CALCULUS
The Mean Value Theorem

Let $y = L(x)$ be the line through $(3, 1)$ and $(9, 13)$.

2 units rise
per unit run

$$\text{slope} = \frac{13-1}{9-3} = 2$$

Start here.
Goal: $(8, ?)$

Problem: Find $L(8) = \overbrace{\underbrace{\left[\frac{13-1}{9-3}\right]}_{\text{slope}}\underbrace{[8-3]}_{\text{run}}}^{\text{rise}} + 1 = 11$

$8 :\rightarrow x$

$$L(x) = \left[\frac{13-1}{9-3}\right][x-3] + 1$$

$$L'(x) = \frac{13-1}{9-3} = 2$$

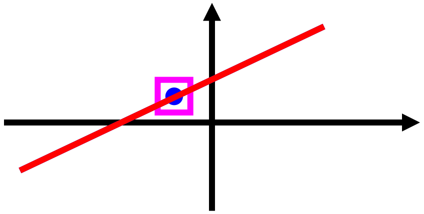
Question: What is the slope
of the tangent line to $y = L(x)$ at $(8, 11)$?

$$L'(8) = \frac{13-1}{9-3} = 2$$

PRELIMINARIES

Note: Any tangent line to a line is just the line itself.

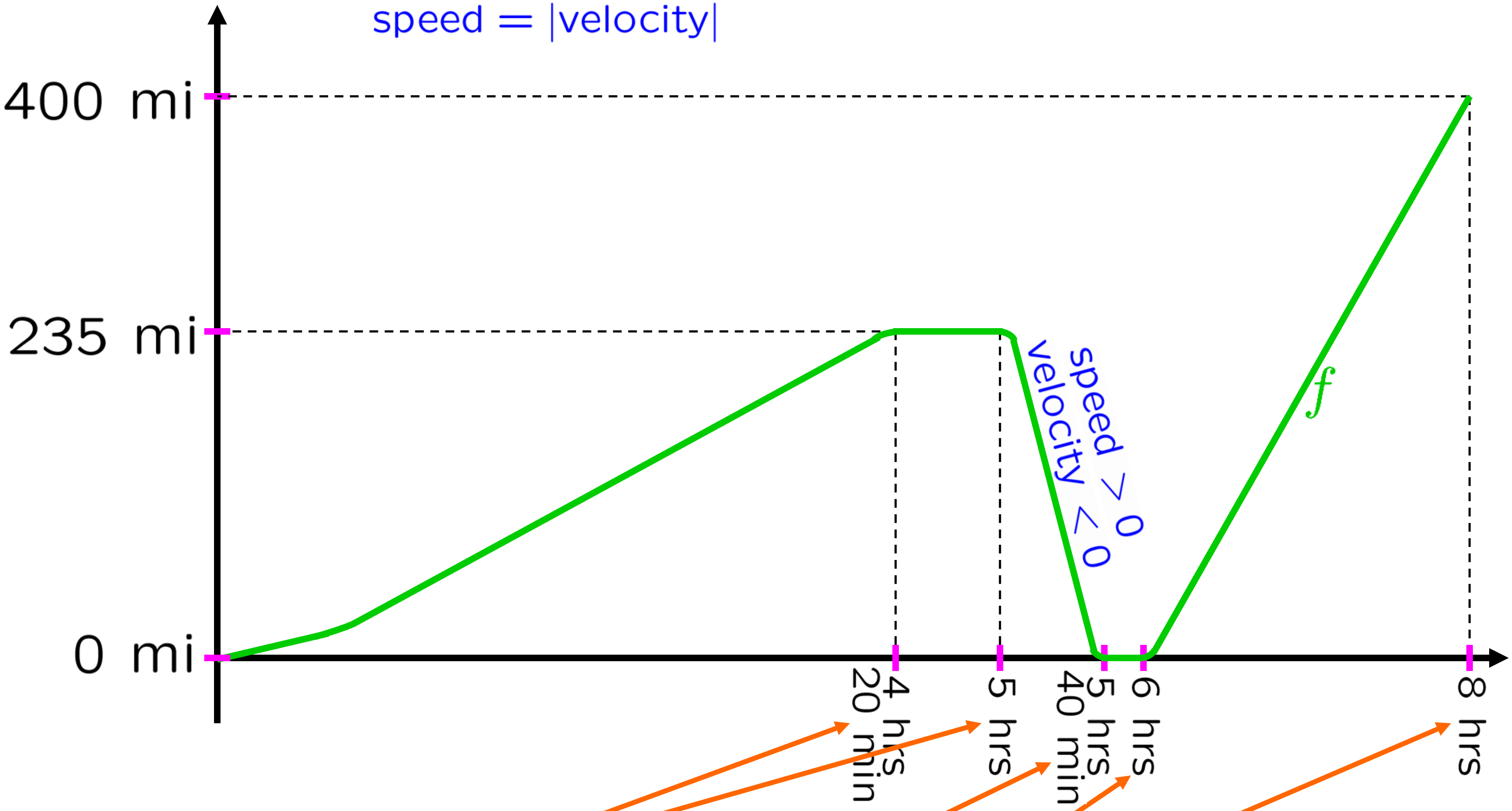
The slope of a tangent line
to a line
is just the slope of the line.



