CALCULUS
Linearity of the derivative,
and derivatives of polynomials

Recall: $f'(x) \overset{\text{def'n}}{:=} \lim\limits_{h\to 0} \dfrac{[f(x+h)]-[f(x)]}{h}$

cf. §3.2, p. 48 DERIVATIVE RESPECTS SCALAR MULT.

Let p and a be scalars and let f be a function.

Assume that f is differentiable at a.

Then pf is differentiable at a, and $(pf)'(a) = p\cdot[f'(a)]$.

$$\left[\frac{d}{dx}[p\cdot(f(x))]\right]_{x:\to a} = p\cdot\left[\frac{d}{dx}[f(x)]\right]_{x:\to a}$$

$(pf)' = p\cdot f'$

$$\frac{d}{dx}[p\cdot(f(x))] = p\cdot\left[\frac{d}{dx}[f(x)]\right]$$

Proof: $(pf)'(a) = \lim\limits_{h\to 0}\dfrac{[(pf)(a+h)]-[(pf)(a)]}{h}$

$$= \lim_{h\to 0}\frac{p\cdot[f(a+h)]-p\cdot[f(a)]}{h} = \lim_{h\to 0} p\cdot\left[\frac{[f(a+h)]-[f(a)]}{h}\right]$$

$$= p\cdot\left[\lim_{h\to 0}\frac{[f(a+h)]-[f(a)]}{h}\right] = p\cdot[f'(a)]$$ QED

Recall: $f'(x) \overset{\text{def'n}}{:=} \lim_{h \to 0} \frac{[f(x+h)] - [f(x)]}{h}$

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$(pf)' = p \cdot f'$ $\qquad$ $\frac{d}{dx}[p \cdot (f(x))] = p \cdot \left[\frac{d}{dx}[f(x)]\right]$

cf. §3.1, p. 176 DERIVATIVE RESPECTS ADDITION

Let a be a scalar and let f and g be functions.

Assume that f and g are differentiable at a.

Then $f+g$ is differentiable at a,

and $(f+g)'(a) = [f'(a)] + [g'(a)]$.

Proof is similar. (Exercise)

Recall: $f'(x) \overset{\text{def'n}}{:=} \lim_{h \to 0} \frac{[f(x+h)] - [f(x)]}{h}$

cf. §3.2, p. 48 DERIVATIVE RESPECTS SCALAR MULT.

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LINEARITY OF DIFFERENTIATION:

Let a, p, q be a scalars and let f and g be functions.

Assume that f and g are differentiable at a.

Then $pf + qg$ is differentiable at a, Recall: $f'(x) \overset{\text{def'n}}{:=} \lim_{h\to 0} \frac{[f(x+h)]-[f(x)]}{h}$

cf. §3.2, p. 48 DERIVATIVE RESPECTS SCALAR MULT.

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Recall: $f'(x) \overset{\text{def'n}}{:=} \lim_{h\to 0} \frac{[f(x+h)] - [f(x)]}{h}$

e.g.: $\frac{d}{dx}\left[\frac{4}{\sqrt{x}} - \frac{3}{x}\right] = \frac{d}{dx}[4x^{-1/2} - 3x^{-1}]$

$= 4\left(-\frac{1}{2}x^{-1/2-1}\right) - 3\left((-1)x^{-1-1}\right)$

$\frac{d}{dx}[x^n] = nx^{n-1}$

SKILL diff. of lin. comb. of power functions

$= -2x^{-3/2} + 3x^{-2}$

SKILL differentiation of polynomials

e.g.: $\frac{d}{dx}[6x^3 - 4x^2 + 2x + 6] = \frac{d}{dx}[6x^3 - 4x^2 + 2x + 6(1)]$

$= 6(3x^{3-1}) - 4(2x^{2-1}) + 2(1) + 6(0) = 18x^2 - 8x + 2$

LINEARITY OF DIFFERENTIATION:

Let a, p, q be a scalars and let f and g be functions.

Assume that f and g are differentiable at a.

