CALCULUS
The product rule

Let $u := t^5$ and let $v := e^t$.

Goal: $\frac{d}{dt}[t^5 e^t]$

$\triangle u = (t + \triangle t)^5 - t^5$

$\triangle v = e^{t+\triangle t} - e^t$

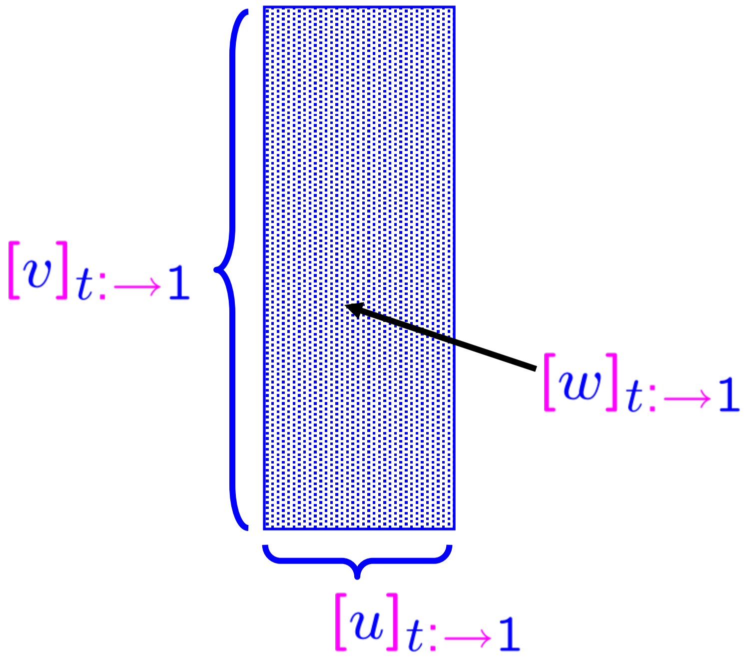
Let $w := uv$, so $w = t^5 e^t$.

$\triangle w = (t + \triangle t)^5 e^{t+\triangle t} - t^5 e^t$

$t :\rightarrow 1$

$\triangle t :\rightarrow 0.03$

$t + \triangle t :\rightarrow 1.03$



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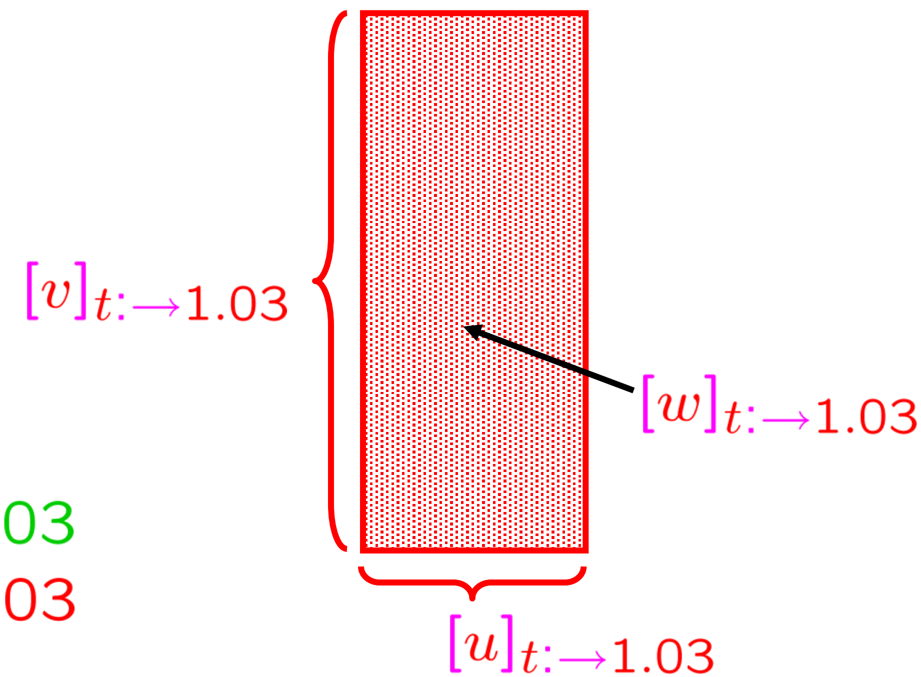