Next: Trip
to Chicago

velocity = rate of change in position w.r.t. time

speed = |velocity|

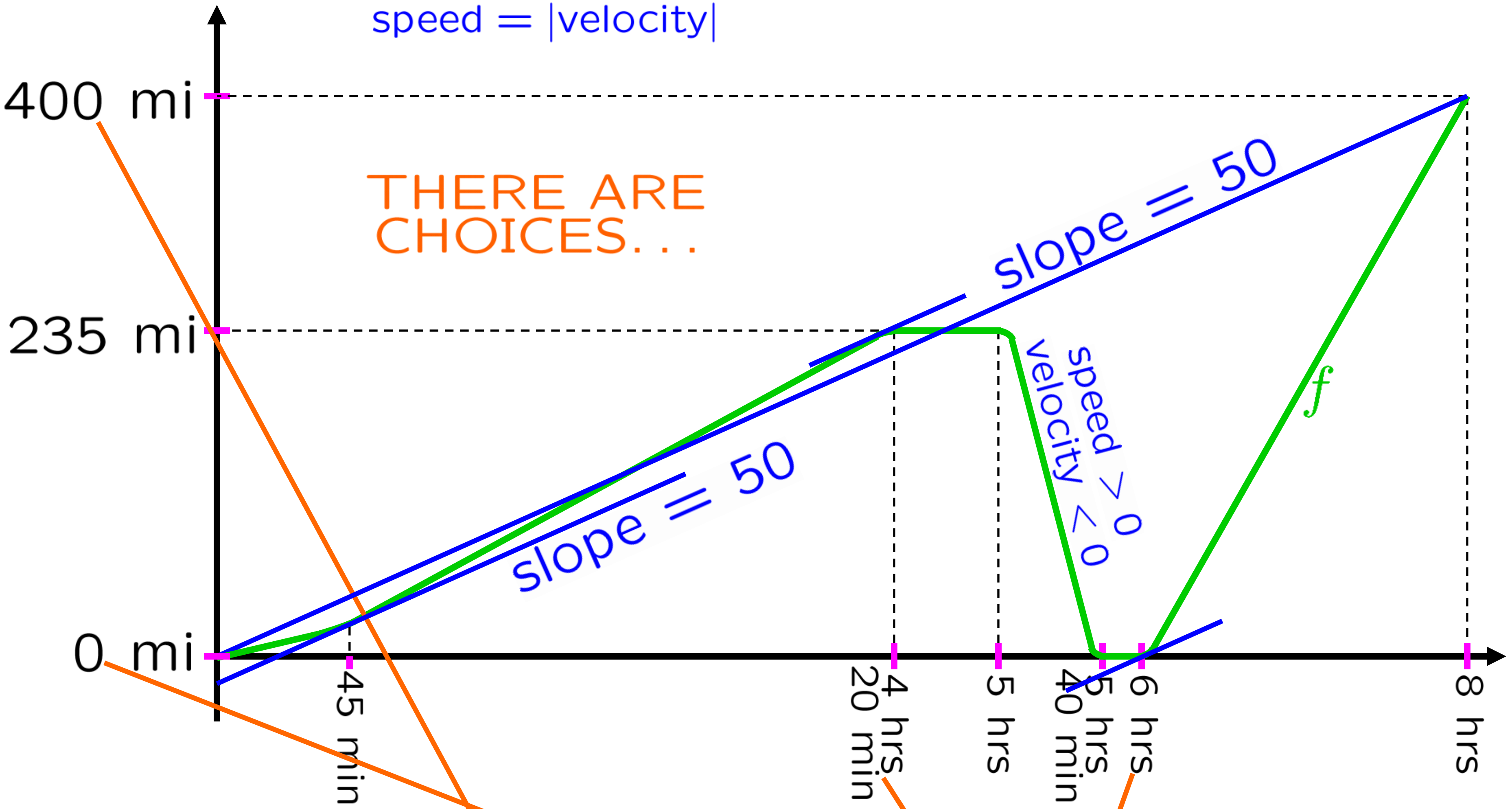


0 hrs, leave Minneapolis
4 hrs 20 mins, start break
5 hrs, notice problem,
head back
5 hrs 40 mins, arrive Minneapolis

6 hrs, leave Minneapolis
8 hrs, arrive Chicago

velocity = rate of change in position w.r.t. time

speed = |velocity|



Average velocity over the eight hrs:

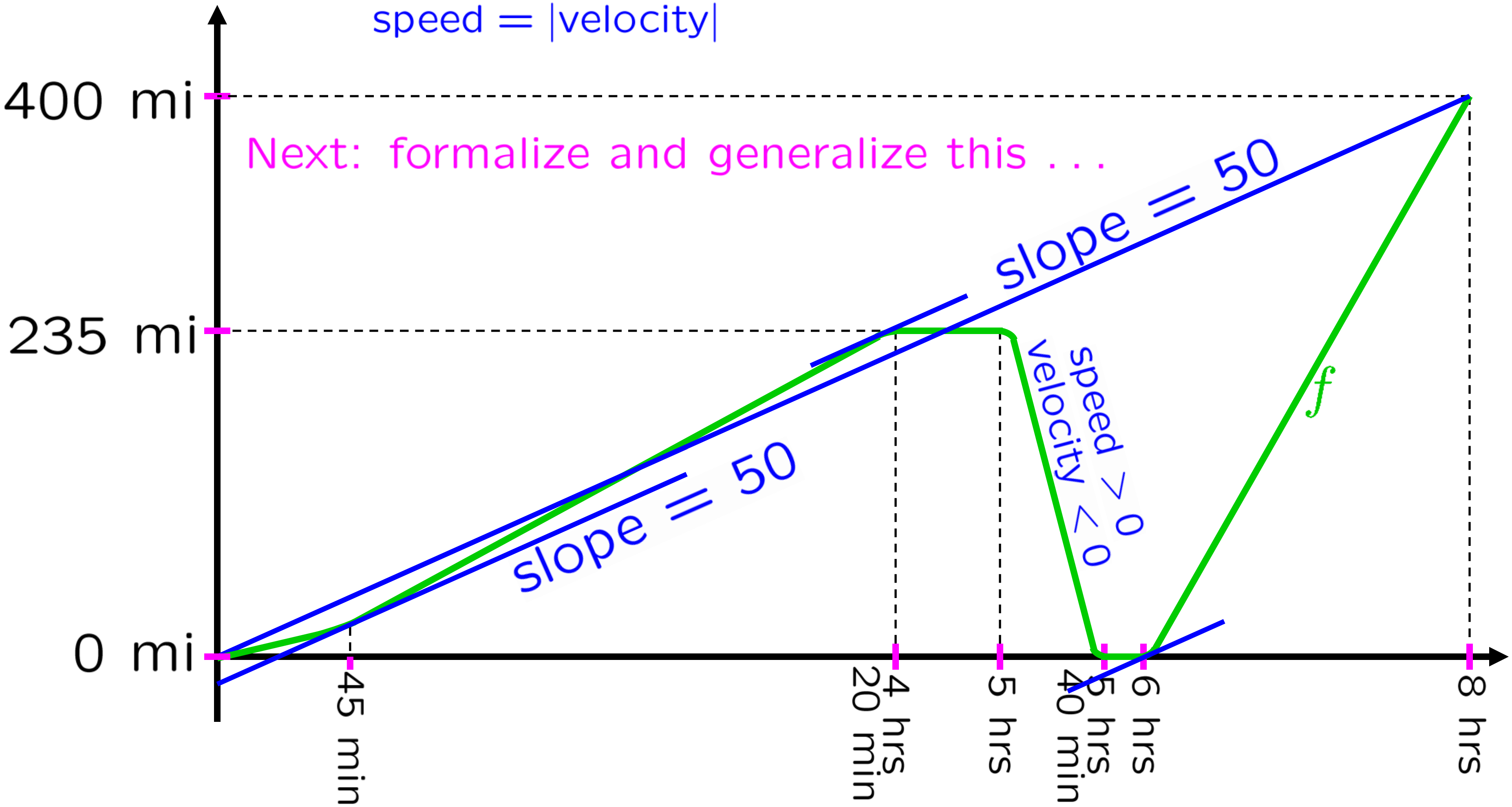
$$\frac{[f(8)] - [f(0)]}{8 - 0} = \frac{400 - 0}{8} = 50$$

Average velocity is 50 mph from 0 hrs to 8 hrs.

instantaneous velocity is 50 mph at some time (45 min)
(4 hrs 19 min 30 sec)
(6 hrs 5 sec)

velocity = rate of change in position w.r.t. time

speed = |velocity|



Next: formalize and generalize this . . .

Expect: Every avg. velocity is an instantaneous velocity.

Expect: Every sec. slope is a tangent slope.

Average velocity is 50 mph from 0 hrs to 8 hrs.

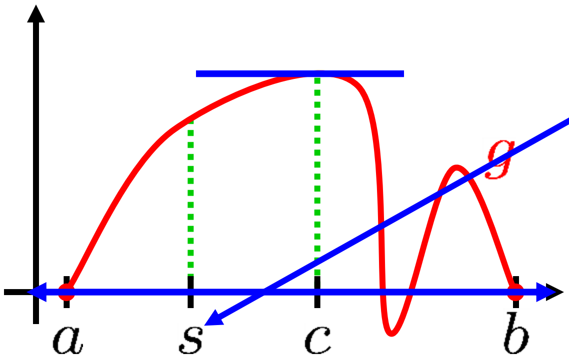
instantaneous velocity is 50 mph at some time (45 min)
(4 hrs 19 min 30 sec)
(6 hrs 5 sec)

Let $a, b \in \mathbb{R}$ and assume that $a < b$.

Fact: Assume that g is continuous on $[a, b]$,
that g is differentiable on (a, b),
that $g(a) = g(b) = 0$
and that $\exists s \in (a, b)$ s.t. $g(s) > 0$.
Then $\exists c \in (a, b)$ s.t. $g'(c) = 0$.

"TAME" HYPOTHESES

Proof:



ExtrValThm: Choose $c \in [a, b]$ s.t. g attains a global maximum at c.

$g(c) \geq g(s) > 0$

$g(a) = g(b) = 0$

$c \neq a$ and $c \neq b$, so g is differentiable at c.
($g'(c)$ exists.)

Fermat: $g'(c) = 0$. QED

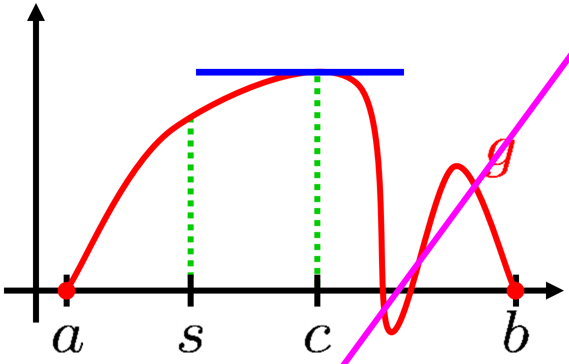
Every sec. slope is a tangent slope.

