CALCULUS
l'Hôpital's rule

Problem: $\displaystyle\lim_{x\to 3}\frac{5+x}{5-x} = 4$ ■

Form: $\dfrac{8}{2}$

One of the limit laws implies:

When the form is $\dfrac{8}{2}$,

the answer is 4.

When the *form* determines the answer,

we call that form a "**determinate form**".

"$56/7 = 8$", etc.

"$8/2 = 4$", which, in this context, means...

$\dfrac{8}{2}$ is a determinate form, and it determines an answer of 4.

Theorem: $f(x) \to 8$, as $x \to a$
and $g(x) \to 2$, as $x \to a$

$\Rightarrow \quad \dfrac{f(x)}{g(x)} \to 4$, as $x \to a$

Theorem: $f(x) \to 56$
and $g(x) \to 7$

$\Rightarrow \quad \dfrac{f(x)}{g(x)} \to 8$

(OR $x \to a^+$
OR $x \to a^-$
OR $x \to \infty$
OR $x \to -\infty$)

Problem: $\lim_{x\to 3} \frac{5+x}{5-x} = 4$ ■

Form: $\frac{8}{2}$

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When the *form* determines the answer, "56/7 = 8", etc.

we call that form a "**determinate form**".

"8/2 = 4", which, in this context, means. . .

$\frac{8}{2}$ is a determinate form, and it determines an answer of 4.

Problem: $\lim_{x\to 0} \frac{1}{x^2}$

$\lim_{x\to 0} 1 = 1$

$\lim_{x\to 0} x^2 = 0^+$

Form: $\frac{1}{0}$

Form: $\frac{1}{0^+}$

Problem: $\displaystyle\lim_{x\to 3}\frac{5+x}{5-x}=4$ ■

Form: $\displaystyle\frac{8}{2}$

One of the limit laws implies:

When the form is $\displaystyle\frac{8}{2}$,

the answer is 4.

When the *form* determines the answer, "$56/7=8$", etc.

we call that form a "**determinate form**".

"$8/2=4$", which, in this context, means...

$\displaystyle\frac{8}{2}$ is a determinate form, and it determines an answer of 4.

Problem: $\displaystyle\lim_{x\to 0}\frac{1}{x^2}=\infty$ ■

Form: $\displaystyle\frac{1}{0+}$

$\displaystyle\frac{1}{0.001}=1000$

When the form is $\displaystyle\frac{1}{0+}$,

the answer is $+\infty$.

$\displaystyle\frac{1}{0+}$ is a determinate form,

and it determines an answer of $+\infty$.

Colloquially, we write: "$1/(0^+)=+\infty$"

Problem: $\lim\limits_{x\to 0} \frac{1}{-x^2} = -\infty$ ■ Form: $\frac{1}{0^-}$ & $\frac{1}{0}$ $\boxed{\text{“}1/(0^-) = -\infty\text{”}}$

Problem: $\lim\limits_{x\to 0^-} \frac{1}{x} = -\infty$ ■

Problem: $\lim\limits_{x\to 0^+} \frac{1}{x} = \infty$ ■

Problem: $\lim\limits_{x\to 0} \frac{1}{x}$ does not exist

Form: $\frac{1}{0}$

When the form is $\frac{1}{0}$,

the answer could be ∞,
or it could be $-\infty$,
or it could be "does not exist".

Problem: $\lim\limits_{x\to 0} \frac{1}{x^2} = \infty$ ■

Form: $\frac{1}{0+}$ & $\frac{1}{0}$

$\frac{1}{0.001} = 1000$

When the form is $\frac{1}{0+}$,

the answer is $+\infty$.

$\frac{1}{0+}$ is a determinate form,
and it determines an answer of $+\infty$.

Colloquially, we write: $\boxed{\text{“}1/(0^+) = +\infty\text{”}}$

Problem: $\lim_{x\to 0} \frac{1}{-x^2} = -\infty$ ■

Problem: $\lim_{x\to 0} \frac{1}{x^2} = \infty$ ■

Problem: $\lim_{x\to 0^-} \frac{1}{x} = -\infty$ ■

Problem: $\lim_{x\to 0^+} \frac{1}{x} = \infty$ ■

Problem: $\lim_{x\to 0} \frac{1}{x}$ does not exist

Form: $\frac{1}{0}$

When the form is $\frac{1}{0}$,

the answer could be ∞,
or it could be $-\infty$,
or it could be "does not exist".

When the form does not determine the answer,
we call the form an "**indeterminate form**".

Problem: $\lim_{x\to 0} \frac{1}{x^2} = \infty$ ■

Problem: $\lim_{x\to 0} \frac{1}{-x^2} = -\infty$ ■

Problem: $\lim_{x\to 0^+} \frac{1}{x^2} = \infty$ ■

two-sided

Problem: $\lim_{x\to 0^-} \frac{1}{x} = -\infty$ ■

Problem: $\lim_{x\to 0^+} \frac{1}{x} = \infty$ ■

Problem: $\lim_{x\to 0} \frac{1}{x}$ does not exist

Form: $\frac{1}{0}$

When the form is $\frac{1}{0}$,

the answer could be ∞,
or it could be $-\infty$,
or it could be "does not exist".