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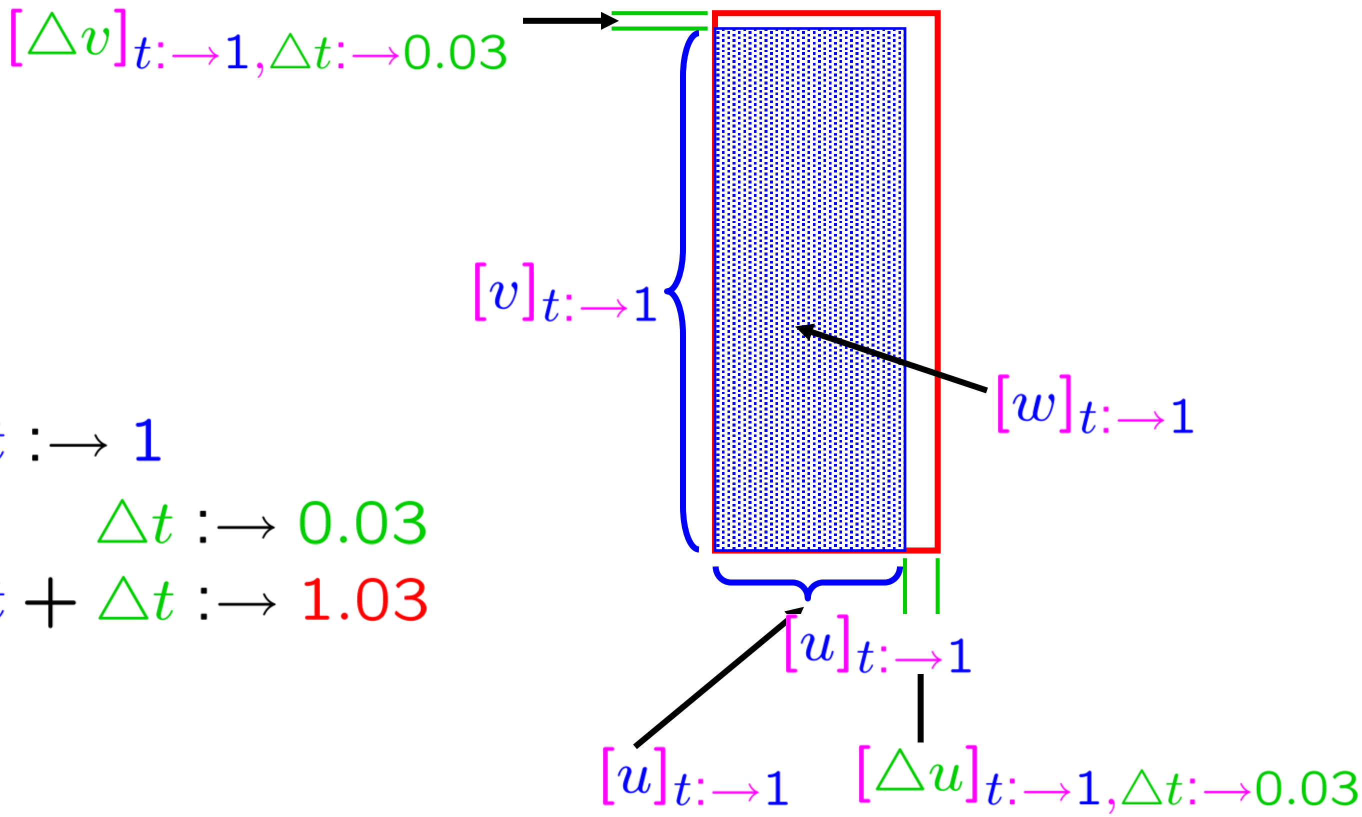