Let $a, b \in \mathbb{R}$ and assume that $a < b$.

Fact: Assume that g is continuous on $[a, b]$,
that g is differentiable on (a, b),
that $g(a) = g(b) = 0$
and that $\exists s \in (a, b)$ s.t. $g(s) \boxed{>} 0$.
Then $\exists c \in (a, b)$ s.t. $g'(c) = 0$.

a similar argument works with $<$

Proof:



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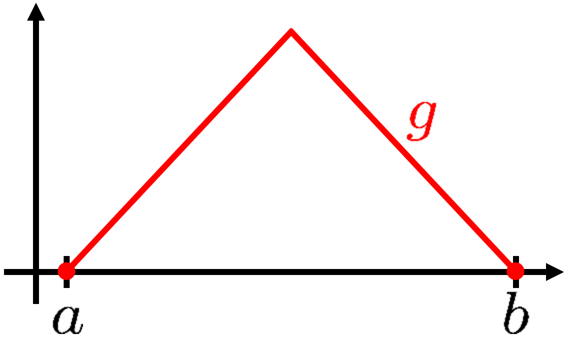
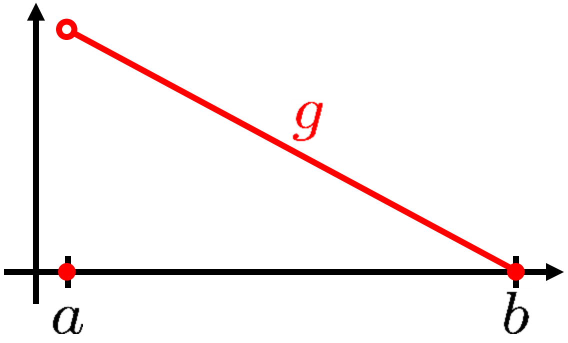
Differentiability needed:



Continuity needed:



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Fact: Assume that g is continuous on $[a, b]$,
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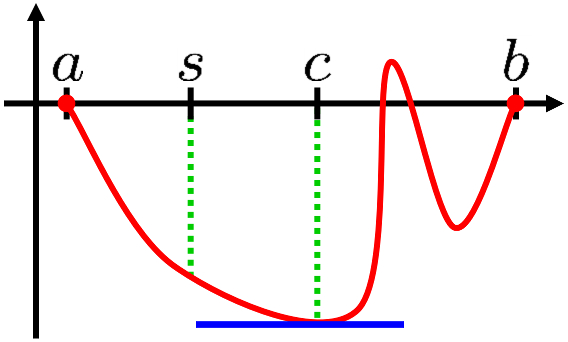
Then $\exists c \in (a, b)$ s.t. $g'(c) = 0$.

Proof:



ExtrValThm: Choose $c \in [a, b]$
s.t. g attains a global minimum at c.

$g(c) \leq g(s) < 0$ $g(a) = g(b) = 0$

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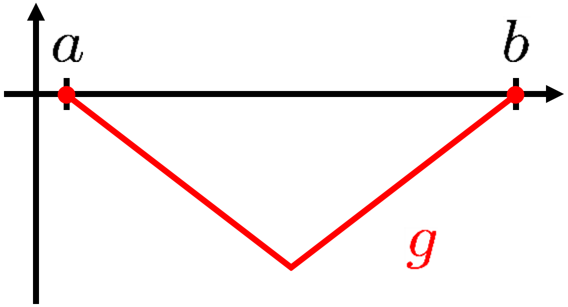

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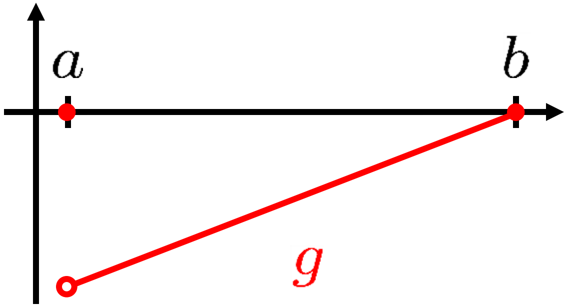

Let $a, b \in \mathbb{R}$ and assume that $a < b$.

Fact: Assume that g is continuous on $[a, b]$,
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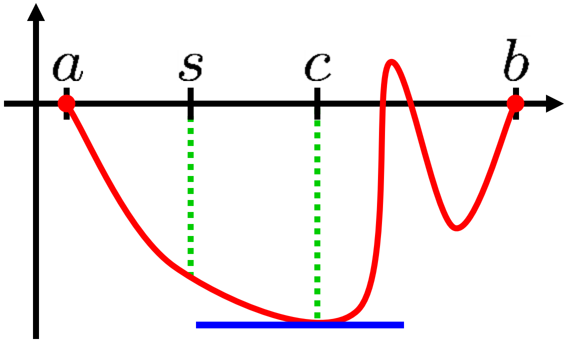
$\boxed{\text{and that } \exists s \in (a, b) \text{ s.t. } g(s) < 0}$. needed?
Then $\exists c \in (a, b)$ s.t. $g'(c) = 0$.

Proof:



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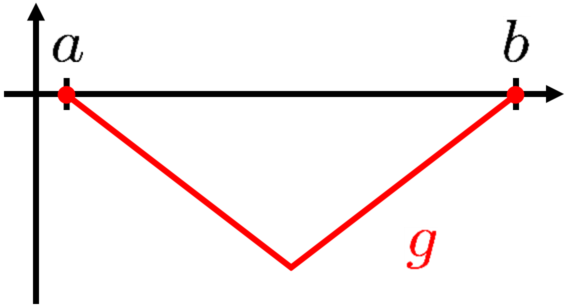

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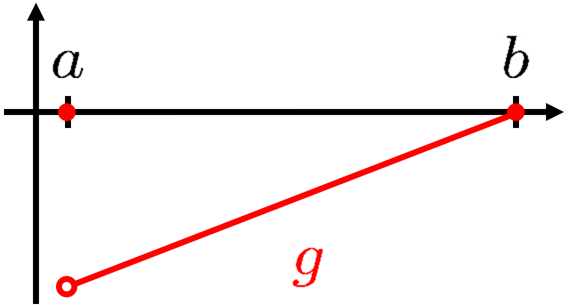

Let $a, b \in \mathbb{R}$ and assume that $a < b$.

Fact: Assume that g is continuous on $[a, b]$,
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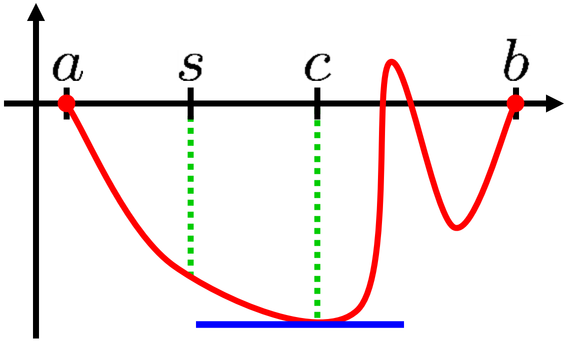
and that $\exists s \in (a, b)$ s.t. $g(s) < 0$. needed? no. . .
Then $\exists c \in (a, b)$ s.t. $g'(c) = 0$.

Proof:



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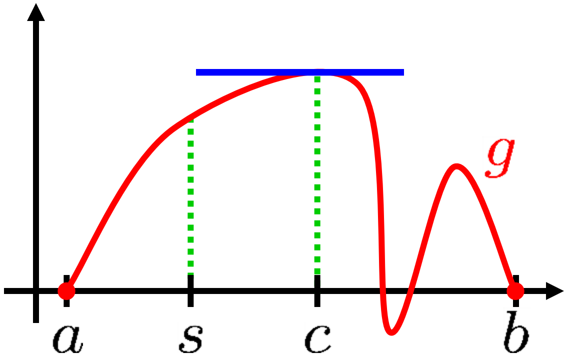
Fact: Assume that g is continuous on $[a, b]$,
that g is differentiable on (a, b),
and that $g(a) = g(b) = 0$.
Then $\exists c \in (a, b)$ s.t. $g'(c) = 0$.

