CALCULUS
Differentiation problems
without techniques of differentiation

Example: $f(x) = x^3 + 7x - 8$

$[f(x+h)] - [f(x)] = ??$

Find the derivative of f using the def'n of derivative. What are the domains of f and f'?

SKILL

find deriv from the def'n

$$(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$$

$(a+b)^0$ 1

$(a+b)^1$ 1 1

$(a+b)^2$ 1 2 1

$(a+b)^3$ 1 3 3 1

$(a+b)^4$ 1 4 6 4 1

$(a+b)^5$ 1 5 10 10 5 1

etc.

Example: $f(x) = x^3 + 7x - 8 \cdot 1$

$[f(x+h)] - [f(x)] = ??$

Find the derivative of f using the def'n of derivative. What are the domains of f and f'?

$\text{dom}[f] = \mathbb{R} = \text{dom}[f']$ ■

SKILL
find deriv from the def'n

$(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$

ADD

$(x+h)^3 - x^3 = 3x^2h + 3xh^2 + h^3$

$(+7)\times$ → $(x+h) - x = h$

$(-8)\times$ → $1 - 1 = 0$

$[f(x+h)] - [f(x)] = 3x^2h + 3xh^2 + h^3 + 7h - 8 \cdot 0$

$$\frac{[f(x+h)] - [f(x)]}{h} \overset{h \neq 0}{=} 3x^2 + 3xh + h^2 + 7$$

$$\overset{h \to 0}{\to} 3x^2 + 7$$

Example: a. Find dy/dx, when $y = x^2 - 2x^3$.

b. Find an eq'n of the tangent line to the curve $y = x^2 - 2x^3$ at the point $(1, -1)$.

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a. $\triangle y = [(x + \triangle x)^2 - 2(x + \triangle x)^3] - [x^2 - 2x^3]$

$= [(\cancel{x^2} + 2x(\triangle x) + (\triangle x)^2)$

$-2(\cancel{x^3} + 3x^2(\triangle x) + 3x(\triangle x)^2 + (\triangle x)^3)]$

$- [\cancel{x^2} - \cancel{2x^3}]$

DIVIDE BY $\triangle x$

$\frac{\triangle y}{\triangle x} \overset{\triangle x \neq 0}{=} 2x + (\triangle x) - 2(3x^2 + 3x(\triangle x) + (\triangle x)^2)$

$\to 0$ $\to 0$ $\to 0$

$\triangle x \to 0$

$\frac{dy}{dx} = 2x - 2(3x^2)$

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a. $\dfrac{dy}{dx} = 2x - 2(3x^2) = 2x - 6x^2$

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Example: a. Find dy/dx, when $y = x^2 - 2x^3$.

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a. $\dfrac{dy}{dx} = 2x - 2(3x^2) = 2x - 6x^2$

SLOPES OF ALL TANGENT LINES

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b. $\left[\dfrac{dy}{dx}\right]_{x:\to 1} = 2 - 6 = -4$

ONLY NEED ONE

SKILL
find slope of tan line

WRONG: $y - (-1) = (2x - 6x^2)(x - 1)$

NOT EVEN LINEAR

RIGHT: $y - (-1) = (-4)(x - 1)$ ■

SKILL
find eq'n of tan line

first, $f'(7)\ldots$

EXAMPLE: Find f' if $f(x) = \dfrac{2+x}{3-x}$.

0/0 indeterminate form — rational in h

$$\frac{[f(7+h)]-[f(7)]}{h} = \frac{\frac{2+(7+h)}{3-(7+h)} - \frac{2+7}{3-7}}{h}$$