When the *form* does not determine the answer,
we call that form an "**indeterminate form**".

$\frac{1}{0}$ is a SLIGHTLY indeterminate form...

NOTE: Most one-sided 1/0 problems in this course are $1/(0^+)$ or $1/(0^-)$.

Problem: $\lim_{x\to 0^+} x^{\boxed{1/x^2}} = 0$ ■

SKILL $(0^+)^\infty$

Form: $(0^+)^{\boxed{\infty}}$

$(0.001)^{1000000} \approx 0$

"$(0^+)^\infty = 0$"

$f(x) = x$

$g(x) = 1/x^2$

$[f(x)]^{g(x)}$

NOT $\frac{f(x)}{g(x)}$

Problem: $\lim_{x\to 1} \frac{x^3+2x-3}{\ln x}$

Form: $\frac{0}{0}$

When the form is $\frac{0}{0}$,

the answer could be *anything*.

$\frac{0}{0}$ is a VERY indeterminate form.

Goal: Techniques for handling indeterminate forms. . .

When the *form* does not determine the answer,
we call that form an "**indeterminate form**".

$\frac{1}{0}$ is a SLIGHTLY indeterminate form. . .

NOTE: Most one-sided $1/0$ problems in this course are $1/(0^+)$ or $1/(0^-)$.

Problem: $\lim_{x\to 0^+} x^{1/x^2} = 0$ ■

SKILL
$(0^+)^\infty$

$(0.001)^{1000000} \approx 0$

Form: $(0^+)^\infty$

"$(0^+)^\infty = 0$"

Problem: $\lim_{x\to1}\frac{x^3+2x-3}{\ln x}$

When the form is $\frac{0}{0}$,

Form: $\frac{0}{0}$

the answer could be *anything*.

$\frac{0}{0}$ is a VERY indeterminate form.

Goal: Techniques for handling indeterminate forms...

$$\lim_{x\to1}\frac{x^3+2x-3}{\ln x} = \lim_{x\to1}\frac{(x^3+2x-3)-\overbrace{(1^3+2\cdot1-3)}^{0}}{(\ln x)-\underbrace{(\ln 1)}_{0}}$$

Problem: $\lim_{x\to 1} \frac{x^3+2x-3}{\ln x}$

Form: $\frac{0}{0}$

When the form is $\frac{0}{0}$,

the answer could be *anything*.

$\frac{0}{0}$ is a VERY indeterminate form.

Goal: Techniques for handling indeterminate forms...

DIVIDE NUMERATOR AND DENOMINATOR BY $x-1$

$$\lim_{x\to 1} \frac{x^3+2x-3}{\ln x} = \lim_{x\to 1} \frac{(x^3+2x-3)-(1^3+2\cdot 1-3)}{(\ln x)-(\ln 1)}$$

$$= \lim_{x\to 1} \frac{\dfrac{(x^3+2x-3)-(1^3+2\cdot 1-3)}{x-1}}{\dfrac{(\ln x)-(\ln 1)}{x-1}}$$

Problem: $\lim_{x\to 1} \frac{x^3+2x-3}{\ln x}$

Form: $\frac{0}{0}$

Goal: Techniques for handling indeterminate forms...

$$\lim_{x\to 1} \frac{(x^3+2x-3)-(1^3+2\cdot 1-3)}{x-1}$$

Goal: Techniques for handling indeterminate forms...

x^3+2x-3

$$\lim_{x\to 1} \frac{(\ln x)-(\ln 1)}{x-1}$$

$$\lim_{x\to 1} \frac{x^3+2x-3}{\ln x} = \lim_{x\to 1} \frac{\dfrac{(x^3+2x-3)-(1^3+2\cdot 1-3)}{x-1}}{\dfrac{(\ln x)-(\ln 1)}{x-1}}$$

Problem: $\displaystyle\lim_{x\to 1}\frac{x^3+2x-3}{\ln x}$

Form: $\dfrac{0}{0}$

Goal: Techniques for handling indeterminate forms...

$$\lim_{x\to 1}\frac{(x^3+2x-3)-(1^3+2\cdot 1-3)}{x-1} = \left[\frac{d}{dx}\left(x^3+2x-3\right)\right]_{x:\to 1}$$

$$f'(a) = \lim_{h\to 0}\frac{[f(a+h)]-[f(a)]}{h}$$

$x := a+h$

$x-a=h$

$h\to 0 \Leftrightarrow x\to a$

$$= \lim_{x\to a}\frac{[f(x)]-[f(a)]}{x-a}$$

$a := 1$

$f(x) := x^3+2x-3$

$$\lim_{x\to 1}\frac{(\ln x)-(\ln 1)}{x-1}$$

$$\lim_{x\to 1}\frac{x^3+2x-3}{\ln x} = \lim_{x\to 1}\frac{\dfrac{(x^3+2x-3)-(1^3+2\cdot 1-3)}{x-1}}{\dfrac{(\ln x)-(\ln 1)}{x-1}}$$