Proof: Easy if $g = 0$ on (a, b),
so we may assume $\exists s \in (a, b)$ s.t. $g(s) \neq 0$.
Done if $g(s) < 0$.

Let $a, b \in \mathbb{R}$ and assume that $a < b$.

Fact: Assume that g is continuous on $[a, b]$,
that g is differentiable on (a, b),
that $g(a) = g(b) = 0$
and that $\exists s \in (a, b)$ s.t. $g(s) > 0$.
Then $\exists c \in (a, b)$ s.t. $g'(c) = 0$.

Proof:



ExtrValThm: Choose $c \in [a, b]$
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Fact: Assume that g is continuous on $[a, b]$,
that g is differentiable on (a, b),
and that $g(a) = g(b) = 0$.
Then $\exists c \in (a, b)$ s.t. $g'(c) = 0$.

Proof: Easy if $g = 0$ on (a, b),
so we may assume $\exists s \in (a, b)$ s.t. $g(s) \neq 0$.
Done if $g(s) < 0$. Done if $g(s) > 0$. QED

Let $a, b \in \mathbb{R}$ and assume that $a < b$.

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cf. §6.5, p. 134 (TH'M 6.26) MEAN VALUE THEOREM:

Assume that f is continuous on $[a, b]$,
and that f is differentiable on (a, b).

"TAME" HYPOTHESES

Fact: Assume that g is continuous on $[a, b]$,
that g is differentiable on (a, b),
and that $g(a) = g(b) = 0$.

Fact: Then $\exists c \in (a, b)$ is s.t. $g'(c) = 0$. on $[a, b]$,
that g is differentiable on (a, b),
and that $g(a) = g(b) = 0$.

Then $\exists c \in (a, b)$ s.t. $g'(c) = 0$.

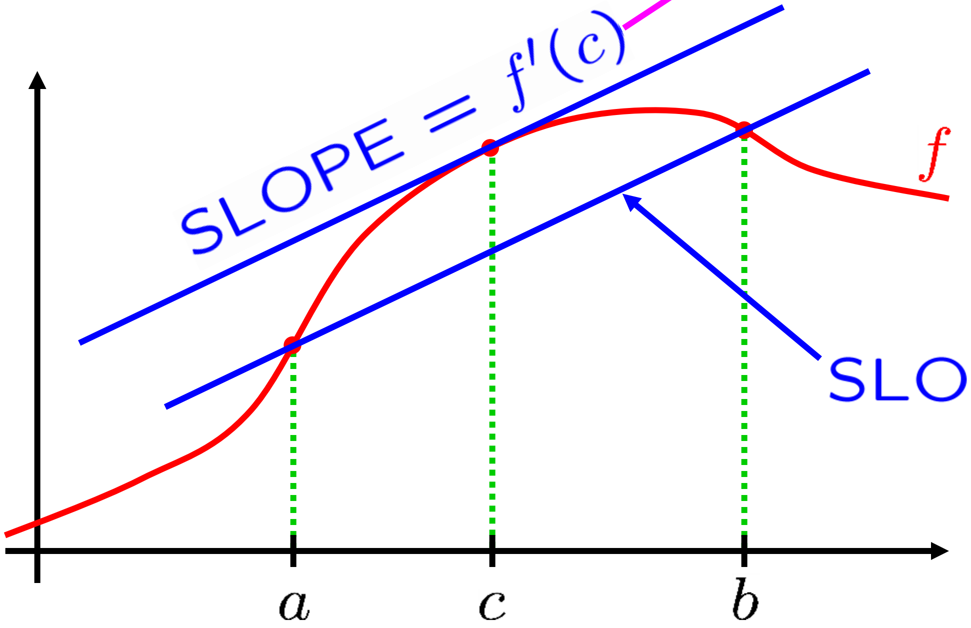
Let $a, b \in \mathbb{R}$ and assume that $a < b$.

cf. §6.5, p. 134 (TH'M 6.26) MEAN VALUE THEOREM:

Assume that f is continuous on $[a, b]$,
and that f is differentiable on (a, b).

ONLY "TAME" HYPOTHESES

Then $\exists c \in (a, b)$ such that $f'(c) = \dfrac{(f(b)) - (f(a))}{b - a}$.



SOMETIMES THERE ARE CHOICES. . .

SLOPE = $\dfrac{(f(b)) - (f(a))}{b - a}$

Every sec. slope is a tangent slope.

Fact: Assume that g is continuous on $[a, b]$,
that g is differentiable on (a, b),
$\boxed{\text{and that } g(a) = g(b) = 0}$.
Then $\exists c \in (a, b)$ s.t. $g'(c) = 0$.

NO hypothesis like this on f in MVT

Let $a, b \in \mathbb{R}$ and assume that $a < b$.

cf. §6.5, p. 134 (TH'M 6.26) MEAN VALUE THEOREM:

Assume that f is continuous on $[a, b]$,
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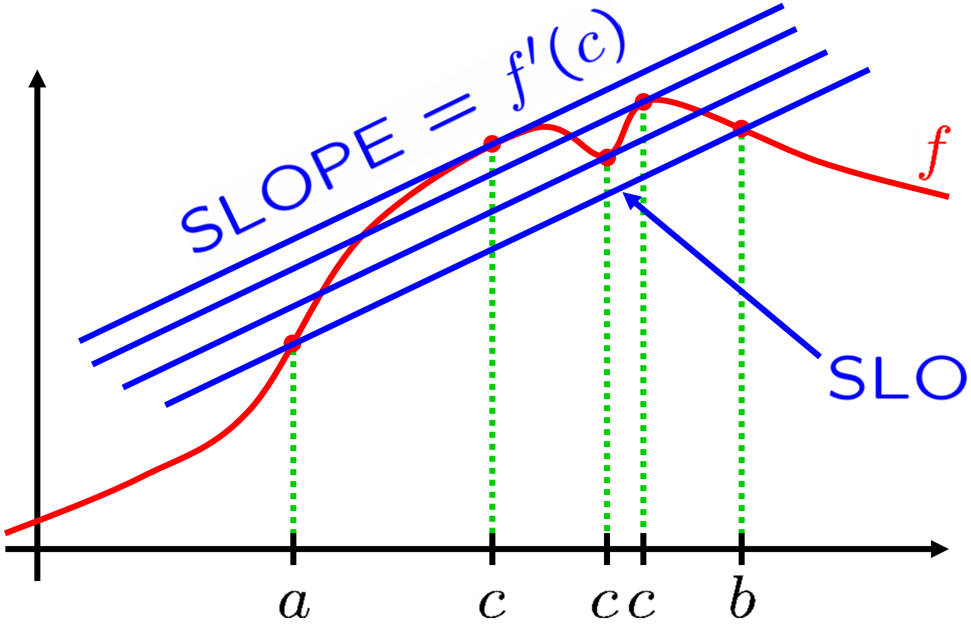
Then $\exists c \in (a, b)$ such that $f'(c) = \dfrac{(f(b)) - (f(a))}{b - a}$.

pf in a special case. . .



SOMETIMES
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Every sec. slope is a tangent slope.

Fact: Assume that g is continuous on $[a, b]$,
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and that $g(a) = g(b) = 0$.
Then $\exists c \in (a, b)$ s.t. $g'(c) = 0$.

Let $a, b \in \mathbb{R}$ and assume that $a < b$.

---

cf. §6.5, p. 134 (TH'M 6.26) MEAN VALUE THEOREM:

Assume that f is continuous on $[a, b]$,
and that f is differentiable on (a, b).

Then $\exists c \in (a, b)$ such that $f'(c) = \dfrac{(f(b)) - (f(a))}{b - a}$.

pf in a special case. . .

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e.g.: Suppose $a = 3$, $b = 9$, $f(3) = 1$ and $f(9) = 13$.