$7 :\to x$
$-4 :\to 3-x$
$4 :\to x-3$
$9 :\to 2+x$

common denominator: $[-4-h][-4]$

$$= \frac{\frac{9+h}{-4-h} - \frac{9}{-4}}{h} = \frac{\frac{[9+h][-4]}{[-4-h][-4]} - \frac{[-4-h][9]}{[-4-h][-4]}}{h}$$

$$= \frac{\frac{[9+h][-4]-[-4-h][9]}{[-4-h][-4]}}{h} = \frac{[9+h][-4]-[-4-h][9]}{h[-4-h][-4]}$$

$$= \frac{[-36-4h]-[-36-9h]}{4h[h+4]} = \frac{5h}{4h[h+4]}$$

$9-4$

$(2+x)-(x-3)$

$=5$

$$\overset{h\neq 0}{=} \frac{5}{4[h+4]} \overset{h\to 0}{\to} \frac{5}{4^2}$$

$$f'(7) = \frac{5}{4^2} \qquad \boxed{f'(x) = \frac{5}{(x-3)^2}}$$

Example: Find $\dfrac{dy}{dx}$, when $y = \dfrac{3x}{(x+4)^2}$.

$u := x + \triangle x$

$$\triangle y = \frac{3u}{(u+4)^2} - \frac{3x}{(x+4)^2} = \frac{3u(x+4)^2 - 3x(u+4)^2}{(u+4)^2(x+4)^2}$$

$$= \frac{3u[x^2 + 8x + 16] - 3x[u^2 + 8u + 16]}{(u+4)^2(x+4)^2}$$

$$(\triangle y)(u+4)^2(x+4)^2 = [3ux^2 + 48u] - [3xu^2 + 48x]$$

Example: Find $\dfrac{dy}{dx}$, when $y = \dfrac{3x}{(x+4)^2}$.

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$$= \frac{3u[x^2 + 8x + 16] - 3x[u^2 + 8u + 16]}{(u+4)^2(x+4)^2}$$

$$(\triangle y)(u+4)^2(x+4)^2 = [3ux^2 + 48u] - [3xu^2 + 48x]$$

$$= 3[ux^2 - u^2x] + 48[u - x]$$

$$= 3[(x+\triangle x)x^2 - (x+\triangle x)^2 x] + 48[(x+\triangle x) - x]$$

$$= 3[(x + \triangle x)x^2 - (x^2 + 2x(\triangle x) + (\triangle x)^2)x] + 48[\triangle x]$$

$$= 3[\triangle x][x^2 - (2x + (\triangle x))x] + 48[\triangle x]$$

$\triangle x$

SKILL
find deriv from the def'n

Example: Find $\dfrac{dy}{dx}$, when $y = \dfrac{3x}{(x+4)^2}$.

$u := x + \triangle x$

$(\triangle y)(u+4)^2(x+4)^2$

$= 3[\triangle x][x^2 - (2x + (\triangle x))x] + 48[\triangle x]$

DIVIDE BY $\triangle x$

$\dfrac{\triangle y}{\triangle x}(x + \triangle x + 4)^2(x+4)^2$

$(\triangle y)(u+4)^2(x+4)^2$

$= 3[\triangle x][x^2 - (2x + (\triangle x))x] + 48[\triangle x]$

Example: Find $\dfrac{dy}{dx}$, when $y = \dfrac{3x}{(x+4)^2}$.

$u := x + \triangle x$

$(\triangle y)(u+4)^2(x+4)^2$

DIVIDE BY $\triangle x$ $= 3[\triangle x][x^2 - (2x + (\triangle x))x] + 48[\triangle x]$

$$\frac{\triangle y}{\triangle x}(x+\triangle x+4)^2(x+4)^2 \overset{\triangle x \neq 0}{=} 3[x^2 - (2x + (\triangle x))x] + 48$$

$$\frac{\triangle y}{\triangle x} \overset{\triangle x \neq 0}{=} \frac{3[x^2 - (2x + (\triangle x))x] + 48}{(x+\triangle x+4)^2(x+4)^2}$$

LET $\triangle x \to 0$ $\to$ $\dfrac{3[x^2 - (2x + (0))x] + 48}{(x+0+4)^2(x+4)^2}$

$$\frac{dy}{dx} = \frac{3[x^2 - 2x^2] + 48}{(x+4)^2(x+4)^2}$$

SKILL
find deriv from the def'n

Example: Find $\dfrac{dy}{dx}$, when $y = \dfrac{3x}{(x+4)^2}$.

