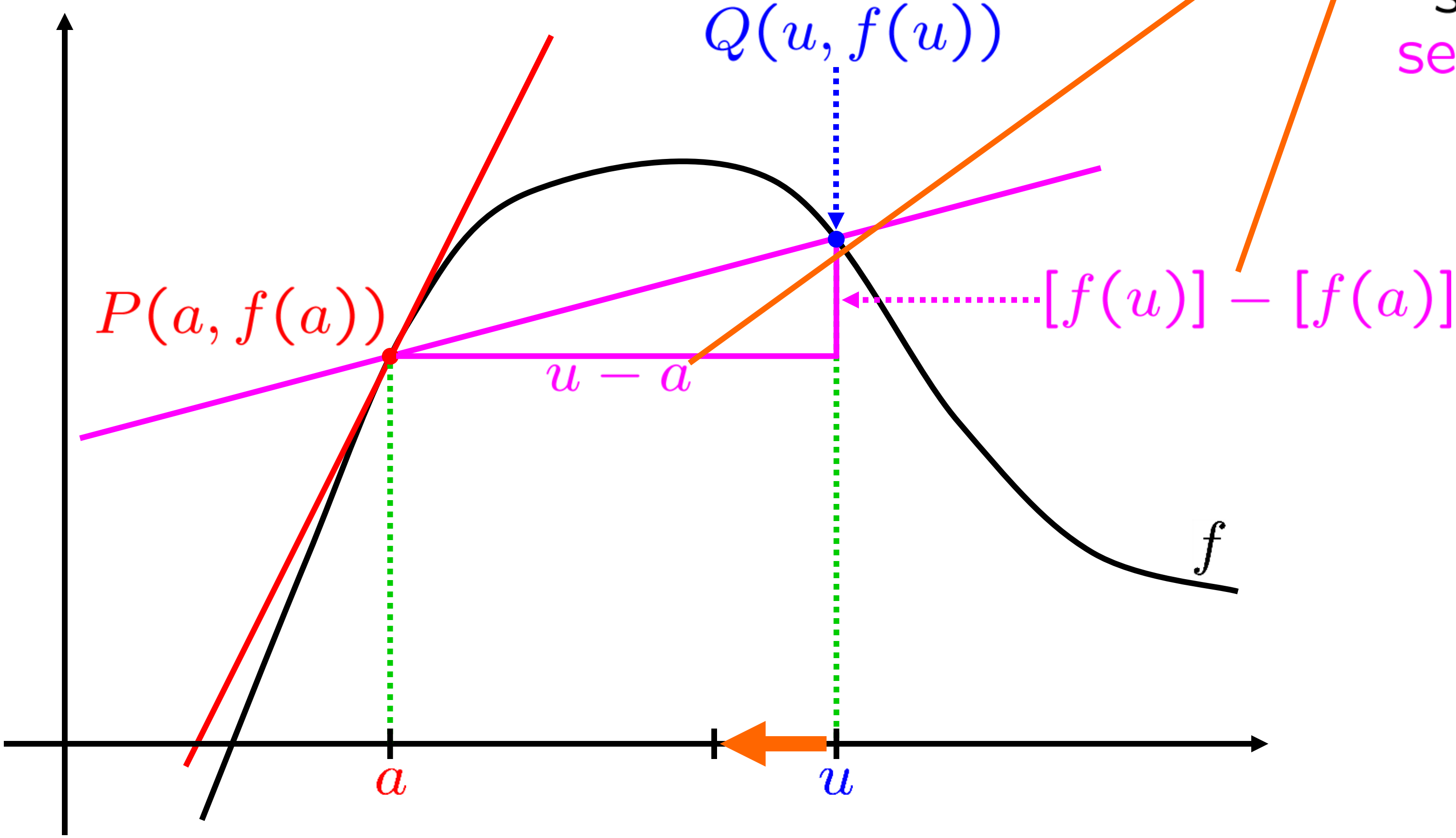
CALCULUS
Derivatives and rates of change

