neither $f(a)$ nor $f(b)$.

(That is, either $f(a) < V < f(b)$ or $f(b) < V < f(a)$.)

Then $\exists c \in (a, b)$ s.t. $f(c) = V$.

Continuity is important. . .



c is not necessarily unique

cf. §2.5, p. 42, THEOREM 2.22

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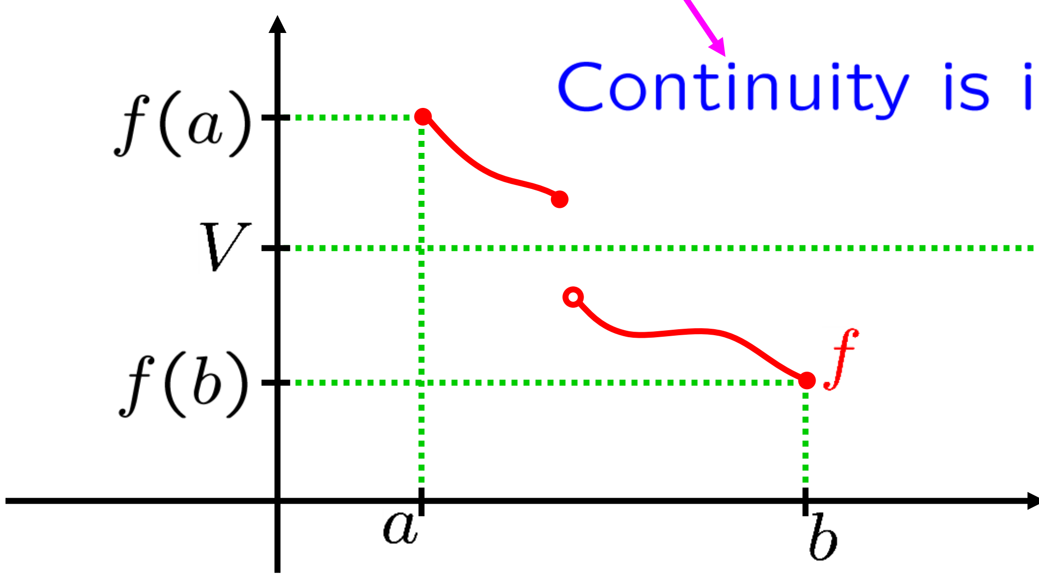
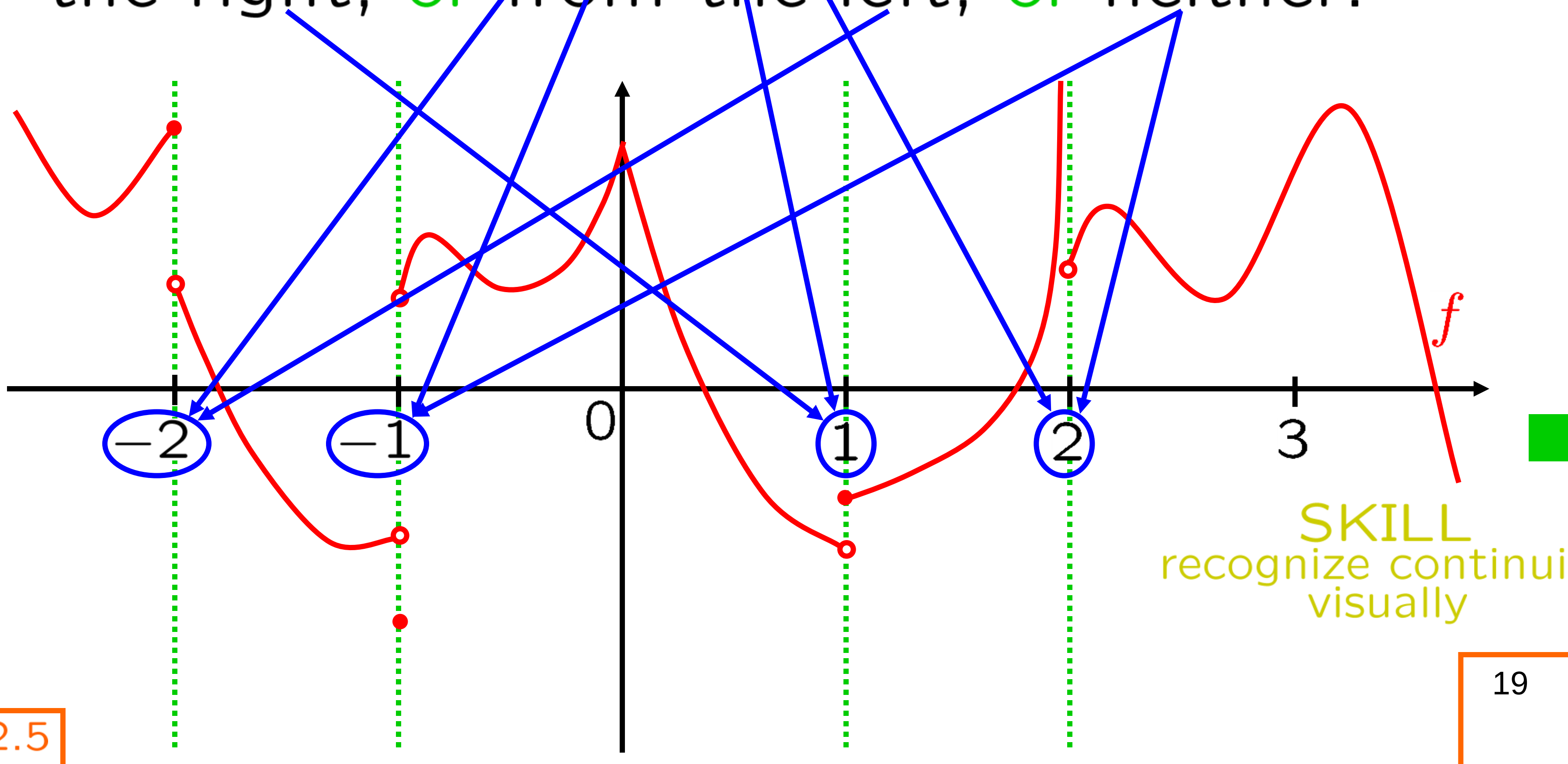