$$\frac{dy}{dx} = \frac{3[\overbrace{x^2 - 2x^2}^{-x^2}] + 48}{(x+4)^2(x+4)^2}$$

$$\frac{dy}{dx} = \frac{3[x^2 - 2x^2] + 48}{(x+4)^2(x+4)^2}$$

Example: Find $\dfrac{dy}{dx}$, when $y = \dfrac{3x}{(x+4)^2}$.

$$\frac{dy}{dx} = \frac{3[\overbrace{x^2 - 2x^2}^{-x^2}] + 48}{\underbrace{(x+4)^2(x+4)^2}_{(x+4)^4}} = \frac{-3x^2 + 48}{(x+4)^4}$$

Example: $g(t) = \dfrac{1}{\sqrt{t}}$

SKILL
find deriv from the def'n

Find the derivative of g using the def'n of derivative. What are the domains of g and g'?

$$[g(t+h)] - [g(t)] = \left[\frac{1}{\sqrt{t+h}}\right] - \left[\frac{1}{\sqrt{t}}\right]$$

$$= \frac{\sqrt{t} - \sqrt{t+h}}{\sqrt{t+h}\sqrt{t}} \frac{\sqrt{t} + \sqrt{t+h}}{\sqrt{t} + \sqrt{t+h}}$$

$$= \frac{t - (t+h)}{\sqrt{t+h}\sqrt{t}(\sqrt{t} + \sqrt{t+h})}$$

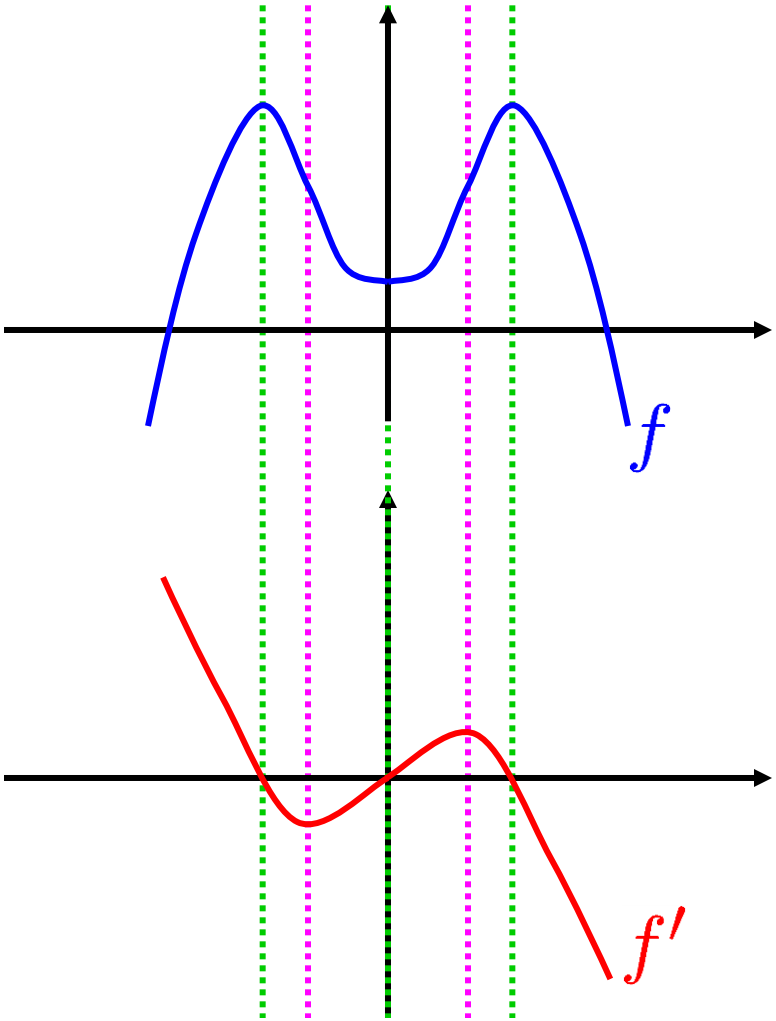
$$\frac{[g(t+h)] - [g(t)]}{h} \overset{h \neq 0}{=} \frac{-1}{\sqrt{t+h}\sqrt{t}(\sqrt{t} + \sqrt{t+h})}$$

$$\overset{h \to 0}{\to} \frac{-1}{\sqrt{t+0}\sqrt{t}(\sqrt{t} + \sqrt{t+0})} = \frac{-1}{2t\sqrt{t}}$$