Problem: $\lim_{x\to 1} \frac{x^3+2x-3}{\ln x}$

Form: $\frac{0}{0}$

Goal: Techniques for handling indeterminate forms...

$$\lim_{x\to 1} \frac{(x^3+2x-3)-(1^3+2\cdot 1-3)}{x-1} = \left[\frac{d}{dx}\left(x^3+2x-3\right)\right]_{x:\to 1}$$

$$f'(a) = \lim_{h\to 0} \frac{[f(a+h)]-[f(a)]}{h}$$

$$= \lim_{x\to a} \frac{[f(x)]-[f(a)]}{x-a}$$

$$= [3x^2+2]_{x:\to 1} = 5$$

$a := 1$

$f(x) := x^3+2x-3$

$$\lim_{x\to 1} \frac{(\ln x)-(\ln 1)}{x-1} = \left[\frac{d}{dx}(\ln x)\right]_{x:\to 1} = \left[\left(\frac{1}{x}, x>0\right)\right]_{x:\to 1} = 1$$

$$\lim_{x\to 1} \frac{x^3+2x-3}{\ln x} = \lim_{x\to 1} \frac{\dfrac{(x^3+2x-3)-(1^3+2\cdot 1-3)}{x-1}}{\dfrac{(\ln x)-(\ln 1)}{x-1}}$$

Problem: $\displaystyle\lim_{x\to 1}\frac{x^3+2x-3}{\ln x}$

Form: $\dfrac{0}{0}$

Goal: Techniques for handling indeterminate forms...

$$\lim_{x\to 1}\frac{(x^3+2x-3)-(1^3+2\cdot 1-3)}{x-1}=5$$

$$\lim_{x\to 1}\frac{(\ln x)-(\ln 1)}{x-1}=1$$

$$=5$$

$$\lim_{x\to 1}\frac{(\ln x)-(\ln 1)}{x-1}=\lim_{x\to 1}\frac{\dfrac{(x^3+2x-3)-(1^3+2\cdot 1-3)}{x-1}}{\dfrac{(\ln x)-(\ln 1)}{x-1}}=1$$

$$\lim_{x\to 1}\ln x$$

$$\lim_{x\to 1}\frac{x^3+2x-3}{\ln x}=\lim_{x\to 1}\frac{\dfrac{(x^3+2x-3)-(1^3+2\cdot 1-3)}{x-1}}{\dfrac{(\ln x)-(\ln 1)}{x-1}}=5$$

Problem: $\lim_{x\to 1} \dfrac{x^3+2x-3}{\ln x} = 5$ ■

Form: $\dfrac{0}{0}$

Goal: Techniques for handling indeterminate forms...

$$\lim_{x\to 1} \frac{(x^3+2x-3)-(1^3+2\cdot 1-3)}{x-1} = 5$$

$$\lim_{x\to 1} \frac{(\ln x)-(\ln 1)}{x-1} = 1$$

$$\lim_{x\to 1} \frac{x^3+2x-3}{\ln x} = \lim_{x\to 1} \frac{\dfrac{(x^3+2x-3)-(1^3+2\cdot 1-3)}{x-1}}{\dfrac{(\ln x)-(\ln 1)}{x-1}}$$

$$= \quad \frac{5}{1} \quad = \quad 5$$

Problem: $\lim_{x\to 1} \frac{x^3+2x-3}{\ln x} = 5$ ■

Form: $\frac{0}{0}$

Goal: Techniques for handling indeterminate forms...

Key point:

$$\lim_{x\to 1} \frac{x^3+2x-3}{\ln x} = \lim_{x\to 1} \frac{[d/dx][x^3+2x-3]}{[d/dx][\ln x]}$$

$$= \lim_{x\to 1} \frac{3x^2+2}{1/x}$$

$$= \frac{3(1)^2+2}{1/1}$$

$$= \frac{5}{1} = 5$$

Problem: $\lim_{x\to 1} \frac{x^3+2x-3}{\ln x} = 5$ ■

Form: $\frac{0}{0}$

Goal: Techniques for handling indeterminate forms...