Next: Problems

Exercise:

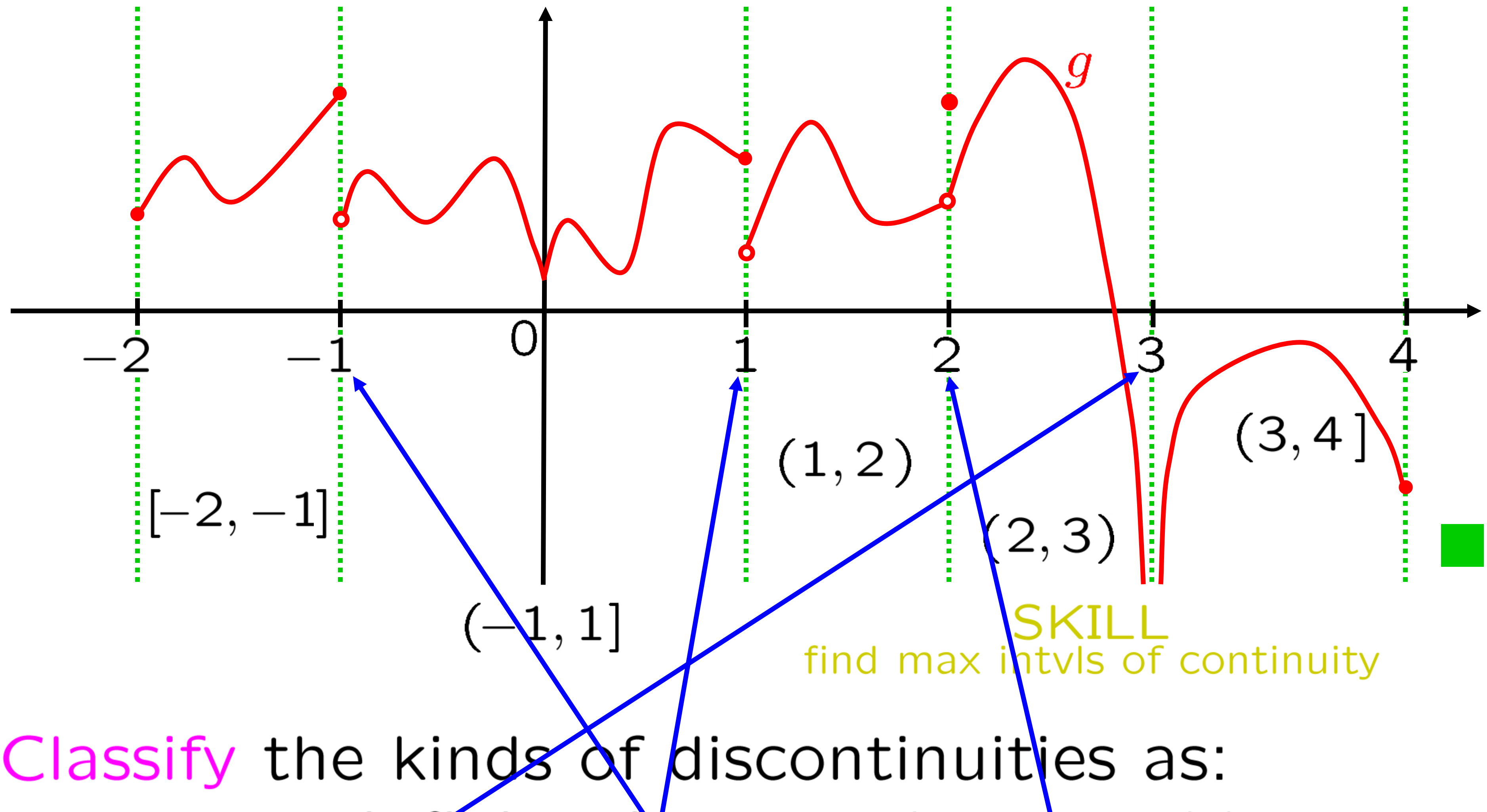
(a) From the graph of f, state the numbers at which f is discontinuous and explain why.