Want: $\exists c \in (a, b)$ such that $f'(c) = \dfrac{13 - 1}{9 - 3}$.

The curve $y = f(x)$ goes through $(3, 1)$ and $(9, 13)$.
Let $y = L(x)$ be the line through $(3, 1)$ and $(9, 13)$.
Then $L(3) = 1$ and $L(9) = 13$.

Define $g : [a, b] \to \mathbb{R}$ by $\boxed{g(x) = [f(x)] - [L(x)]}$.

DIFFERENTIATE

$$g(3) = [f(3)] - [L(3)] = 1 - 1 = 0$$
$$g(9) = [f(9)] - [L(9)] = 13 - 13 = 0$$

Choose $c \in (a, b)$ s.t. $g'(c) = 0$.

Fact: Assume that g is continuous on $[a, b]$,
that g is differentiable on (a, b),
and that $\boxed{g(a) = g(b) = 0}$.

Then $\exists c \in (a, b)$ s.t. $g'(c) = 0$.

Let $a, b \in \mathbb{R}$ and assume that $a < b$.

cf. §6.5, p. 134 (TH'M 6.26) **MEAN VALUE THEOREM**:

Assume that f is continuous on $[a, b]$,
and that f is differentiable on (a, b).

Then $\exists c \in (a, b)$ such that $f'(c) = \dfrac{(f(b)) - (f(a))}{b - a}$.

pf in a special case. . .

e.g.: Suppose $a = 3$, $b = 9$, $f(3) = 1$ and $f(9) = 13$.

Want: $\exists c \in (a, b)$ such that $f'(c) = \dfrac{13 - 1}{9 - 3}$.

The curve $y = f(x)$ goes through $(3, 1)$ and $(9, 13)$.

Let $y = L(x)$ be the line through $(3, 1)$ and $(9, 13)$.

Then $L(3) = 1$ and $L(9) = 13$.

Define $g : [a, b] \to \mathbb{R}$ by $\boxed{g(x) = [f(x)] - [L(x)]}$.

DIFFERENTIATE

$$g(3) = [f(3)] - [L(3)] = 1 - 1 = 0$$

$$g(9) = [f(9)] - [L(9)] = 13 - 13 = 0$$

Choose $c \in (a, b)$ s.t. $\begin{array}{c} g'(c) \\ \| \\ [f'(c)] - [L'(c)] \end{array} = 0.$ $\qquad \begin{array}{c} g'(x) \\ \| \\ [f'(x)] - [L'(x)] \end{array}$

$$f'(c) = L'(c)$$

Let $a, b \in \mathbb{R}$ and assume that $a < b$.

cf. §6.5, p. 134 (TH'M 6.26) MEAN VALUE THEOREM:

Assume that f is continuous on $[a, b]$,
and that f is differentiable on (a, b).

Then $\exists c \in (a, b)$ such that $f'(c) = \dfrac{(f(b)) - (f(a))}{b - a}$.

pf in a special case. . .

e.g.: Suppose $a = 3$, $b = 9$, $f(3) = 1$ and $f(9) = 13$.

Want: $\exists c \in (a, b)$ such that $f'(c) = \dfrac{13 - 1}{9 - 3}$. ☺

The curve $y = f(x)$ goes through $(3, 1)$ and $(9, 13)$.

Let $y = L(x)$ be the line through $(3, 1)$ and $(9, 13)$.

Then $L(3) = 1$ and $L(9) = 13$.

Define $g : [a, b] \to \mathbb{R}$ by $g(x) = [f(x)] - [L(x)]$.

$$g(3) = [f(3)] - [L(3)] = 1 - 1 = 0$$

$$g(9) = [f(9)] - [L(9)] = 13 - 13 = 0$$

Choose $c \in (a, b)$ s.t. $g'(c) = 0$.

The slope of a tangent line
to a line
is just the slope of the line.

$$L'(x) = \frac{13 - 1}{9 - 3}$$

$3 :\to a$ $\quad$ $1 :\to f(a)$

$9 :\to b$ $\quad$ $13 :\to f(b)$

$$f'(c) = L'(c) = \frac{13 - 1}{9 - 3}$$

Let $a, b \in \mathbb{R}$ and assume that $a < b$.

---

cf. §6.5, p. 134 (TH'M 6.26) MEAN VALUE THEOREM:

Assume that f is continuous on $[a, b]$,
and that f is differentiable on (a, b).

Then $\exists c \in (a, b)$ such that $f'(c) = \dfrac{(f(b)) - (f(a))}{b - a}$.

Proof:

Want: $\exists c \in (a, b)$ such that $f'(c) = \dfrac{(f(b)) - (f(a))}{b - a}$.

The curve $y = f(x)$ goes through $(a, f(a))$ and $(b, f(b))$.

Let $y = L(x)$ be the line through $(a, f(a))$ and $(b, f(b))$.

Then $L(a) = f(a)$ and $L(b) = f(b)$.

Define $g : [a, b] \to \mathbb{R}$ by $g(x) = [f(x)] - [L(x)]$.

$$g(a) = [f(a)] - [L(a)] = 0$$

$$g(b) = [f(b)] - [L(b)] = 0$$

Choose $c \in (a, b)$ s.t. $g'(c) = 0$.

The slope of a tangent line
to a line
is just the slope of the line.

$$L'(x) = \frac{(f(b)) - (f(a))}{b - a}$$

$$f'(c) = L'(c) = \frac{(f(b)) - (f(a))}{b - a}$$

QED

Let $a, b \in \mathbb{R}$ and assume that $a < b$.

---

cf. §6.5, p. 134 (TH'M 6.26) **MEAN VALUE THEOREM**:

Assume that f is continuous on $[a, b]$,
and that f is differentiable on (a, b).

Then $\exists c \in (a, b)$ such that $$f'(c) = \frac{\boxed{(f(b)) - (f(a))}}{b - a}.$$

$= 0$

CONSEQUENSES ∴

THEOREM (ONE-TO-ONE TEST):

If $f'(x) \neq 0$, for all x in an interval I,
then f is one-to-one on I.

cf. §6.5, p. 134 (TH'M 6.25) **ROLLE'S THEOREM**:

Assume that f is continuous on $[a, b]$,
that f is differentiable on (a, b)
and that $f(a) = f(b)$.

Then $\exists c \in (a, b)$ such that $f'(c) = 0$.

Every sec. slope is a
tangent slope.

Idea: If some secant line is horizontal,
then some tangent line is horizontal.

Spp

Let $a, b \in \mathbb{R}$ and assume that $a < b$.

cf. §6.5, p. 134 (TH'M 6.26) MEAN VALUE THEOREM:

Assume that f is continuous on $[a, b]$,
and that f is differentiable on (a, b).

Then $\exists c \in (a, b)$ such that $f'(c) = \dfrac{(f(b)) - (f(a))}{b - a}$.

THEOREM (ONE-TO-ONE TEST):

If $\underline{f'(x) \boxed{\neq} 0, \text{ for all } x \text{ in an interval } I,}$
then f is one-to-one on I.

works for any kind of interval (open, closed, half-open) (bdd, unbdd)

Proof: Let $a, b \in I$. Want: $f(a) \neq f(b)$.

Assume $a < b$. Assume $f(a) = f(b)$.

Want: Contradiction.

Choose $c \in (a, b)$ such that $f'(c) = \dfrac{(f(b)) - (f(a))}{b - a}$.

$$\|$$

0

Contradiction. QED

Every sec. slope is a tangent slope.

Idea: If no tangent line is horizontal,
then no secant line is horizontal.

Spp

Let $a, b \in \mathbb{R}$ and assume that $a < b$.

cf. §6.5, p. 134 (TH'M 6.26) MEAN VALUE THEOREM:

Assume that f is continuous on $[a, b]$,
and that f is differentiable on (a, b).

Then $\exists c \in (a, b)$ such that $f'(c) = \dfrac{(f(b)) - (f(a))}{b - a}$.