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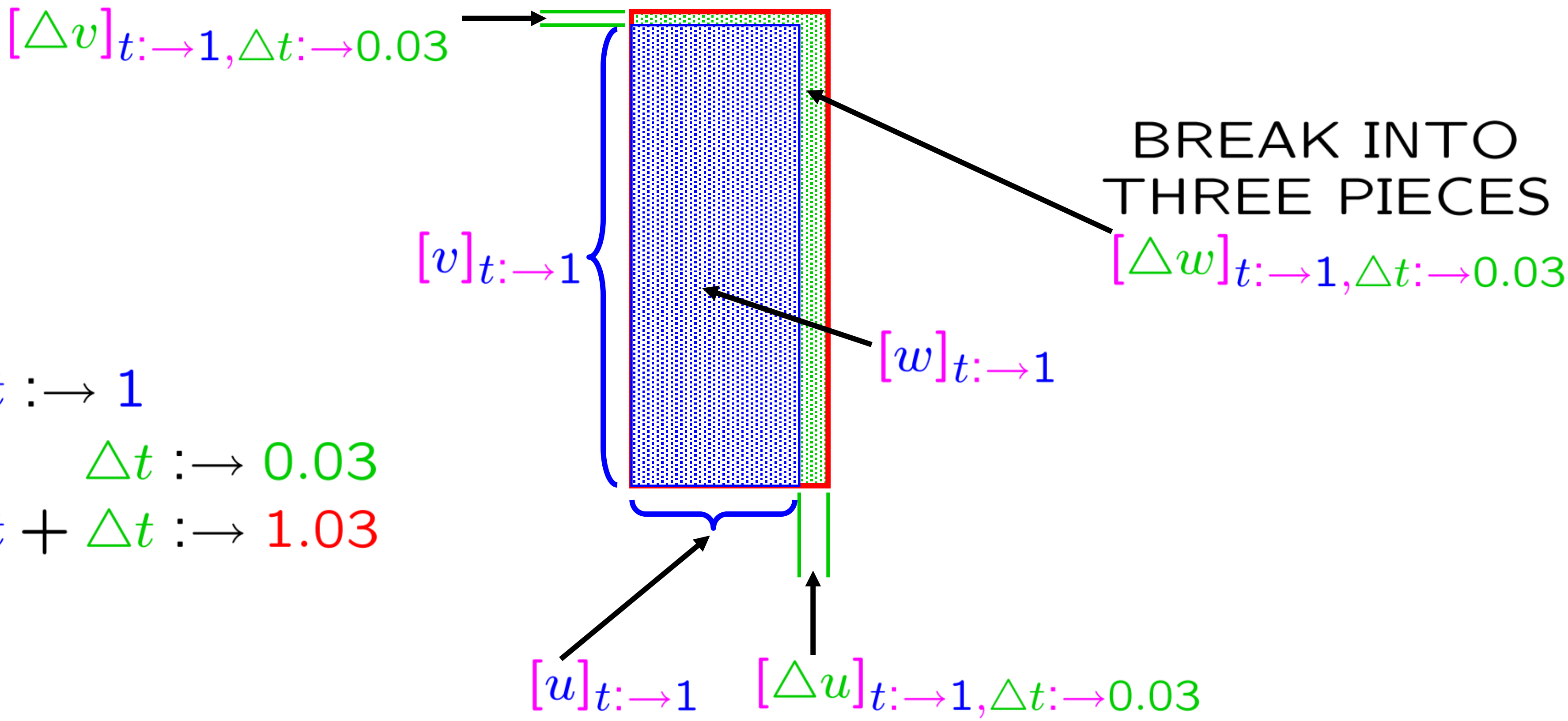