(b) For each of the numbers stated in part (a), determine whether f is continuous from the right, or from the left, or neither.



SKILL
recognize continuity visually

Exercise: From the graph of g, state the maximal intervals on which g is continuous.



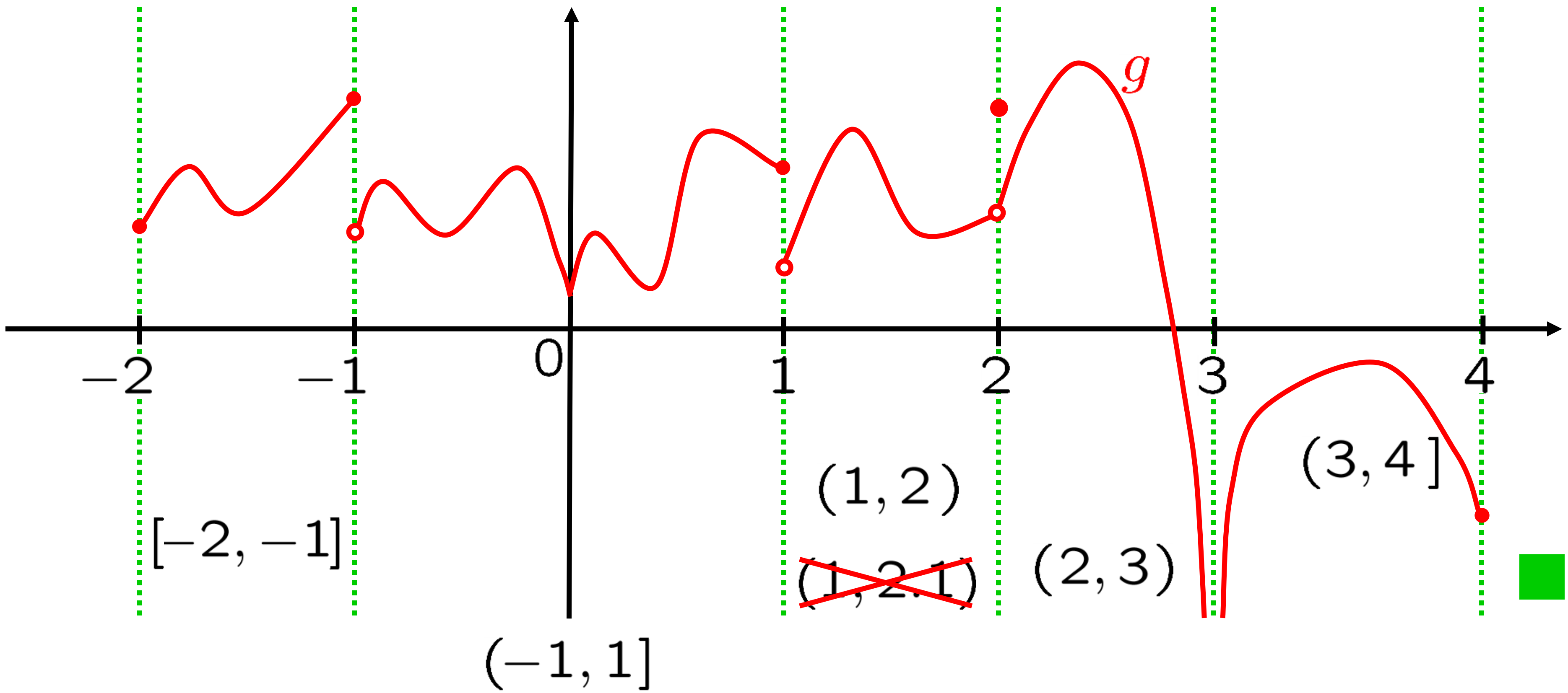
SKILL
find max intvls of continuity

Classify the kinds of discontinuities as:
infinite, jump and removable.

SKILL
recognize types of discontinuities visually

Spp

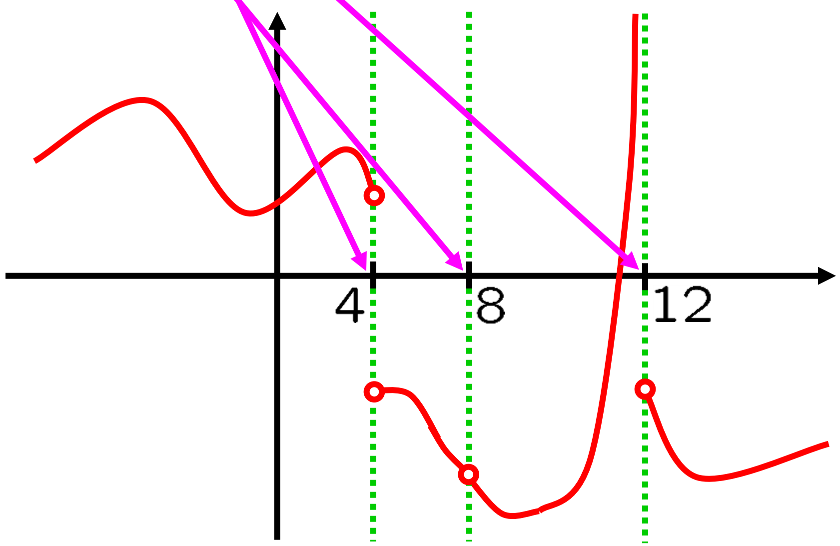
Exercise: From the graph of g, state the maximal intervals on which g is continuous.



Note: Even though $(-1, 1]$ is the longest of the intervals shown above, they are all said to be **maximal intervals of continuity**, because:

none can be made larger, without losing continuity.

Exercise: Sketch the graph of a function that has a jump discontinuity at $x = 4$, a removable discontinuity at $x = 8$ and an infinite discontinuity at $x = 12$, but is continuous elsewhere.



There are many other examples. . .