$$\left.\begin{array}{c}\text{avg rate of chg in output}\\ \text{per unit chg in input}\\ \text{between } a \text{ and } u\end{array}\right\} = \frac{[f(x)]_{x:\to a}^{x:\to u}}{[x]_{x:\to a}^{x:\to u}} = \frac{[f(u)] - [f(a)]}{u - a}$$

$$\|$$

slope of secant line



$$\left.\begin{array}{c}\text{avg rate of chg in output}\\ \text{per unit chg in input}\\ \text{between } a \text{ and } u\end{array}\right\} = \frac{[f(x)]_{x:\to a}^{x:\to u}}{[x]_{x:\to a}^{x:\to u}} = \frac{[f(u)] - [f(a)]}{u-a}$$

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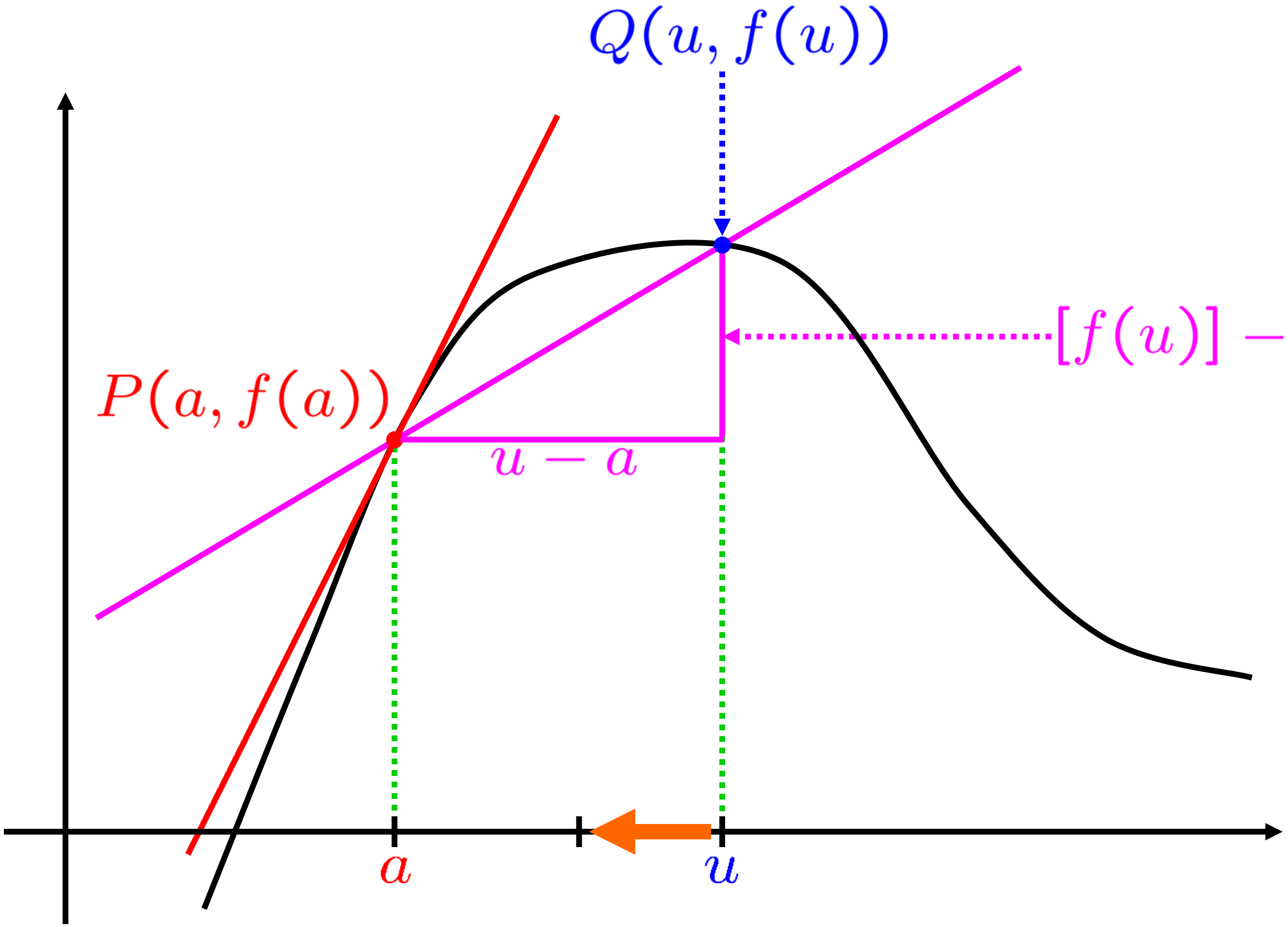
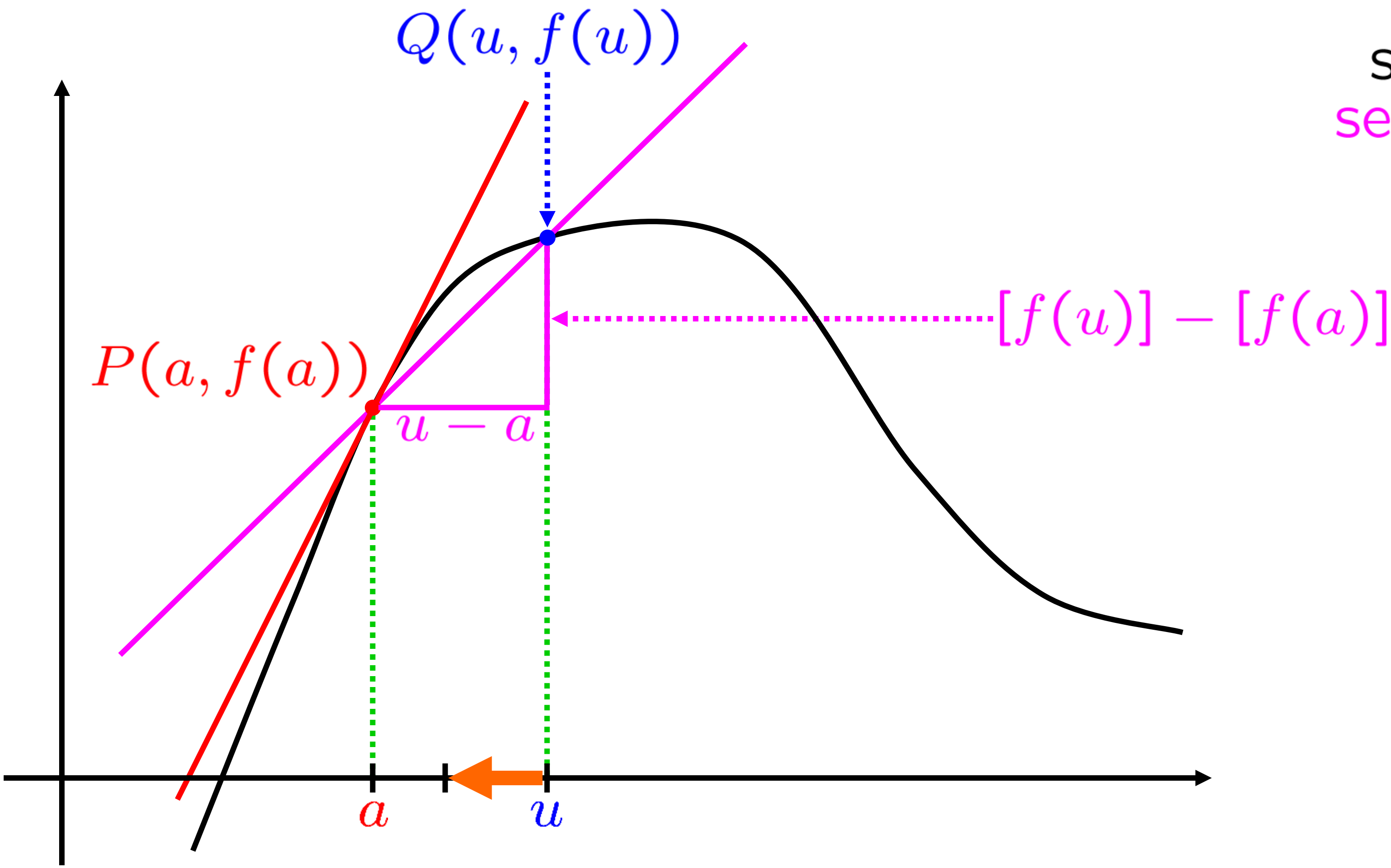