WARNING: Check that the form is a **l'Hôpital** form:

$$\frac{0}{0},\ \frac{\infty}{\infty},\ \frac{-\infty}{\infty},\ \frac{\infty}{-\infty} \text{ and } \frac{-\infty}{-\infty}$$

Non-e.g.: $\lim_{x\to 1} \frac{x^2+7}{x^2} \neq \lim_{x\to 1} \frac{[d/dx][x^2+7]}{[d/dx][x^2]}$

$$\lim_{x\to 1} \frac{x^2+7}{x^2} = \frac{(1)^2+7}{(1)^2} = \frac{8}{1} = 8$$

$$\lim_{x\to 1} \frac{[d/dx][x^2+7]}{[d/dx][x^2]} = \lim_{x\to 1} \frac{2x}{2x} = \frac{2}{2} = 1$$

Problem: $\displaystyle\lim_{x\to 1} \frac{x^3+2x-3}{\ln x} = 5$ ■

Form: $\dfrac{0}{0}$

Goal: Techniques for handling indeterminate forms...

WARNING: Check that the form is a **l'Hôpital** form:

$$\frac{0}{0}, \frac{\infty}{\infty}, \frac{-\infty}{\infty}, \frac{\infty}{-\infty} \text{ and } \frac{-\infty}{-\infty}$$

WARNING: Given one of these kinds of indeterminate forms, don't differentiate the quotient, using the Quotient Rule.

Instead, differentiate the numerator and denominator, then divide the results, then try taking the limit.

If you get to an answer (even ∞ or $-\infty$) then the original limit has the same answer.

If you get "does not exist", cross it all out look for another method. ☹

cf. §4.8, p. 81 TH'M 4.11 (L'Hôpital's Rule): Let $a \in \mathbb{R}$.

Suppose ONE of the following five possibilities happens:

$$
\begin{array}{llcccl}
 & \Big[\ \text{both} & \lim\limits_{x\to a} f(x) = 0 & \text{and} & \lim\limits_{x\to a} g(x) = 0 & \Big]\ \frac{0}{0} \\
\text{OR} & \Big[\ \text{both} & \lim\limits_{x\to a} f(x) = \infty & \text{and} & \lim\limits_{x\to a} g(x) = \infty & \Big]\ \frac{\infty}{\infty} \\
\text{OR} & \Big[\ \text{both} & \lim\limits_{x\to a} f(x) = -\infty & \text{and} & \lim\limits_{x\to a} g(x) = \infty & \Big]\ \frac{-\infty}{\infty} \\
\text{OR} & \Big[\ \text{both} & \lim\limits_{x\to a} f(x) = \infty & \text{and} & \lim\limits_{x\to a} g(x) = -\infty & \Big]\ \frac{\infty}{-\infty} \\
\text{OR} & \Big[\ \text{both} & \lim\limits_{x\to a} f(x) = -\infty & \text{and} & \lim\limits_{x\to a} g(x) = -\infty & \Big].\ \frac{-\infty}{-\infty}
\end{array}
$$

Suppose $\boxed{\lim\limits_{x\to a} \frac{f'(x)}{g'(x)}}$ exists. (Can be finite, ∞ or $-\infty$.)

Then $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \boxed{\lim\limits_{x\to a} \frac{f'(x)}{g'(x)}}$.

WARNING: If $\lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$ does not exist, then you can't conclude anything. ☹

cf. §4.8, p. 81 TH'M 4.11 (L'Hôpital's Rule): Let $a \in \mathbb{R}$.

Suppose ONE of the following five possibilities happens:

$$\begin{array}{lllll}
 & \Big[\text{ both } & \lim\limits_{x \to a} f(x) = 0 & \text{and} & \lim\limits_{x \to a} g(x) = 0 \Big] \\
\text{OR} & \Big[\text{ both } & \lim\limits_{x \to a} f(x) = \infty & \text{and} & \lim\limits_{x \to a} g(x) = \infty \Big] \\
\text{OR} & \Big[\text{ both } & \lim\limits_{x \to a} f(x) = -\infty & \text{and} & \lim\limits_{x \to a} g(x) = \infty \Big] \\
\text{OR} & \Big[\text{ both } & \lim\limits_{x \to a} f(x) = \infty & \text{and} & \lim\limits_{x \to a} g(x) = -\infty \Big] \\
\text{OR} & \Big[\text{ both } & \lim\limits_{x \to a} f(x) = -\infty & \text{and} & \lim\limits_{x \to a} g(x) = -\infty \Big].
\end{array}$$

Suppose $\lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$ exists. (Can be finite, ∞ or $-\infty$.)

$x \to a^+$?

Then $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$.

e.g.:
$$a := 0$$
$$f(x) := x^2[\sin(1/x)]$$
$$g(x) := x$$

WARNING: If $\lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$ does not exist,

then you can't conclude anything.

cf. §4.8, p. 81 TH'M 4.11 (L'Hôpital's Rule): Let $a \in \mathbb{R}$.