SKILL
sketch types of discontinuities

Exercise: In a certain country,

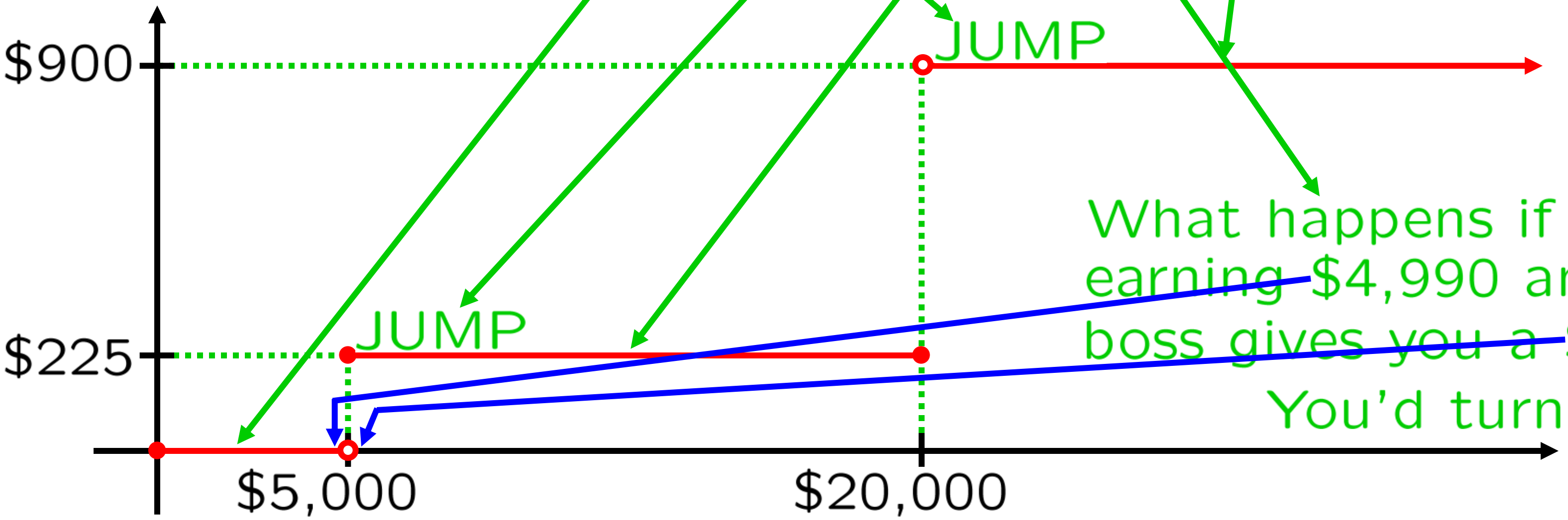
income below $5,000 is not taxed;

income between $5,000 and $20,000 results in a $225 tax bill;

income above $20,000 results in a $900 tax bill.

(a) Sketch a graph of taxes paid as a function of income.

(b) Discuss the discontinuities of this function and their significance to workers.



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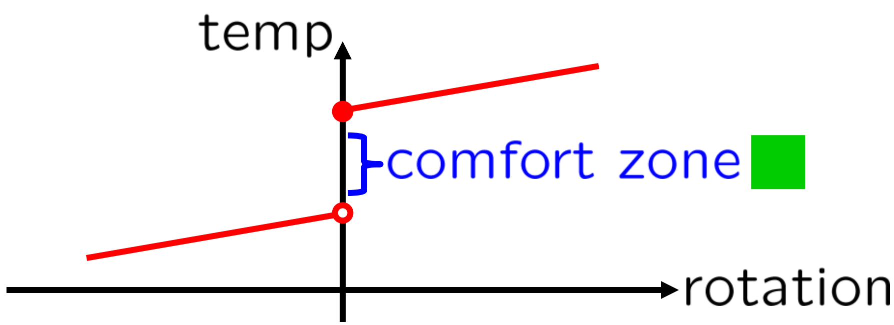
Discussion: The horror of the discontinuous shower temperature control . . .



Exercise: Find the values of a and b that make f continuous everywhere.

$$f(x) = \begin{cases} \dfrac{x^2-9}{x+3}, & \text{if } x < -3 \\ ax^2 - bx + 9, & \text{if } -3 \le x \le 2 \\ 4x + 5a + b, & \text{if } x > 2 \end{cases}$$

SKILL
force continuity

$(x+3)(x-3)$ $\quad -3$

$$\lim_{x\uparrow -3} f(x) = \lim_{x\uparrow -3} \frac{x^2-9}{x+3} = \lim_{x\uparrow -3} (x-3) = -6$$

$$\lim_{x\downarrow -3} f(x) = \lim_{x\downarrow -3} (ax^2 - bx + 9) = 9a + 3b + 9$$

-3

$$9a + 3b + 9 \;\|\; f(-3)$$

$$-6 = 9a + 3b + 9$$

Exercise: **Find** the values of a and b that make f continuous everywhere.

$$f(x) = \begin{cases} \dfrac{x^2 - 9}{x + 3}, & \text{if } x < -3 \\ ax^2 - bx + 9, & \text{if } -3 \le x \le 2 \\ 4x + 5a + b, & \text{if } x > 2 \end{cases}$$