$$\left.\begin{array}{c}\text{avg rate of chg in output}\\ \text{per unit chg in input}\\ \text{between } a \text{ and } u\end{array}\right\} = \frac{[f(x)]_{x:\to a}^{x:\to u}}{[x]_{x:\to a}^{x:\to u}} = \frac{[f(u)]-[f(a)]}{u-a}$$

$$||$$

slope of
secant line

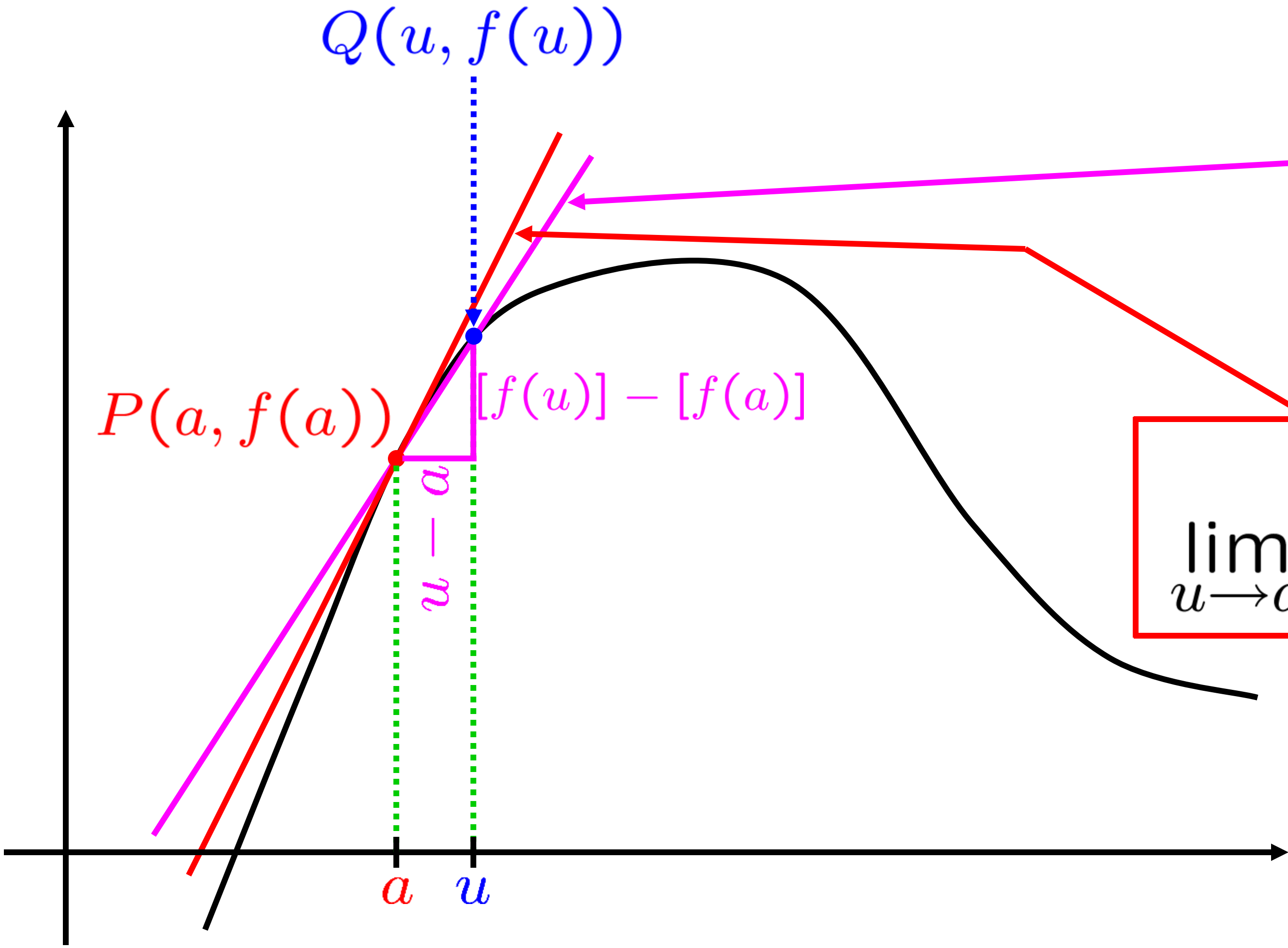


§2.1

$$\left.\begin{array}{c}\text{avg rate of chg in output}\\ \text{per unit chg in input}\\ \text{between } a \text{ and } u\end{array}\right\} = \frac{[f(x)]^{x:\to u}_{x:\to a}}{[x]^{x:\to u}_{x:\to a}} = \frac{[f(u)] - [f(a)]}{u-a}$$

$$\|$$

slope of
secant line



$$\lim_{u \to a} \frac{[f(u)] - [f(a)]}{u-a}$$

$$\|$$

slope of
tangent line

$$\frac{[f(u)] - [f(a)]}{u - a} = $$

slope of secant line between $(a, f(a))$ and $(u, f(u))$

$$\frac{[f(u)] - [f(a)]}{u - a}$$

slope of secant line

$u :\to a + h$, with $h \to 0$

$$\lim_{u \to a} \frac{[f(u)] - [f(a)]}{u - a} = $$

slope of tangent line at $\lim_{u \to a}$

$$\frac{[f(u)] - [f(a)]}{u - a}$$

slope of tangent line

$$\frac{[f(a+h)]-[f(a)]}{(\cancel{a}+h)-\cancel{a}} =$$ slope of secant line between $(a, f(a))$ and $(a+h, f(a+h))$

$u :\longrightarrow a+h$, with $h \to 0$

$$\lim_{h\to 0} \frac{[f(a+h)]-[f(a)]}{(\cancel{a}+h)-\cancel{a}} =$$ slope of tangent line at $(a, f(a))$