Suppose ONE of the following five possibilities happens:

$$\left[\text{ both } \lim_{x\to a^+} f(x) = 0 \text{ and } \lim_{x\to a^+} g(x) = 0 \right]$$

$$\text{OR} \left[\text{ both } \lim_{x\to a^+} f(x) = \infty \text{ and } \lim_{x\to a^+} g(x) = \infty \right]$$

$$\text{OR} \left[\text{ both } \lim_{x\to a^+} f(x) = -\infty \text{ and } \lim_{x\to a^+} g(x) = \infty \right]$$

$$\text{OR} \left[\text{ both } \lim_{x\to a^+} f(x) = \infty \text{ and } \lim_{x\to a^+} g(x) = -\infty \right]$$

$$\text{OR} \left[\text{ both } \lim_{x\to a^+} f(x) = -\infty \text{ and } \lim_{x\to a^+} g(x) = -\infty \right].$$

Suppose $\displaystyle\lim_{x\to a^+} \frac{f'(x)}{g'(x)}$ exists. (Can be finite, ∞ or $-\infty$.)

$x \to a^-$?

Then $\displaystyle\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}$.

e.g.: $a := 0$

$f(x) := x^2[\sin(1/x)]$

$g(x) := x$

WARNING: If $\displaystyle\lim_{x\to a^+} \frac{f'(x)}{g'(x)}$ does not exist,

then you can't conclude anything. ☹

cf. §4.8, p. 81 TH'M 4.11 (L'Hôpital's Rule): Let $a \in \mathbb{R}$.

Suppose ONE of the following five possibilities happens:

$$\left[\text{ both } \lim_{x\to a^-} f(x) = 0 \text{ and } \lim_{x\to a^-} g(x) = 0 \right]$$

$$\text{OR } \left[\text{ both } \lim_{x\to a^-} f(x) = \infty \text{ and } \lim_{x\to a^-} g(x) = \infty \right]$$

$$\text{OR } \left[\text{ both } \lim_{x\to a^-} f(x) = -\infty \text{ and } \lim_{x\to a^-} g(x) = \infty \right]$$

$$\text{OR } \left[\text{ both } \lim_{x\to a^-} f(x) = \infty \text{ and } \lim_{x\to a^-} g(x) = -\infty \right]$$

$$\text{OR } \left[\text{ both } \lim_{x\to a^-} f(x) = -\infty \text{ and } \lim_{x\to a^-} g(x) = -\infty \right].$$

Suppose $\displaystyle\lim_{x\to a^-} \frac{f'(x)}{g'(x)}$ exists. (Can be finite, ∞ or $-\infty$.)

$x \to \infty$?

Then $\displaystyle\lim_{x\to a^-} \frac{f(x)}{g(x)} = \lim_{x\to a^-} \frac{f'(x)}{g'(x)}$.

e.g.: $a := 0$
$f(x) := x^2[\sin(1/x)]$
$g(x) := x$

WARNING: If $\displaystyle\lim_{x\to a^-} \frac{f'(x)}{g'(x)}$ does not exist,

then you can't conclude anything. ☹

cf. §4.8, p. 81 TH'M 4.11 (L'Hôpital's Rule):

Suppose ONE of the following five possibilities happens:

[both $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to\infty} g(x) = 0$]

OR [both $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$]

OR [both $\lim_{x\to\infty} f(x) = -\infty$ and $\lim_{x\to\infty} g(x) = \infty$]

OR [both $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = -\infty$]

OR [both $\lim_{x\to\infty} f(x) = -\infty$ and $\lim_{x\to\infty} g(x) = -\infty$].

Suppose $\lim_{x\to\infty} \dfrac{f'(x)}{g'(x)}$ exists. (Can be finite, ∞ or $-\infty$.)

Then $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = \lim_{x\to\infty} \dfrac{f'(x)}{g'(x)}$.

WARNING: If $\lim_{x\to\infty} \dfrac{f'(x)}{g'(x)}$ does not exist, then you can't conclude anything. ☹

cf. §4.8, p. 81 TH'M 4.11 (L'Hôpital's Rule):

Suppose ONE of the following five possibilities happens:

$\left[\text{ both } \lim_{x\to\infty} f(x) = 0 \text{ and } \lim_{x\to\infty} g(x) = 0 \right]$

OR $\left[\text{ both } \lim_{x\to\infty} f(x) = \infty \text{ and } \lim_{x\to\infty} g(x) = \infty \right]$

OR $\left[\text{ both } \lim_{x\to\infty} f(x) = -\infty \text{ and } \lim_{x\to\infty} g(x) = \infty \right]$

OR $\left[\text{ both } \lim_{x\to\infty} f(x) = \infty \text{ and } \lim_{x\to\infty} g(x) = -\infty \right]$

OR $\left[\text{ both } \lim_{x\to\infty} f(x) = -\infty \text{ and } \lim_{x\to\infty} g(x) = -\infty \right]$.

Suppose $\displaystyle\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ exists. (Can be finite, ∞ or $-\infty$.)

$x \to -\infty$?

Then $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$.

e.g.: $f(x) := x^{-2}\sin(x)$

$g(x) := x^{-1}$

other examples later...