$\text{dom}[g] = (0, \infty) = \text{dom}[g']$ ■

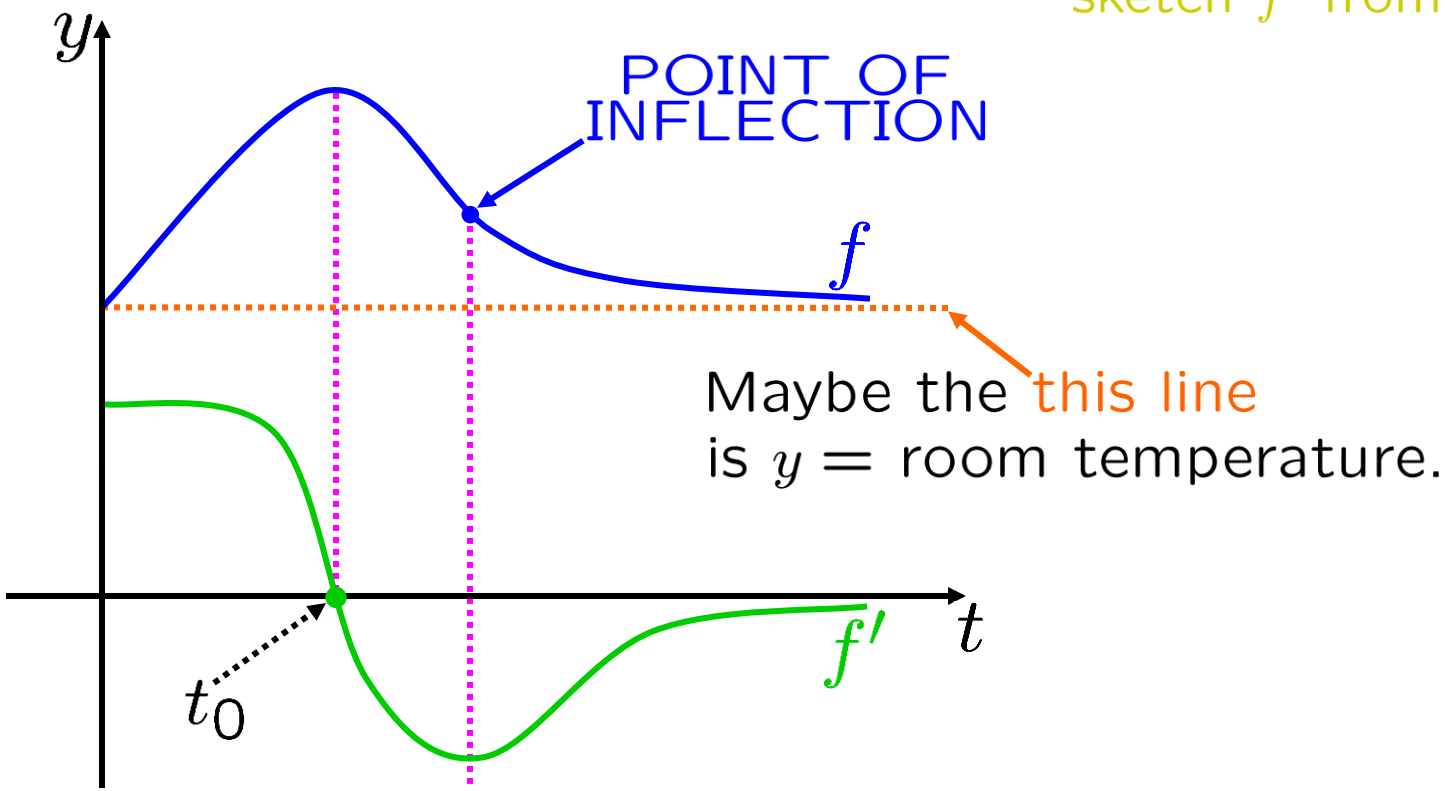
Example: Sketch the graph of f' below the graph of f.