$$\frac{[f(a+h)] - [f(a)]}{h} =$$ slope of secant line between $(a, f(a))$ and $(a+h, f(a+h))$

$$\lim_{h \to 0} \frac{[f(a+h)] - [f(a)]}{h} =$$ slope of tangent line at $(a, f(a))$

DEFINITION 2.18, §2.5, p. 42:

The derivative of a function f **at a number** a,

denoted by $f'(a)$, is $\displaystyle\lim_{h \to 0} \frac{[f(a+h)] - [f(a)]}{h}$.

EXAMPLE: Find the derivative of the function

$f(x) = x^2 - 7x + 4$ at the number 3.

$$f'(3) = \lim_{h \to 0} \frac{[f(3+h)] - [f(3)]}{h}$$

$1, -7, 4$ l.c. of $(3+h)^2, 3+h, 1$ $\quad$ $1, -7, 4$ l.c. of $3^2, 3, 1$

$$\frac{[(3+h)^2 - 7(3+h) + 4] - [3^2 - 7 \cdot 3 + 4]}{h}$$

DEFINITION 2.18, §2.5, p. 42:

$a :\to 3$

The **derivative of a function** f **at a number** a,

denoted by $f'(a)$, is $\displaystyle \lim_{h \to 0} \frac{[f(a+h)] - [f(a)]}{h}$.

EXAMPLE: Find the derivative of the function $f(x) = x^2 - 7x + 4$ at the number 3.

$$f'(3) = \lim_{h\to 0} \underbrace{\frac{[f(3+h)] - [f(3)]}{h}}$$

1, −7, 4 l.c. of $(3+h)^2$, $3+h$, 1   1, −7, 4 l.c. of 3^2, 3, 1

$$\frac{[(3+h)^2 - 7(3+h) + 4] - [3^2 - 7\cdot 3 + 4]}{h}$$

$$\|$$

$$\boxed{1}\frac{[(3+h)^2] - [3^2]}{h} - 7\frac{[3+h] - [3]}{h} + 4\frac{[1] - [1]}{h}$$