WARNING: If $\displaystyle\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ does not exist,

then you can't conclude anything. ☹

cf. §4.8, p. 81 TH'M 4.11 (L'Hôpital's Rule):

Suppose ONE of the following five possibilities happens:

$$\left[\text{both } \lim_{x\to-\infty} f(x) = 0 \quad \text{and} \quad \lim_{x\to-\infty} g(x) = 0\right]$$

$$\text{OR} \left[\text{both } \lim_{x\to-\infty} f(x) = \infty \quad \text{and} \quad \lim_{x\to-\infty} g(x) = \infty\right]$$

$$\text{OR} \left[\text{both } \lim_{x\to-\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x\to-\infty} g(x) = \infty\right]$$

$$\text{OR} \left[\text{both } \lim_{x\to-\infty} f(x) = \infty \quad \text{and} \quad \lim_{x\to-\infty} g(x) = -\infty\right]$$

$$\text{OR} \left[\text{both } \lim_{x\to-\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x\to-\infty} g(x) = -\infty\right].$$

Suppose $\displaystyle\lim_{x\to-\infty} \frac{f'(x)}{g'(x)}$ exists. (Can be finite, ∞ or $-\infty$.)

Then $\displaystyle\lim_{x\to-\infty} \frac{f(x)}{g(x)} = \lim_{x\to-\infty} \frac{f'(x)}{g'(x)}$.

e.g.: $f(x) := x^{-2}\sin(x)$

$g(x) := x^{-1}$

WARNING: If $\displaystyle\lim_{x\to-\infty} \frac{f'(x)}{g'(x)}$ does not exist,

then you can't conclude anything. ☹

NOTE: Most people simply put in the equal sign while working through the problem.
However, if, at the end, you get "does not exist", for the limit of the quotient of the derivatives,
you cannot conclude that the original limit doesn't exist . . . ☹
To show an example of this, we need a fact. . .

EXAMPLE: Find $\lim_{x\to\infty} \frac{1-e^{2x}}{x}$.

"$1 - e^{2\cdot\infty} = 1 - \infty = -\infty$"

$e^{2\cdot 1000}$ is very positive.

First step: Check the form. $\frac{-\infty}{\infty}$

Second step: Check whether l'Hôpital's rule leads to an answer.

SKILL l'Hôpital

$[d/dx][1 - e^{2x}] = -2e^{2x}$

"$e^{2\cdot\infty} = \infty$"

$$\lim_{x\to\infty} \frac{1-e^{2x}}{x} \stackrel{\text{l'H}}{=} \lim_{x\to\infty} \frac{-2e^{2x}}{1} = -\infty \quad ■$$

"$\frac{-2\cdot\infty}{1} = -\infty$"

$[d/dx][x] = 1$

Last step: Put in the remaining equal sign.

NOTE: Most people simply put in the equal sign while working through the problem.
However, if, at the end, you get "does not exist", for the limit of the quotient of the derivatives,
you cannot conclude that the original limit doesn't exist . . . ☹
To show an example of this, we need a fact. . .

$$-1 \leq \sin \leq 1$$

$$-\frac{1}{10^6} \leq \frac{\sin(10^6)}{10^6} \leq \frac{1}{10^6}$$

$$\approx 0$$

e.g.: $\displaystyle\lim_{x\to\infty} \frac{\sin x}{x} = 0$ ■

Next: example where limit *after* l'H DNE, but limit *before* l'H exists. . .

"$\dfrac{\text{bdd}}{\infty} = 0$"

lim could be $\displaystyle\lim_{x\to a}$, $\displaystyle\lim_{x\to a^+}$, $\displaystyle\lim_{x\to a^-}$, $\displaystyle\lim_{x\to\infty}$ or $\displaystyle\lim_{x\to-\infty}$.

Fact: f bdd, $\lim g(x) = \infty \Rightarrow \lim \dfrac{f(x)}{g(x)} = 0$

EXAMPLE: Find $\lim_{x\to\infty} \frac{(\cos(x^5))/x}{(x+4)^{-1/2}}$.

$$\frac{\text{bdd}/\infty}{\infty^{-1/2}} = \frac{0}{0}$$

$$\lim_{x\to\infty} \frac{(x^{-1})(\cos(x^5))}{(x+4)^{-1/2}}$$

$$\frac{A+B}{C} = \frac{A}{C} + \frac{B}{C}$$

$$\overset{\substack{\text{tentative}\\ \text{l'H}}}{=} \lim_{x\to\infty} \frac{\boxed{(x^{-1})(-\sin(x^5))(5x^4)} + \boxed{(-x^{-2})(\cos(x^5))}}{\boxed{-(1/2)(x+4)^{-3/2}}}$$

$$= \lim_{x\to\infty} \frac{\boxed{(x^{-1})(-\sin(x^5))(5x^4)}}{\boxed{-(1/2)(x+4)^{-3/2}}} + \frac{\boxed{(-x^{-2})(\cos(x^5))}}{\boxed{-(1/2)(x+4)^{-3/2}}}$$

---

e.g.: $\lim_{x\to\infty} \frac{\sin x}{x} = 0$ ■

Next: example where limit *after* l'H DNE, but limit *before* l'H exists...