Then $pf + qg$ is differentiable at a,

and $(pf + qg)'(a) = p \cdot [f'(a)] + q \cdot [g'(a)]$.

EXAMPLE:

$$\frac{d}{dx}(x^8 - 12x^5 + 3x^4 + 8x^3 + 7x - 2)$$

$$= 8x^7 - 12(5x^4) + 3(4x^3) + 8(3x^2) + 7 + 0$$

$$= 8x^7 - 60x^4 + 12x^3 + 24x^2 + 7 \quad \blacksquare$$

SKILL differentiation of polynomials

Example: Differentiate the function.

$$G(x) = \frac{2}{3}x^6$$

$$G'(x) = \frac{2}{3}(6x^5)$$

$$= 4x^5 \blacksquare$$

SKILL differentiation of polynomials

Example: Differentiate the function.

$$Q(s) = 7s^{-2/9}$$

9/9

$$Q'(s) = 7\left(-\frac{2}{9}s^{-(2/9)-1}\right)$$

$$= -\frac{14}{9}s^{-11/9} \blacksquare$$

SKILL
differentiation of lin. comb.
of power functions

Example: Differentiate the function.

$$V(w) = kw^{-4}$$

$$V'(w) = -4kw^{-5} \blacksquare$$

SKILL
differentiation of lin. comb.
of power functions

EXAMPLE: Differentiate the function.

$$v = \left(\sqrt[3]{x} + \frac{1}{\sqrt{x}} \right)^3$$

$$v = (x^{1/3} + x^{-1/2})^3$$

$$= (x^{1/3})^3 + 3(x^{1/3})^2(x^{-1/2}) + 3(x^{1/3})(x^{-1/2})^2 + (x^{-1/2})^3$$

$$= x + 3x^{(2/3)-(1/2)} + 3x^{(1/3)-1} + x^{-3/2}$$

$$= x + 3x^{1/6} + 3x^{-2/3} + x^{-3/2}$$

EXAMPLE: Differentiate the function.

$$v = \left(\sqrt[3]{x} + \frac{1}{\sqrt{x}} \right)^3$$

$$v = x + 3x^{1/6} + 3x^{-2/3} + x^{-3/2}$$

SKILL
expand to lin. comb. of power functions

$$= x + 3x^{1/6} + 3x^{-2/3} + x^{-3/2}$$

EXAMPLE: Differentiate the function.

$$v = \left(\sqrt[3]{x} + \frac{1}{\sqrt{x}} \right)^3$$

$$v = x + 3x^{1/6} + 3x^{-2/3} + x^{-3/2}$$

SKILL
expand to lin. comb.
of power functions

$$\frac{dv}{dx} = 1 + 3(1/6)x^{-5/6} + 3(-2/3)x^{-5/3} + (-3/2)x^{-5/2}$$

SKILL
differentiation of lin. comb.
of power functions

$$= 1 + (1/2)x^{-5/6} - 2x^{-5/3} - (3/2)x^{-5/2} \blacksquare$$

EXAMPLE: For what values of x does the graph of $f(x) = x^3 + 6x^2 - 15x + 9$ have a horizontal tangent?

SKILL
find horizontal tangent

$$
\begin{aligned}
f'(x) &= 3x^2 + 12x - 15 \\
&= 3(x^2 + 4x - 5) \\
&= 3(x + 5)(x - 1)
\end{aligned}
$$