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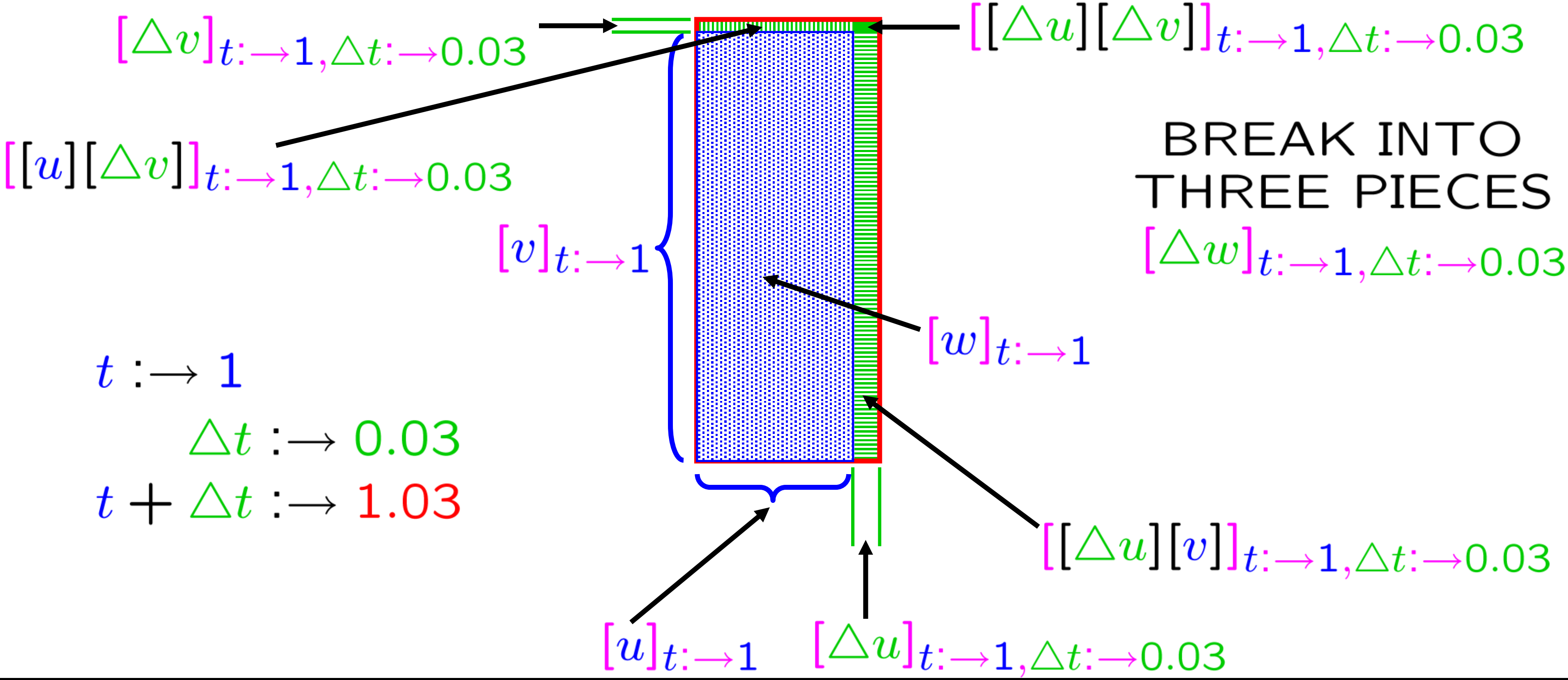

BREAK INTO THREE PIECES

$[\triangle w]_{t:\to 1, \triangle t:\to 0.03}$

$t :\to 1$

$\triangle t :\to 0.03$

$t + \triangle t :\to 1.03$

$$[\triangle w]_{t:\to 1, \triangle t:\to 0.03} = [[u][\triangle v] + [\triangle u][v] + [\triangle u][\triangle v]]_{t:\to 1, \triangle t:\to 0.03}$$

$$\triangle w = [u][\triangle v] + [\triangle u][v] + [\triangle u][\triangle v]$$

Let $u := t^5$ and let $v := e^t$.

$\triangle u = (t + \triangle t)^5 - t^5$

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Goal: $\frac{d}{dt}[t^5 e^t]$

Let $w := uv$, so $w = t^5 e^t$.

$\triangle w = (t + \triangle t)^5 e^{t+\triangle t} - t^5 e^t$

ANOTHER PROOF:

$\triangle u = [u]_{t:\to t+\triangle t} - u$

$\triangle v = [v]_{t:\to t+\triangle t} - v$

$u + \triangle u = [u]_{t:\to t+\triangle t}$

$v + \triangle v = [v]_{t:\to t+\triangle t}$

$$\triangle w = [uv]_{t:\to t+\triangle t} - uv$$

$$= [u + \triangle u][v + \triangle v] - uv$$

EXPAND TO FOUR TERMS

$$= uv + u[\triangle v] + [\triangle u]v + [\triangle u][\triangle v] - uv$$

$$= u[\triangle v] + [\triangle u]v + [\triangle u][\triangle v]$$

DIVIDE BY $\triangle t$

$$\frac{\triangle w}{\triangle t} = u\left[\frac{\triangle v}{\triangle t}\right] + \left[\frac{\triangle u}{\triangle t}\right]v + \left[\frac{\triangle u}{\triangle t}\right]\left[\frac{\triangle v}{\triangle t}\right][\triangle t]$$

$$[\triangle w]_{t:\to 1, \triangle t:\to 0.03} = [[u][\triangle v] + [\triangle u][v] + [\triangle u][\triangle v]]_{t:\to 1, \triangle t:\to 0.03}$$

$$\triangle w = [u][\triangle v] + [\triangle u][v] + [\triangle u][\triangle v]$$

Let $u := t^5$ and let $v := e^t$.