SKILL
sketch f' from f



Example: The graph below shows the temperature of a certain liquid, as a function of time.
Freehand a graph of its derivative.
Any conclusions?

SKILL
sketch f' from f

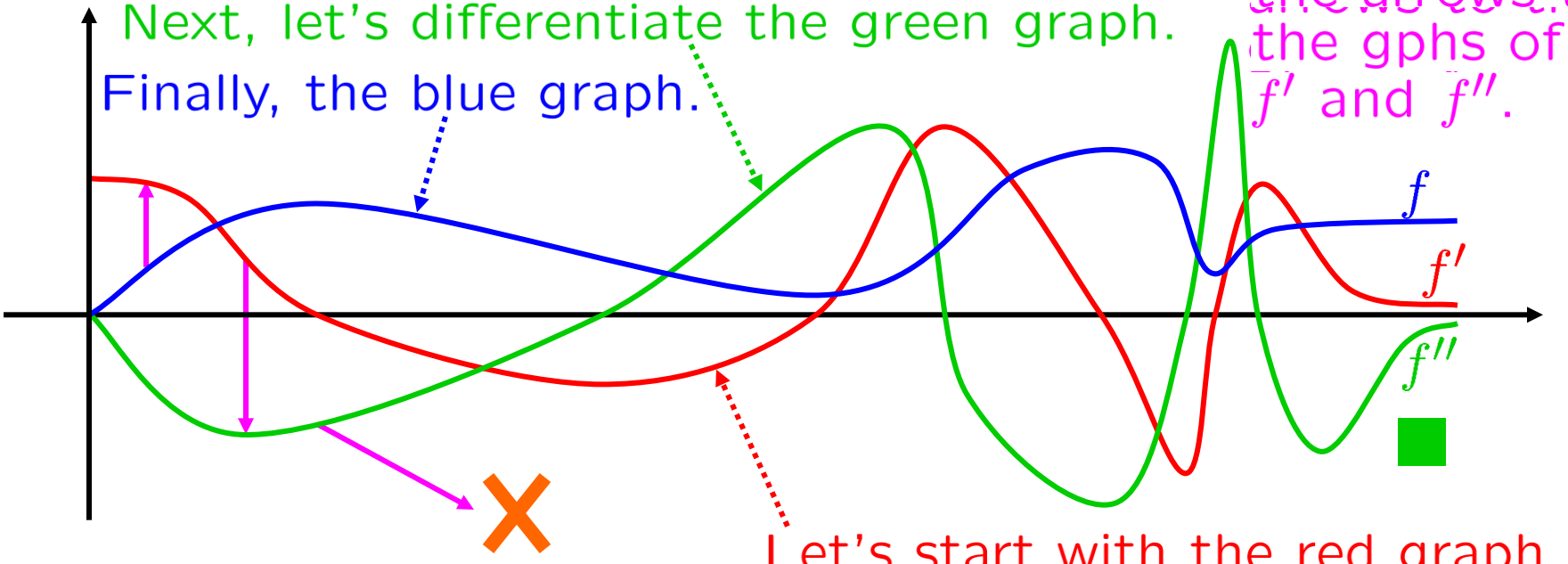


Maybe we heated the liquid until just before time t_0, and then slowly turned the heat off. ■

Example: One of these graphs is f.
Another is f'. The third is f''.
Figure out which is which.

$f'' := (f')'$

Now, move up forward along the arrows to the gphs of f' and f''.



Strategy:

For each graph, "eyeball" its derivative.
If its derivative, is one of the other graphs,
put in an arrow from the graph
to its derivative's.

SKILL
§2.4 sort fn & derivs

SKILL

Find deriv from the def'n

Whitman problems §2.4, p. 39, #1-5

SKILL

gph f' from f

Whitman problems §2.4, p. 39, #6-7

SKILL

Sketch gph with specs: bdd/contin/diff

Whitman problems §2.5, p. 43, #1

SKILL

Recognize & bound a bdd fn

Whitman problems §2.5, p. 43, #2-4

SKILL

Check continuity

Whitman problems §2.5, p. 43, #5

SKILL

Find a root

Whitman problems §2.5, p. 43, #6-7