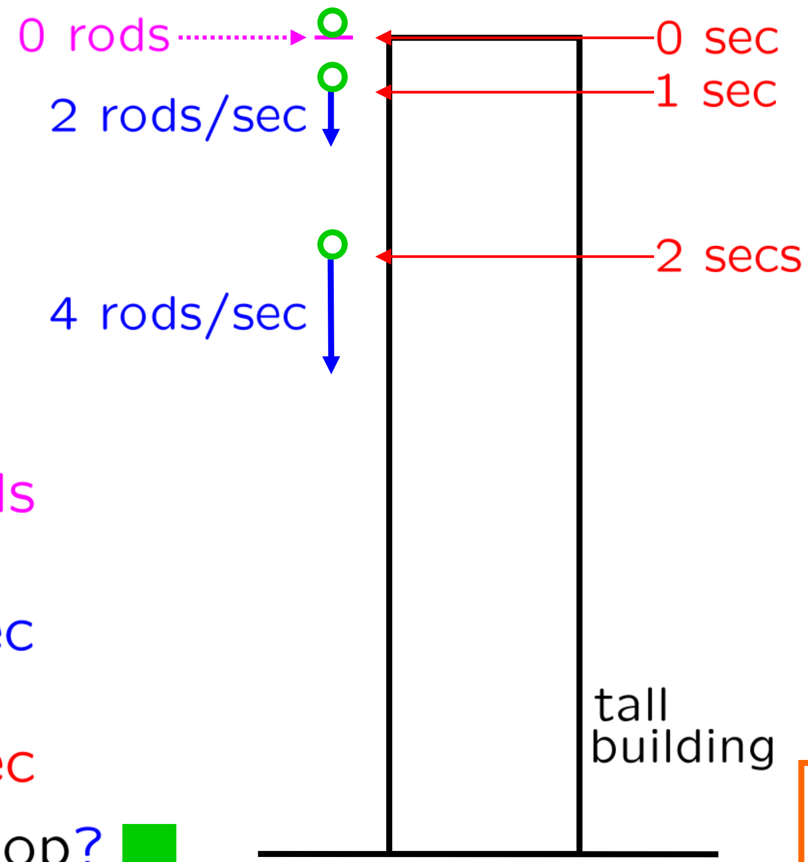
At $x = -5$ and at $x = 1$. ■

MOTION ALONG A LINE

velocity := (position)$^\bullet$
acceleration := (velocity)$^\bullet$
jerk := (acceleration)$^\bullet$
snap := (jerk)$^\bullet$
crackle := (snap)$^\bullet$
pop := (crackle)$^\bullet$
etc., etc., etc.

Overdot denotes d/dt.
That is, for any expression X of t,
$$\dot{X} := (d/dt)(X);$$
also,
$$(X)^\bullet := (d/dt)(X).$$

cf. EXAMPLE 3
§2.1, pp. 85-86



position at time t:
t^2 rods

velocity at time t:
$2t$ rods/sec

acceleration:
2 rods/sec/sec

jerk, snap, crackle, pop? ■

MOTION ALONG A LINE

velocity := $(\text{position})^\bullet$
acceleration := $(\text{velocity})^\bullet$
jerk := $(\text{acceleration})^\bullet$
snap := $(\text{jerk})^\bullet$
crackle := $(\text{snap})^\bullet$
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Overdot denotes d/dt.
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also,
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Example: A train pulls out of the station. After t seconds, it has traveled
$$(0.003)t^5 + (0.02)t^4 - (0.1)t^3 + 2t^2 + 3t \text{ m}$$
Find its velocity, acceleration, jerk, snap, crackle and pop at time t.

Sol'n: velocity $= (0.015)t^4 + (0.08)t^3 - (0.3)t^2 + 4t + 3$ m/s

acceleration $= (0.06)t^3 + (0.24)t^2 - (0.6)t + 4$ m/s^2

jerk $= (0.18)t^2 + (0.48)t - (0.6)$ m/s^3

snap $= (0.36)t + (0.48)$ m/s^4

crackle $= 0.36$ m/s^5

pop $= 0$ m/s^6 ■

EXAMPLE: Find a second-degree polynomial P such that

SKILL
find poly, given jet

$$P(7) = 3,\ P'(7) = 10 \text{ and } P''(7) = -8.$$

$P(x) = ax^2 + bx + c$	$3 = a7^2 + b7 + c$
$P'(x) = 2ax + b$	$10 = 2a7 + b$
$P''(x) = 2a$	$-8 = 2a$