THEOREM (CONSTANT TEST):

If $f'(x) = 0$, for all x in an interval I,
then f is constant on I.

works for any kind of interval
(open, closed, half-open)
(bdd, unbdd)

Proof: Let $a, b \in I$. Want: $f(a) = f(b)$.

Assume $a < b$. Assume $f(a) \neq f(b)$.

Want: Contradiction.

$$\text{Choose } c \in (a, b) \text{ such that } f'(c) = \underset{\underset{0}{\nparallel}}{\frac{(f(b)) - (f(a))}{b - a}}.$$

Every sec. slope is a tangent slope.

Contradiction. QED

Idea: If every tangent line is horizontal,
then every secant line is horizontal.

Let $a, b \in \mathbb{R}$ and assume that $a < b$.

---

cf. §6.5, p. 134 (TH'M 6.26) MEAN VALUE THEOREM:

Assume that f is continuous on $[a, b]$,

and that f is differentiable on (a, b).

Then $\exists c \in (a, b)$ such that $f'(c) = \dfrac{(f(b)) - (f(a))}{b - a}$.

THEOREM (CONSTANT TEST):

If $f'(x) \boxed{\equiv} 0$, for all x in an interval I,

then f is constant on I.

works for any kind of interval (open, closed, half-open) (bdd, unbdd)

---

cf. §6.5, p. 136 (TH'M 6.28):

If $g'(x) = h'(x)$, for all x in an interval I,

then $g - h$ is constant on I;

that is, $\exists k \in \mathbb{R}$ s.t. $\forall x \in I$,

$g(x) = (h(x)) + k$.

works for any kind of interval (open, closed, half-open) (bdd, unbdd)

Proof: Let $f := g - h$.

Then $\forall x \in I$, $\quad f'(x) = (g'(x)) - (h'(x)) = 0$.

So f is constant on I. Choose $k \in \mathbb{R}$ s.t. $f = k$ on I.

That is, $\forall x \in I$, $\quad f(x) = k$.

That is, $\forall x \in I$, $\quad (g(x)) - (h(x)) = k$. QED

Let $a, b \in \mathbb{R}$ and assume that $a < b$.

cf. §6.5, p. 134 (TH'M 6.26) MEAN VALUE THEOREM:

Assume that f is continuous on $[a, b]$,
and that f is differentiable on (a, b).

Then $\exists c \in (a, b)$ such that $f'(c) = \dfrac{(f(b)) - (f(a))}{b - a}$.

INCREASING TEST:

If $f'(x) \boxed{>} 0$, for all x in an interval I,
then f is increasing on I.

works for any kind of interval
(open, closed, half-open)
(bdd, unbdd)

Proof: Let $a, b \in I$. Want: $f(a) < f(b)$.

Assume $a < b$. Assume $f(a) \geq f(b)$.

Want: Contradiction.

Choose $c \in (a, b)$ such that $f'(c) = \dfrac{(f(b)) - (f(a))}{b - a}$.

$\wedge|$

0

Contradiction. QED

Every sec. slope is a tangent slope.

Idea: If every tangent line runs uphill,
then every secant line runs uphill.

Spp

Let $a, b \in \mathbb{R}$ and assume that $a < b$.

---

cf. §6.5, p. 134 (TH'M 6.26) MEAN VALUE THEOREM:

Assume that f is continuous on $[a, b]$,
and that f is differentiable on (a, b).

Then $\exists c \in (a, b)$ such that $f'(c) = \dfrac{(f(b)) - (f(a))}{b - a}$.

DECREASING TEST:

If $f'(x) < 0$, for all x in an interval I,
then f is decreasing on I.

works for any kind of interval (open, closed, half-open) (bdd, unbdd)

Proof: Let $a, b \in I$. Want: $f(a) > f(b)$.

Assume $a < b$. Assume $f(a) \leq f(b)$.

Want: Contradiction.

Choose $c \in (a, b)$ such that $f'(c) = \dfrac{(f(b)) - (f(a))}{b - a}$.

$$\vee\!\mid$$

$$0$$

Contradiction. QED

Every sec. slope is a tangent slope.

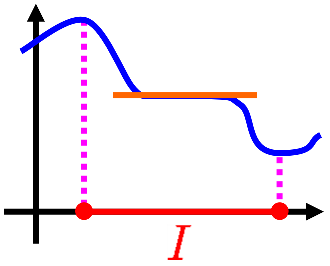
Idea: If every tangent line runs downhill,
then every secant line runs downhill.

Spp

NONINCREASING TEST:

works for any kind of interval
(open, closed, half-open)
(bdd, unbdd)

If $f'(x) \boxed{\le} 0$, for all x in an interval I,
then f is $\boxed{\text{nonincreasing}}$ on I.
semi-decreasing
no secant line runs uphill



Pf is similar.

DECREASING TEST:

works for any kind of interval
(open, closed, half-open)
(bdd, unbdd)

If $f'(x) < 0$, for all x in an interval I,
then f is decreasing on I.

Proof: Let $a, b \in I$. Want: $f(a) > f(b)$.
Assume $a < b$. Assume $f(a) \le f(b)$.
Want: Contradiction.

Choose $c \in (a, b)$ such that $f'(c) = \dfrac{(f(b)) - (f(a))}{b - a}$.

∨|

0

Every sec. slope is a tangent slope.

Contradiction. QED

Idea: If every tangent line runs downhill,
then every secant line runs downhill.

Spp

NONINCREASING TEST:

If $f'(x) \leq 0$, for all x in an interval I,
then f is nonincreasing on I.

semi-decreasing
no secant line runs uphill

works for any kind of interval
(open, closed, half-open)
(bdd, unbdd)

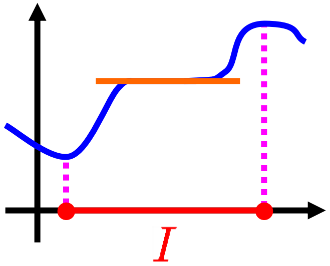
Converse for the NONINCREASING TEST...

NONDECREASING TEST:

If $f'(x) \boxed{\geq} 0$, for all x in an interval I,
then f is nondecreasing on I.

semi-increasing
no secant line runs downhill

works for any kind of interval
(open, closed, half-open)
(bdd, unbdd)



Converse...

Pf is similar.

THEOREM:

If f is nondecreasing and differentiable on an open interval I

then $\boxed{f'(x)} \geq 0$, for all $x \in I$.

limit of slopes of secant lines
slopes of secant lines ≥ 0 QED

Spp

NONINCREASING TEST:

If $f'(x) \leq 0$, for all x in an interval I,
then f is nonincreasing on I.

works for any kind of interval
(open, closed, half-open)
(bdd, unbdd)

Converse for the NONINCREASING TEST...

THEOREM:

If f is nonincreasing and differentiable on an open interval I

then $\boxed{f'(x)} \leq 0$, for all $x \in I$.

limit of slopes of secant lines
slopes of secant lines ≤ 0 QED

THEOREM:

If f is nondecreasing and differentiable on an open interval I

then $\boxed{f'(x)} \geq 0$, for all $x \in I$.

limit of slopes of secant lines
slopes of secant lines ≥ 0 QED

Spp NO perfect converse for the INCREASING TEST...

NONINCREASING TEST:

If $f'(x) \leq 0$, for all x in an interval I,
then f is nonincreasing on I.

works for any
kind of interval
(open, closed,
half-open)
(bdd, unbdd)

THEOREM:

If f is nonincreasing and differentiable
on an open interval I
then $f'(x) \leq 0$, for all $x \in I$.

INCREASING TEST:

If $f'(x) > 0$, for all x in an interval I,
then f is increasing on I.

works for any
kind of interval
(open, closed,
half-open)
(bdd, unbdd)

Spp NO perfect converse for the INCREASING TEST. . .