EXAMPLE: Find the derivative of the function

$f(x) = x^2 - 7x + 4$ at the number 3.

$$f'(3) = \lim_{h\to 0} \frac{[f(3+h)] - [f(3)]}{h} = -1$$

$$\frac{[(3+h)^2 - 7(3+h) + 4] - [3^2 - 7\cdot 3 + 4]}{h}$$

$$\|$$

$[9 + 6h + h^2]$

$$1\frac{[(3+h)^2] - [3^2]}{h} - 7\frac{[3+h] - [3]}{h} + 4\frac{[1] - [1]}{h}$$

$6 + h$ $(h \neq 0)$ $(h \neq 0)$ 1 $(h \neq 0)$

$(h \neq 0)$ $6 + h - 7 \longrightarrow -1$

§2.5

Suggestion: Try $a :\longrightarrow 3$, then $a :\longrightarrow 4$, then general a.

EXAMPLE: Find the derivative of the function

$$f(x) = x^2 - 7x + 4 \text{ at the number } a.$$

Just finished $a :\rightarrow 3$,
you do $a :\rightarrow 4$...

$$f'(a) = \lim_{h\to 0} \frac{[f(a+h)] - [f(a)]}{h}$$

1, −7, 4 l.c. of $(a+h)^2, a+h, 1$

1, −7, 4 l.c. of $a^2, a, 1$

$$\frac{[(a+h)^2 - 7(a+h) + 4] - [a^2 - 7a + 4]}{h}$$

$$\|$$

$$\boxed{1}\,\frac{[(a+h)^2] - [a^2]}{h} - 7\,\frac{[a+h] - [a]}{h} + 4\,\frac{[1] - [1]}{h}$$

EXAMPLE: Find the derivative of the function

$f(x) = x^2 - 7x + 4$ at the number a.

$$f'(a) = \lim_{h\to 0} \underbrace{\frac{[f(a+h)] - [f(a)]}{h}} = 2a - 7 \blacksquare$$

$$\frac{[(a+h)^2 - 7(a+h) + 4] - [a^2 - 7a + 4]}{h}$$

$$\|$$

$$[a^2 + 2ah + h^2]$$

$$\boxed{1}\underbrace{\frac{[(a+h)^2] - [a^2]}{h}}_{2a + h \ (h \neq 0)} - 7\underbrace{\frac{[a+h] - [a]}{h}}_{(h \neq 0)\ 1} + 4\frac{[1] - [1]}{h} \quad (h \neq 0)$$

$$(h \neq 0)\ 2a + h - 7 \longrightarrow 2a - 7$$

EXAMPLE: Find the derivative of the function $f(x) = x^2 - 7x + 4$ at the number a.

$$f'(a) = \lim_{h \to 0} \frac{[f(a+h)] - [f(a)]}{h} = 2a - 7$$ ■

EXAMPLE: Find the equation of the tangent line to $y = x^2 - 7x + 4$ at $(3, -8)$.

$$f'(3) = 2 \cdot 3 - 7 = -1 \qquad y - (-8) = (-1)(x - 3)$$

EXAMPLE: Find the derivative of the function $f(x) = x^2 - 7x + 4$ at the number a.

$$f'(a) = \lim_{h \to 0} \frac{[f(a+h)] - [f(a)]}{h} = 2a - 7$$ ■

THIS HAS ALL SLOPES OF ALL TANGENT LINES.

EXAMPLE: Find the equation of the tangent line to $y = x^2 - 7x + 4$ at $(3, -8)$.

$$f'(3) = 2 \cdot 3 - 7 = -1$$

WE ONLY NEED THIS ONE.

$$y - (-8) = (-1)(x - 3)$$

$$y = -8 + (-1)(x - 3)$$

■

SKILL

pt/slope to eqn

WRONG: $y - (-8) = (2a - 7)(x - 3)$

Example: Let $f(x) = \dfrac{x^2 - 4}{x + 7}$. Find $f'(a)$.