$$10 = 2(-4)7 + b = -56 + b$$

$$3 = (-4)7^2 + (66)7 + c$$

$$= [(-4)7 + (66)]7 + c$$

$$= [38]7 + c = 266 + c$$

$a = -4$ $\quad b = 66$ $\quad c = -263$

$$P(x) = -4x^2 + 66x - 263 \blacksquare$$

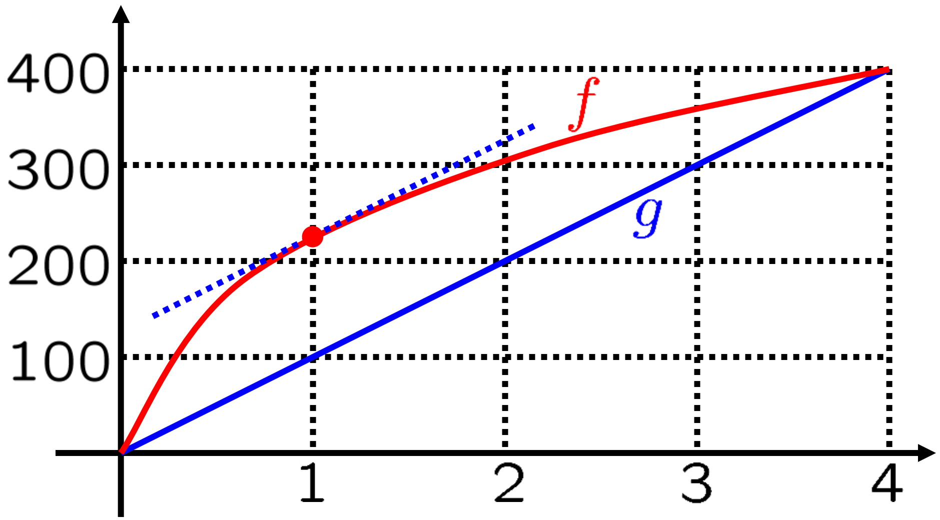
Example: Two cars (one blue, one red) race. The graphs of position *vs.* time for each are shown below.

a. At what time is their distance apart greatest?

$t_0 = ??$ At time t_0. At time 1.

b. At what time do they have the same velocity?

At time 1. ∎



$\forall t \neq 1,\ f'(t) \neq g'(t)$

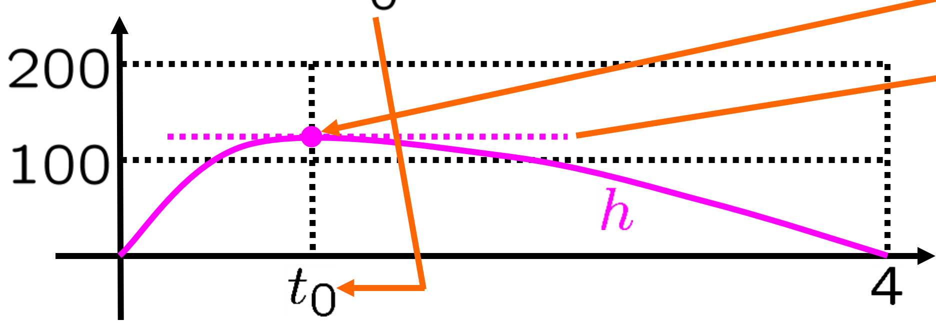
$f'(1) = g'(1)$

$h(t) := (f(t)) - (g(t))$

$h'(t) = (f'(t)) - (g'(t))$

$t_0 :=$ time when h is maximized

$h'(t_0) = 0$



$h'(1) = (f'(1)) - (g'(1)) = 0$

$\forall t \neq 1,\ h'(t) \neq 0$

$t_0 = 1$

§2.4

SKILL
linearity to find derivs
Whitman problems
§3.2, p. 50, #1-6

SKILL
eq'n tangent line
Whitman problems
§3.2, p. 50, #7-8

SKILL
words: position to acceleration
Whitman problems
§3.2, p. 50, #9

SKILL
gph of multiple and deriv
Whitman problems
§3.2, p. 50, #10

SKILL
difference differentiable
Whitman problems
§3.2, p. 50, #11

SKILL
deriv of general poly
Whitman problems
§3.2, p. 50, #12

SKILL
find poly with properties
Whitman problems
§3.2, p. 50, #13

STOP