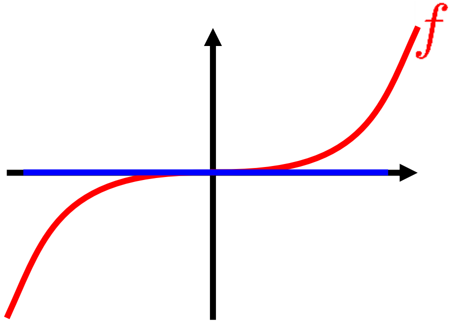
WARNING: f is increasing and differentiable on an open interval I

$$\not\Rightarrow \quad \boxed{f'(x)} > 0, \text{ for all } x \in I.$$

limit of slopes of secant lines

slopes of secant lines > 0 $\not\Rightarrow$ limit of slopes of secant lines > 0

$f(x) = x^3$ is increasing on $I = (-\infty, \infty)$, but $f'(0) = 0$.



An increasing function can "level off for an instant".

f is increasing and differentiable on an open interval I

$$\Rightarrow \quad f'(x) \boxed{\geq} 0, \text{ for all } x \in I.$$

INCREASING TEST:

If $f'(x) > 0$, for all x in an interval I, then f is increasing on I.

works for any kind of interval (open, closed, half-open) (bdd, unbdd)

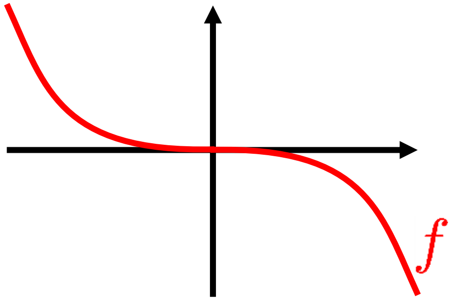
Spp NO perfect converse for the DECREASING TEST. . .

WARNING: f is decreasing and differentiable on an open interval I

$\not\Rightarrow \quad f'(x) < 0$, for all $x \in I$.

Next: problems. . .

$f(x) = -x^3$ is decreasing on $I = (-\infty, \infty)$, but $f'(0) = 0$.



A decreasing function can "level off for an instant".

f is decreasing and differentiable on an open interval I

$\Rightarrow \quad f'(x) \leq 0$, for all $x \in I$.

DECREASING TEST:

If $f'(x) < 0$, for all x in an interval I,
then f is decreasing on I.

works for any kind of interval
(open, closed, half-open)
(bdd, unbdd)

Spp NO perfect converse for the DECREASING TEST. . .

EXAMPLE: Suppose that $f(0) = -3$

and that $f'(t) \le 5$, $\forall t \in [0,2]$.

How large can $f(2)$ possibly be?

Answer: 7 ■

SKILL

maximize value, given deriv bd

By MVT, choose $c \in (0,2)$ s.t.

$$\left(\frac{(f(2)) - (-3)}{2} = \frac{(f(2)) - (f(0))}{2-0} = f'(c) \le 5\right) \times 2$$

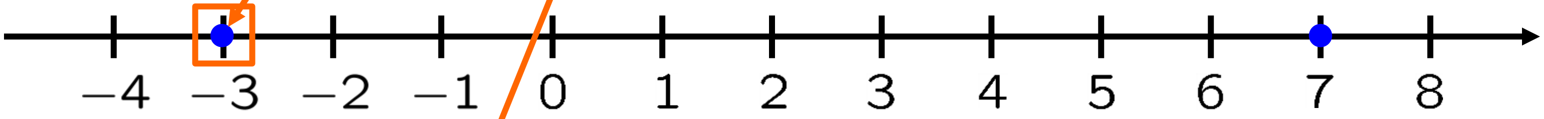
$$\left((f(2)) - (-3) \le 10\right) + (-3)$$

$$f(2) \le 10 + (-3) = 7$$

Note: If $f(t) = -3 + 5t$,

then $f'(t) = 5$,

and $f(2) = 7$.



§6.5 Speed limit: 5

Where can we get to at $t = 2$?

EXAMPLE: Assume that $\frac{d}{dx}[(\arctan x) + (\operatorname{arccot} x)] = 0$.

Prove the identity $(\arctan x) + (\operatorname{arccot} x) = \frac{\pi}{2}$. ☺

$$(\arctan + \operatorname{arccot})'(x) = 0$$

$\arctan + \operatorname{arccot}$ is constant on $\mathbb{R}$.

$$(\arctan + \operatorname{arccot})(1) = \left(\tfrac{\pi}{4}\right) + \left(\tfrac{\pi}{4}\right) = \tfrac{\pi}{2}$$

$\arctan + \operatorname{arccot} = \frac{\pi}{2}$ on $\mathbb{R}$

$$(\arctan x) + (\operatorname{arccot} x) = \tfrac{\pi}{2} \quad ■$$

$$\sin\left(\tfrac{\pi}{4}\right) = \tfrac{\sqrt{2}}{2} = \cos\left(\tfrac{\pi}{4}\right)$$

$$\tan\left(\tfrac{\pi}{4}\right) = 1 = \cot\left(\tfrac{\pi}{4}\right)$$

$$\arctan(1) = \tfrac{\pi}{4} = \operatorname{arccot}(1)$$

SKILL
calculus proves
algebra/trig identity

THEOREM (CONSTANT TEST):

If $f'(x) = 0$, for all x in an interval I,
then f is constant on I.

works for any
kind of interval
(open, closed,
half-open)
(bdd, unbdd)

EXAMPLE: Verify that the fn $f(x) = x^4 - 23x^2 + 42x + 5$ satisfies the three hypotheses of Rolle's Theorem on $[0,3]$.

Find all c that satisfy the conclusion of Rolle's Theorem.

$f(x) = x^4 - 23x^2 + 42x + 5$ is contin. on $[0,3]$,
because polynomials are continuous.

$$f(0) = 5 \qquad f(3) = 81 - (23 \cdot 9) + 126 + 5$$
$$= 212 - 207 = 5 = f(0)$$

FACTOR

$f'(x) = \boxed{4x^3 - 46x + 42}$ is defined on $(0,3)$,
because polynomials defined everywhere.

$$f'(1) = 4 - 46 + 42 = 0$$

$$f'(c) = 0$$
$$c \in [0,3]$$

$$f'(x) = 4x^3 - 46x + 42$$ ← divisible by $x - 1$

$$= (x-1)(4x^2 + 4x - 42)$$ ← factor out 4

$$= 4(x-1)\left(x^2 + x - \frac{21}{2}\right)$$ ← use the quadratic formula

$$f'(x) = 0 \quad \text{iff} \quad x \in \left\{ 1, \underbrace{\frac{-1 \pm \sqrt{1 + 4 \cdot \frac{21}{2}}}{2}} \right\}$$

$$\frac{-1 \pm \sqrt{43}}{2} \doteq -0.5 \pm 3.2787$$

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$$f'(c) = 0$$
$$c \in [0, 3]$$

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$f'(x) = 4x^3 - 46x + 42$ is defined on $(0, 3)$,
because polynomials defined everywhere.

$$f'(c) = 0$$
$$c \in [0, 3]$$

$$f'(x) = 0 \quad \text{iff} \quad x \in \left\{ 1, \underbrace{\frac{-1 \pm \sqrt{1 + 4 \cdot \frac{21}{2}}}{2}}_{\frac{-1 \pm \sqrt{43}}{2} \doteq -0.5 \pm 3.2787} \right\}$$

$$\frac{-1 - \sqrt{43}}{2} < 0 \qquad 1, \frac{-1 + \sqrt{43}}{2} \in [0, 3]$$

$$c = 1 \quad \text{or} \quad c = \frac{-1 + \sqrt{43}}{2} \blacksquare$$

EXAMPLE: Let $f(x) = 3 + \tan x$. Show that $f(0) = f(\pi)$, ☺

but that there is no number $c \in (0, \pi)$ such that $f'(c) = 0$. ☺

Why does this not contradict Rolle's Theorem?

---

tan is π-periodic, so $\tan 0 = \tan \pi$.

$$f(0) = 3 + \tan 0 = 3 + \tan \pi = f(\pi)$$

$f' = 0 + \tan' = \sec^2 = \dfrac{1}{\cos^2}$ is never equal to 0.

$f = 3 + \tan = 3 + \dfrac{\sin}{\cos}$ is not continuous at $\pi/2$,

so f is not continuous on $[0, \pi]$,

and f is not differentiable on $(0, \pi)$. ■

SKILL

Tame hypotheses are important.