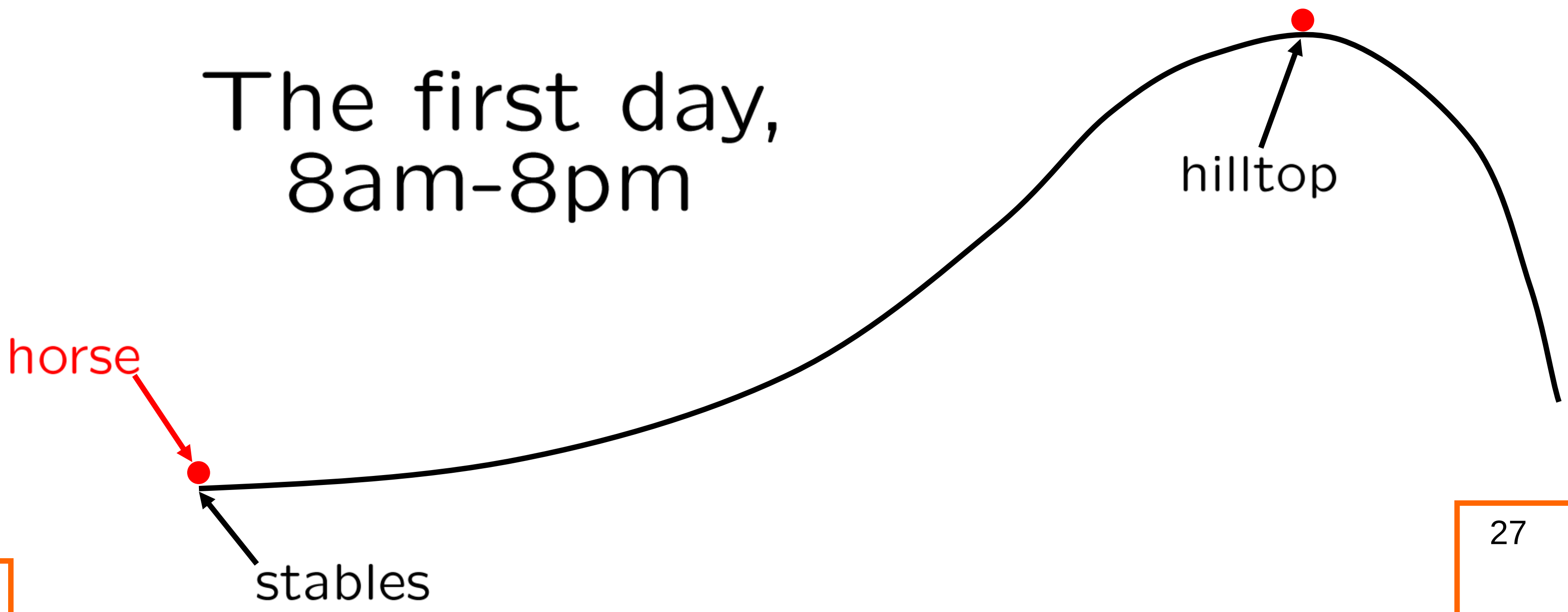
SKILL
force continuity

$$\lim_{x \uparrow 2} f(x) = \lim_{x \uparrow 2} (ax^2 - bx + 9) = 4a - 2b + 9 = f(2)$$

$$\lim_{x \downarrow 2} f(x) = \lim_{x \downarrow 2} (4x + 5a + b) = 8 + 5a + b$$