Goal: $\dfrac{d}{dt}[t^5 e^t]$

$\Delta u = (t + \Delta t)^5 - t^5$ $\qquad$ $\Delta v = e^{t+\Delta t} - e^t$

Let $w := uv$, so $w = t^5 e^t$. $\qquad$ $\Delta w = (t+\Delta t)^5 e^{t+\Delta t} - t^5 e^t$

$\Delta u = [u]_{t:\to t+\Delta t} - u$ $\qquad$ $\Delta v = [v]_{t:\to t+\Delta t} - v$

$u + \Delta u = [u]_{t:\to t+\Delta t}$ $\qquad$ $v + \Delta v = [v]_{t:\to t+\Delta t}$

$$\begin{aligned}\Delta w &= [uv]_{t:\to t+\Delta t} - uv \\ &= [u+\Delta u][v+\Delta v] - uv \\ &= \cancel{uv} + u[\Delta v] + [\Delta u]v + [\Delta u][\Delta v] - \cancel{uv} \\ &= u[\Delta v] + [\Delta u]v + [\Delta u][\Delta v]\end{aligned}$$

$$\frac{\Delta w}{\Delta t} = u\left[\frac{\Delta v}{\Delta t}\right] + \left[\frac{\Delta u}{\Delta t}\right]v + \left[\frac{\Delta u}{\Delta t}\right]\left[\frac{\Delta v}{\Delta t}\right][\Delta t]$$

LET $\Delta t \to 0$

$$\frac{dw}{dt} = u\left[\frac{dv}{dt}\right] + \left[\frac{du}{dt}\right]v + \cancel{\left[\frac{du}{dt}\right]\left[\frac{dv}{dt}\right][0]}$$

Let $u := t^5$ and let $v := e^t$.

Goal: $\frac{d}{dt}[t^5 e^t]$

$\Delta u = (t + \Delta t)^5 - t^5$

$\Delta v = e^{t+\Delta t} - e^t$

Let $w := uv$, so $w = t^5 e^t$.

$\Delta w = (t + \Delta t)^5 e^{t+\Delta t} - t^5 e^t$

$$\Delta u = [u]_{t :\to t+\Delta t} - u \qquad \Delta v = [v]_{t :\to t+\Delta t} - v$$

$$u + \Delta u = [u]_{t :\to t+\Delta t} \qquad v + \Delta v = [v]_{t :\to t+\Delta t}$$

$$\begin{aligned}\Delta w &= [uv]_{t :\to t+\Delta t} - uv \\ &= [u + \Delta u][v + \Delta v] - uv \\ &= \cancel{uv} + u[\Delta v] + [\Delta u]v + [\Delta u][\Delta v] - \cancel{uv} \\ &= u[\Delta v] + [\Delta u]v + [\Delta u][\Delta v]\end{aligned}$$

$$\frac{dw}{dt} = u\left[\frac{dv}{dt}\right] + \left[\frac{du}{dt}\right]v$$

$$\frac{d}{dt}[t^5 e^t]$$

$$\frac{dw}{dt} = u\left[\frac{dv}{dt}\right] + \left[\frac{du}{dt}\right]v$$

Let $u := t^5$ and let $v := e^t$. Goal: $\frac{d}{dt}[t^5 e^t]$

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$$\frac{dw}{dt} = u\left[\frac{dv}{dt}\right] + \left[\frac{du}{dt}\right]v$$

$$\frac{d}{dt}[t^5 e^t] = t^5[e^t] + [5t^4]e^t$$

Let $u := t^5$ and let $v := e^t$.

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$$\begin{aligned} \Delta w &= [uv]_{t:\to t+\Delta t} - uv \\ &= [u + \Delta u][v + \Delta v] - uv \\ &= \cancel{uv} + u[\Delta v] + [\Delta u]v + [\Delta u][\Delta v] - \cancel{uv} \\ &= u[\Delta v] + [\Delta u]v + [\Delta u][\Delta v] \end{aligned}$$

$$\frac{dw}{dt} = u\left[\frac{dv}{dt}\right] + \left[\frac{du}{dt}\right]v \longleftarrow \text{THE PRODUCT RULE}$$

$$\frac{dw}{dt} = u\left[\frac{dv}{dt}\right] + \left[\frac{du}{dt}\right]v$$ ← THE PRODUCT RULE

$$w := uv$$

$u :\rightarrow f(t)$, $v :\rightarrow g(t)$

$w := uv$

$$\frac{dw}{dt} = u\left[\frac{dv}{dt}\right] + \left[\frac{du}{dt}\right]v$$ ← THE PRODUCT RULE

$$\frac{dw}{dt} = u\left[\frac{dv}{dt}\right] + \left[\frac{du}{dt}\right] v \longleftarrow \text{THE PRODUCT RULE}$$