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Fact: f bdd, $\lim g(x) = \infty \Rightarrow \lim \frac{f(x)}{g(x)} = 0$

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EXAMPLE: Find $\displaystyle\lim_{x\to\infty} \frac{(\cos(x^5))/x}{(x+4)^{-1/2}}$. $\displaystyle\frac{\text{bdd}/\infty}{\infty^{-1/2}} = \frac{0}{0}$

$$\lim_{x\to\infty} \frac{(x^{-1})(\cos(x^5))}{(x+4)^{-1/2}}$$

$$\overset{\substack{\text{tentative}\\ \text{l'H}}}{=} \lim_{x\to\infty} \frac{(x^{-1})(-\sin(x^5))(5x^4) + (-x^{-2})(\cos(x^5))}{-(1/2)(x+4)^{-3/2}}$$

$$= \lim_{x\to\infty} \frac{(\boxed{x^{-1}})(+\sin(x^5))(5x^4)}{+(\boxed{1/2})(x+4)^{-3/2}} + \frac{(+\boxed{x^{-2}})(\cos(x^5))}{+(\boxed{1/2})(x+4)^{-3/2}}$$

$$\frac{x^{-1}}{1/2} = \frac{1}{x}\frac{2}{1} = \frac{2}{x} \qquad \frac{x^{-2}}{1/2} = \frac{1}{x^2}\frac{2}{1} = \frac{2}{x^2}$$

$$= \lim_{x\to\infty} \frac{\boxed{2}(\sin(x^5))(5x^{\overset{3}{\cancel{4}}})}{\boxed{\cancel{x}}(x+4)^{-3/2}} \boxed{\cancel{+\frac{\overset{\text{bdd}}{\boxed{2}(\cos(x^5))}}{\underbrace{\boxed{x^2}(x+4)^{-3/2}}_{\text{tends to }\infty}}}}$$

$$x^2(x+4)^{-3/2} \underset{x\to\infty}{\sim} x^2x^{-3/2} = x^{1/2} \underset{x\to\infty}{\longrightarrow} \infty$$

Fact: f bdd, $\lim g(x) = \infty \Rightarrow \lim \dfrac{f(x)}{g(x)} = 0$

EXAMPLE: Find $\displaystyle\lim_{x\to\infty} \frac{(\cos(x^5))/x}{(x+4)^{-1/2}}$.

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$$\overset{\substack{\text{tentative}\\ \text{l'H}}}{=} \lim_{x\to\infty} \frac{(x^{-1})(-\sin(x^5))(5x^4) + (-x^{-2})(\cos(x^5))}{-(1/2)(x+4)^{-3/2}}$$

$$= \lim_{x\to\infty} \frac{(x^{-1})(+\sin(x^5))(5x^4)}{+(1/2)(x+4)^{-3/2}} + \frac{(+x^{-2})(\cos(x^5))}{+(1/2)(x+4)^{-3/2}}$$

$$= \lim_{x\to\infty} \frac{2(\sin(x^5))(5x^{\cancel{4}\,3})}{\cancel{x}(x+4)^{-3/2}} \boxed{\cancel{+ \overbrace{\frac{2(\cos(x^5))}{x^2(x+4)^{-3/2}}}^{\text{bdd}}}}$$

tends to ∞

$$= \lim_{x\to\infty} [\sin(x^5)][10\underbrace{x^3}_{\infty}\underbrace{(x+4)^{3/2}}_{\infty}]$$

"$10\cdot\infty\cdot\infty = \infty$"

EXAMPLE: Find $\lim_{x\to\infty} \frac{(\cos(x^5))/x}{(x+4)^{-1/2}}$.

$\frac{\text{bdd}/\infty}{\infty^{-1/2}} = \frac{0}{0}$

$$\lim_{x\to\infty} \frac{(x^{-1})(\cos(x^5))}{(x+4)^{-1/2}}$$

$$\overset{\text{tentative}}{\underset{\text{l'H}}{=}} \lim_{x\to\infty} \frac{(x^{-1})(-\sin(x^5))(5x^4) + (-x^{-2})(\cos(x^5))}{-(1/2)(x+4)^{-3/2}}$$

$$= \lim_{x\to\infty} \frac{(x^{-1})(+\sin(x^5))(5x^4)}{+(1/2)(x+4)^{-3/2}} + \frac{(+x^{-2})(\cos(x^5))}{+(1/2)(x+4)^{-3/2}}$$

$$= \lim_{x\to\infty} \frac{2(\sin(x^5))(5x^{\cancel{4}\,3})}{\cancel{x}(x+4)^{-3/2}} \boxed{\cancel{+ \frac{\overbrace{2(\cos(x^5))}^{\text{bdd}}}{\underbrace{x^2(x+4)^{-3/2}}_{\text{tends to } \infty}}}}$$

$$= \lim_{x\to\infty} \underbrace{[\sin(x^5)]}_{\text{osc. btwn } 1 \text{ and } -1} \underbrace{[10x^3(x+4)^{3/2}]}_{\infty} \text{ DNE}$$

"$10 \cdot \infty \cdot \infty = \infty$"

EXAMPLE: Find $\displaystyle\lim_{x\to\infty} \frac{(\cos(x^5))/x}{(x+4)^{-1/2}}$.