$$b := a + h$$

$$\frac{(f(b)) - (f(a))}{h} = \left[\frac{1}{h}\right]\left[\frac{b^2 - 4}{b + 7} - \frac{a^2 - 4}{a + 7}\right]$$

$$= \left[\frac{1}{h}\right]\left[\frac{(a + 7)(b^2 - 4) - (a^2 - 4)(b + 7)}{(a + 7)(b + 7)}\right]$$

$$= \left[\frac{1}{h}\right]\left[\frac{(ab^2 - 4a + 7b^2 \cancel{- 28}) - (a^2b + 7a^2 - 4b \cancel{- 28})}{(a + 7)(b + 7)}\right]$$

$$= \frac{ab^2 - 4a + 7b^2 - a^2b - 7a^2 + 4b}{h(a + 7)(b + 7)}$$

Example: Let $f(x) = \dfrac{x^2 - 4}{x + 7}$. Find $f'(a)$.

$b := a + h$

$b^2 = a^2 + 2ah + h^2$

$$\frac{(f(b)) - (f(a))}{h} = \frac{ab^2 - 4a + 7b^2 - a^2b - 7a^2 + 4b}{h(a+7)(b+7)}$$

$$= \frac{ab^2 - 4a + 7b^2 - a^2b - 7a^2 + 4b}{h(a+7)(b+7)}$$

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$$\|$$

$$\frac{a(a^2+2ah+h^2)-4a+7(a^2+2ah+h^2)-a^2(a+h)-7a^2+4(a+h)}{h(a+7)(a+h+7)}$$

Example: Let $f(x) = \dfrac{x^2-4}{x+7}$. Find $f'(a)$.

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$$\frac{(f(b)) - (f(a))}{h} = \frac{ab^2 - 4a + 7b^2 - a^2b - 7a^2 + 4b}{h(a+7)(b+7)}$$

$$||$$

$$\frac{a(a^2+2ah+h^2) - 4a + 7(a^2+2ah+h^2) - a^2(a+h) - 7a^2 + 4(a+h)}{h(a+7)(a+h+7)}$$

$$||\; h \neq 0$$

$$\frac{a(2a+h) + 7(2a+h) - a^2 + 4}{(a+7)(a+h+7)}$$

$$\downarrow h \to 0$$

$$\frac{a(2a+0) + 7(2a+0) - a^2 + 4}{(a+7)(a+0+7)}$$

Example: Let $f(x) = \dfrac{x^2 - 4}{x + 7}$. Find $f'(a)$.

$b := a + h$ $\qquad\qquad$ $b^2 = a^2 + 2ah + h^2$

$$\frac{(f(b)) - (f(a))}{h} \xrightarrow[h \to 0]{} \frac{a(2a+0) + 7(2a+0) - a^2 + 4}{(a+7)(a+0+7)}$$

$$\Vert$$

$$\frac{2a^2 + 14a - a^2 + 4}{(a+7)^2}$$

$$\downarrow h \to 0$$

$$\frac{a(2a+0) + 7(2a+0) - a^2 + 4}{(a+7)(a+0+7)}$$

Example: Let $f(x) = \dfrac{x^2-4}{x+7}$. Find $f'(a)$.

$b := a + h$

$b^2 = a^2 + 2ah + h^2$

$$\frac{(f(b)) - (f(a))}{h} \xrightarrow[h \to 0]{} \frac{a(2a+0)+7(2a+0)-a^2+4}{(a+7)(a+0+7)}$$

$$= \frac{2a^2+14a-a^2+4}{(a+7)^2}$$

$$= \frac{a^2+14a+4}{(a+7)^2} \quad \blacksquare$$

SKILL
diff rat'l fn

Example: The limit $\lim_{h\to 0} \dfrac{\sqrt[3]{27+h} - \boxed{3}}{h}$ is equal

$\boxed{3} = \sqrt[3]{27}$

to $f'(a)$, for some f and some a.

Find such an f and a.

$f(x) = \sqrt[3]{x}$
$a = 27$

$f(x) = ??$
$a = ??$

$$\frac{\sqrt[3]{27+h} - \sqrt[3]{27}}{h} = \frac{[f(a+h)] - [f(a)]}{h}$$

SKILL
recognize deriv

STOP