$$w := uv$$

$$u :\longrightarrow f(t),\; v :\longrightarrow g(t) \qquad w :\longrightarrow [f(t)][g(t)]$$

$$\frac{d[(f(t))(g(t))]}{dt} = [f(t)]\left[\frac{d[g(t)]}{dt}\right] + \left[\frac{d[f(t)]}{dt}\right][g(t)]$$

$$\frac{d}{dt}[(f(t))(g(t))] = [f(t)]\left[\frac{d}{dt}[g(t)]\right] + \left[\frac{d}{dt}[f(t)]\right][g(t)]$$

$$t :\longrightarrow x$$

$$\frac{d}{dx}[(f(x))(g(x))] = [f(x)]\left[\frac{d}{dx}[g(x)]\right] + \left[\frac{d}{dx}[f(x)]\right][g(x)]$$

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THE PRODUCT RULE

$$\frac{d}{dx}\Big[[f(x)]\,[g(x)]\Big]$$

$$\frac{d}{dx}[(f(x))(g(x))] = [f(x)]\left[\frac{d}{dx}[g(x)]\right] + \left[\frac{d}{dx}[f(x)]\right][g(x)]$$

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THE PRODUCT RULE

"Differentiation by parts"

1st part / 2nd part

$$\frac{d}{dx}\Big[\,[f(x)]\,[g(x)]\,\Big] =$$

$$[f(x)]\left[\frac{d}{dx}(g(x))\right] + \left[\frac{d}{dx}(f(x))\right][g(x)]$$

the 1st part — times — the derivative of the 2nd part — plus — the derivative of the 1st part — times — the 2nd part

$$(fg)' = f\,g' + f'g$$

THE PRODUCT RULE

"Differentiation by parts"

$$\frac{d}{dx}\Big[\,[f(x)]\,[g(x)]\,\Big] =$$

$$[f(x)] \quad \left[\frac{d}{dx}(g(x))\right] + \left[\frac{d}{dx}(f(x))\right] \quad [g(x)]$$

the 1st part times the derivative of the 2nd part plus the derivative of the 1st part times the 2nd part

THE PRODUCT RULE

"Differentiation by parts"

There are variations. . .

$$\frac{d}{du}\left[\quad u^2 \quad e^u \quad\right] =$$

$$[\ u^2\ ] \qquad [\ e^u\ ] \quad + \quad [\ 2u\ ] \qquad [\ e^u\ ]$$

the 1st part — times — the derivative of the 2nd part — plus — the derivative of the 1st part — times — the 2nd part

THE PRODUCT RULE

"Differentiation by parts"

There are variations...

$$\frac{d}{du}\left(u^2\, e^u\right) = 2u\; e^u + u^2\; e^u$$

$$= [\ u^2\ ]\qquad [\ e^u\ ] \quad + \quad [\ 2u\ ] \qquad [\ e^u\ ]$$

the 1st part times the derivative of the 2nd part plus the derivative of the 1st part times the 2nd part

EXAMPLE: Differentiate: $g(x) = [\sqrt[3]{x}][e^x]$

SKILL
prod rule

$$g(x) = [x^{1/3}][e^x]$$

$$g'(x) = [x^{1/3}][e^x] + [(1/3)x^{-2/3}][e^x] \blacksquare$$

EXAMPLE: Differentiate: $h(x) = [x^4][e^{2x}]$

$$e^{2x} = e^x e^x$$

$$h(x) = [x^4][e^x][e^x]$$

$$h'(x) = \left[\frac{d}{dx}[x^4]\right][e^x][e^x] + \left[x^4\right]\left[\frac{d}{dx}[e^x]\right][e^x] + \left[x^4\right][e^x]\left[\frac{d}{dx}[e^x]\right]$$

$$= \left[4x^3\right][e^x][e^x] + \left[x^4\right][e^x][e^x] + \left[x^4\right][e^x][e^x]$$

$$= 4x^3e^{2x} + x^4e^{2x} + x^4e^{2x}$$

$$= 4x^3e^{2x} + 2x^4e^{2x} \quad \blacksquare$$

SKILL
many factor prod rule

SKILL

prod rule

Whitman problems

§3.3, p. 52, #1-4

SKILL

prod rule, sketch, eq'n tan line

Whitman problems

§3.3, p. 52, #5

SKILL

many factor prod rule

Whitman problems

§3.3, p. 52–53, #6-8