Next: why this limit exists . . .

$\dfrac{\text{bdd}/x}{\infty^{-1/2}} \to 0$

$$\lim_{x\to\infty} \frac{(x^{-1})(\cos(x^5))}{(x+4)^{-1/2}}$$

$$\overset{\text{tentative}}{\underset{}{\stackrel{\text{l'H}}{=}}} \lim_{x\to\infty} \frac{(x^{-1})(-\sin(x^5))(5x^4) + (-x^{-2})(\cos(x^5))}{-(1/2)(x+4)^{-3/2}}$$

$$= \lim_{x\to\infty} \frac{(x^{-1})(+\sin(x^5))(5x^4)}{+(1/2)(x+4)^{-3/2}} + \frac{(+x^{-2})(\cos(x^5))}{+(1/2)(x+4)^{-3/2}}$$

$$= \lim_{x\to\infty} \frac{2(\sin(x^5))(5x^{\overset{3}{\cancel{4}}})}{\cancel{x}(x+4)^{-3/2}} \boxed{\cancel{+\frac{2(\cos(x^5))}{x^2(x+4)^{-3/2}}}}$$

bdd

tends to ∞

$$= \lim_{x\to\infty} [\sin(x^5)][10x^3(x+4)^{3/2}] \text{ DNE}$$

Next: why this limit exists . . .

EXAMPLE: Find $\lim_{x\to\infty} \frac{(\cos(x^5))/x}{(\boxed{x+4})^{-1/2}}$.

asymptotics //

SKILL 0/0

$$\lim_{x\to\infty} \frac{(\cos(x^5))/\boxed{x}}{\boxed{x}^{-1/2}} = \lim_{x\to\infty} \frac{\cos(x^5)}{\boxed{\boxed{x}\cdot x^{-1/2}}} = \lim_{x\to\infty} \frac{\overset{\text{bdd}}{\cos(x^5)}}{\underset{\text{tends to } \infty}{\boxed{x^{1/2}}}} = 0$$

"$\frac{\text{bdd}}{\infty} = 0$"

Fact: f bdd, $\lim g(x) = \infty \Rightarrow \lim \frac{f(x)}{g(x)} = 0$

EXERCISE: Show that $\lim_{x\to\infty} \frac{(\sqrt{3x} + \cos(x^5))/x}{(x+4)^{-1/2}} = \sqrt{3}$.

SKILL 0/0

Show also that l'Hôpital's Rule again leads to "does not exist".

That gives no information.

☹

Next: DNE after l'Hôpital's Rule does NOT always mean that the original limit is zero . . .

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EXAMPLE: Find $\displaystyle\lim_{x\to 1}\frac{(\tan(2x))-2x}{x^3}$.

First step: Check the form. DETERMINATE!

Answer: $\displaystyle\frac{(\tan 2)-2}{1} = (\tan 2)-2$ $\blacksquare$

EXAMPLE: Find $\displaystyle\lim_{x\to 0}\frac{(\tan(2x))-2x}{x^3}$.

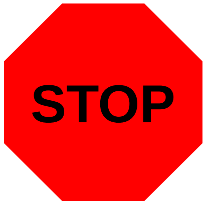
$$\frac{0}{0}\quad \lim_{x\to 0}\frac{(\tan(2x))-2x}{x^3} \overset{\substack{\text{tentative}\\ \text{l'H}}}{=} \lim_{x\to 0}\frac{(\sec^2(2x))(2)-2}{3x^2}\quad \frac{0}{0}$$

sec = 1/ cos
cos 0 = 1

$$\overset{\substack{\text{tentative}\\ \text{l'H}}}{=} \lim_{x\to 0}\frac{\boxed{2}[\overset{1}{\boxed{\cancel{\sec(2x)}}}][(\overset{1}{\boxed{\cancel{\sec(2x)}}})(\tan(2x))][\boxed{2}](\boxed{2})}{6x}$$

$$= \lim_{x\to 0}\frac{\boxed{8}(\tan(2x))}{6x}\quad \frac{0}{0}$$

l'H okay; easier to simplify first



SKILL
l'Hôpital

sec = 1/ cos
cos 0 = 1

$$\overset{\substack{\text{tentative}\\ \text{l'H}}}{=} \lim_{x\to 0}\frac{8(\overset{1}{\boxed{\sec^2(2x)}})(2)}{6} = \frac{16}{6} = \frac{8}{3}\ \blacksquare$$

§4.8