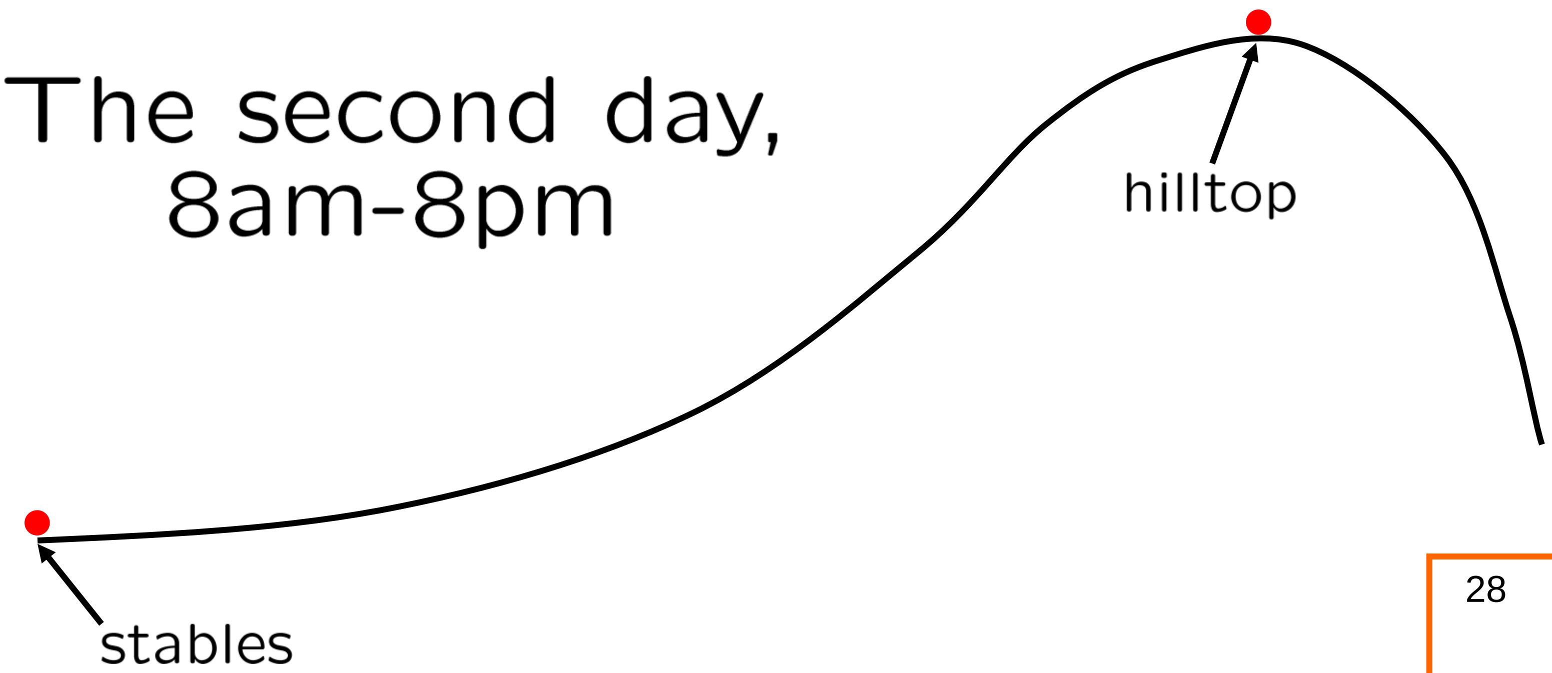
$$\left.\begin{array}{r} 4a - 2b + 9 = 8 + 5a + b \\ -6 = 9a + 3b + 9 \end{array}\right\} \Rightarrow \begin{cases} a = -2 \\ b = 1 \end{cases} \blacksquare$$

Exercise: A horse leaves the stables at 8am and takes an always-climbing path to the top of a hill, arriving at 8pm.



Exercise: A horse leaves the stables at 8am and takes an always-climbing path to the top of a hill, arriving at 8pm. The following day, it starts at 8am at the top and takes the same path back, arriving at the stables at 8pm. Use the Intermediate Value Theorem to show that there is a point on the path that the horse will cross at exactly the same time of day on both days.

The second day,
8am-8pm



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$f(t) :=$ horse's altitude t hours after 8am on the first day

$f(0) = A$ $f(12) = B$

$g(t) :=$ horse's altitude t hours after 8am on the second day

$g(0) = B$ $g(12) = A$

$h(t) := [f(t)] - [g(t)]$

$B :=$ altitude of the hilltop

$A :=$ altitude of the stables

$h(0) = A - B$

$h(12) = B - A$

0 is between $A - B$ and $B - A$.

$\exists t_0 \in (0, 12)$ s.t. $h(t_0) = 0$

§2.5

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$h(t) := [f(t)] - [g(t)]$

$B :=$ altitude of the hilltop

$A :=$ altitude of the stables

$\exists t_0 \in (0, 12)$

s.t. $h(t_0) = 0$

i.e., s.t. $f(t_0) = g(t_0)$

s.t. $h(t_0) = 0$ ■

STOP

§2.5