EXAMPLE: Let $f(x) = (x-5)^{-8}$. Show that there is no $c \in (4,6)$ s.t. $f'(c) = \dfrac{(f(6)) - (f(4))}{6-4}$. ☺

Why does this not contradict the Mean Value Theorem?

$$f(6) = (6-5)^{-8} = 1^{-8} = 1$$

$$f(4) = (4-5)^{-8} = (-1)^{-8} = 1$$

$$\frac{(f(6)) - (f(4))}{6-4} = 0$$

$$f'(x) = (-8)(x-5)^{-9}(1) = \frac{-8}{(x-5)^9}$$ is never 0.

$$f(x) = (x-5)^{-8} = \frac{1}{(x-5)^8}$$ is not defined at $x = 5$,

so f is not continuous on $[4,6]$,

and f is not differentiable on $(4,6)$. ■

Tame hypotheses are important.

EXAMPLE: Show, $\forall k \in \mathbb{R}$, that the equation $x^4 - 5x + k = 0$ has at most one root in the interval $[-1, 1]$.

Let $g(x) = x^4 - 5x + k$.

Want: g has at most one root on $[-1, 1]$. ☺ ∎

We'll show: g is decreasing on $[-1, 1]$. ☺

$$\begin{aligned}
-1 \le x \le 1 \quad &\Rightarrow \quad \left[-1 \le x^3 \le 1 \right] \quad \times \quad 4 \\
&\Rightarrow \quad \left[-4 \le 4x^3 \le 4 \right] \quad - \quad 5 \\
&\Rightarrow \quad -9 \le 4x^3 - 5 \le -1 \\
&\Rightarrow \quad 4x^3 - 5 < 0
\end{aligned}$$

$g'(x) = 4x^3 - 5$ is < 0 on $-1 \le x \le 1$.

By the DECREASING TEST,
g is decreasing on $[-1, 1]$.

EXAMPLE: Using calculus, prove, $\forall x \geq 0$, that

First, check when $x :\to 0$. . . ☺

$$\operatorname{arccot} x = \frac{1}{2} \arccos\left(\frac{x^2-1}{x^2+1}\right).$$

$$\left[\frac{x^2-1}{x^2+1}\right]_{x:\to 0} = -1$$

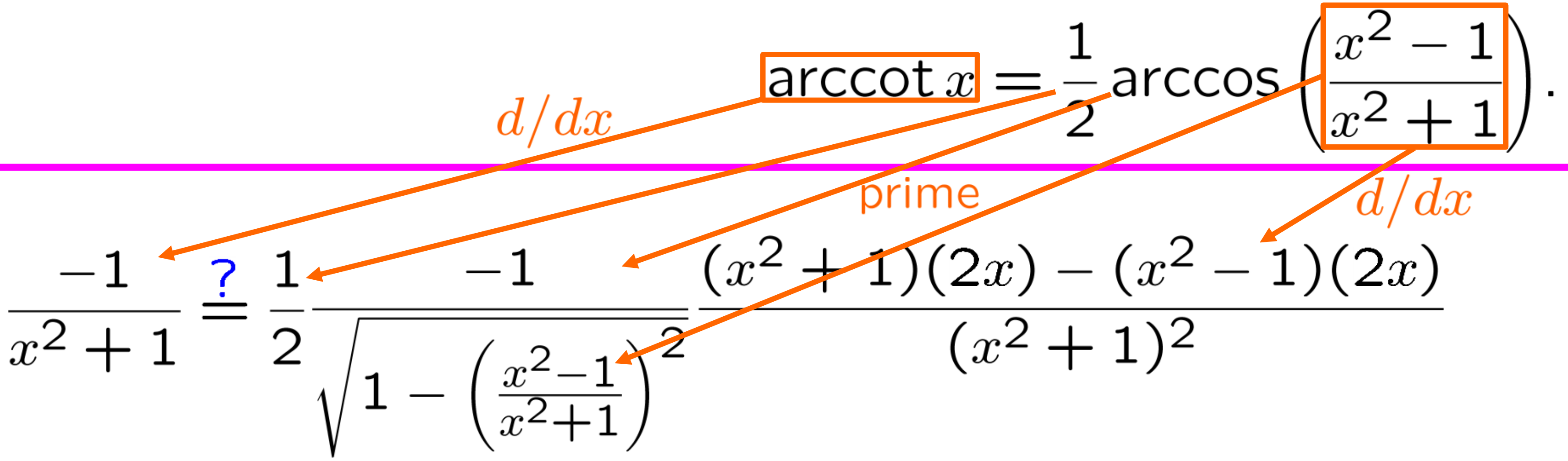
$$\underbrace{\operatorname{arccot} 0}_{\pi/2} \overset{?}{=} \frac{1}{2} \underbrace{\arccos(-1)}_{\pi}$$ ☺

$0 = \operatorname{Cot}(\pi/2)$, so $\operatorname{arccot} 0 = \pi/2$.

$-1 = \operatorname{Cos}(\pi)$, so $\arccos(-1) = \pi$.

Want: $\forall x \geq 0$, $\displaystyle\frac{d}{dx}[\operatorname{arccot} x] = \frac{d}{dx}\left[\frac{1}{2}\arccos\left(\frac{x^2-1}{x^2+1}\right)\right]$

EXAMPLE: Using calculus, prove, $\forall x \geq 0$, that



$$\boxed{\operatorname{arccot} x} = \frac{1}{2} \arccos\left(\boxed{\frac{x^2-1}{x^2+1}}\right).$$

d/dx

prime

d/dx

$$\frac{-1}{x^2+1} \overset{?}{=} \frac{1}{2} \frac{-1}{\sqrt{1-\left(\frac{x^2-1}{x^2+1}\right)^2}} \frac{(x^2+1)(2x)-(x^2-1)(2x)}{(x^2+1)^2}$$

Want: $\forall x \geq 0,\ \frac{d}{dx}[\operatorname{arccot} x] = \frac{d}{dx}\left[\frac{1}{2} \arccos\left(\frac{x^2-1}{x^2+1}\right)\right]$

EXAMPLE: Using calculus, prove, $\forall x \geq 0$, that

$$\operatorname{arccot} x = \frac{1}{2}\arccos\left(\frac{x^2-1}{x^2+1}\right).$$

$$\frac{+1}{x^2+1} \overset{?}{=} \frac{1}{2}\,\frac{+1}{\boxed{\sqrt{1-\left(\frac{x^2-1}{x^2+1}\right)^2}}}\;\frac{\overset{\text{expand}}{\boxed{(x^2+1)(2x)-(x^2-1)(2x)}}}{(x^2+1)^2}$$

$$\frac{1}{\cancel{x^2+1}}\sqrt{1-\left(\frac{x^2-1}{x^2+1}\right)^2} \overset{?}{=} \frac{1}{2}\,\frac{\boxed{(\cancel{2x^3}+2x)+(\cancel{2x^3}+2x)}}{(x^2+1)^{\cancel{2}}}$$

$$\sqrt{1-\boxed{\left(\frac{x^2-1}{x^2+1}\right)^2}} \overset{?}{=} \frac{1}{\cancel{2}}\,\frac{\overset{2}{\cancel{4}}x}{x^2+1}$$

$$\sqrt{1-\boxed{\frac{(x^2-1)^2}{(x^2+1)^2}}} \overset{?}{=} \frac{2x}{x^2+1}$$

$$\left(\frac{a}{b}\right)^2$$

$$\|$$

$$\frac{a^2}{b^2}$$

EXAMPLE: Using calculus, prove, $\forall x \geq 0$, that

$$\operatorname{arccot} x = \frac{1}{2} \arccos\left(\frac{x^2-1}{x^2+1}\right).$$

$$\sqrt{\boxed{1 - \frac{(x^2-1)^2}{(x^2+1)^2}}} \overset{?}{=} \frac{2x}{x^2+1}$$

$$\downarrow$$

$$1 - \frac{(x^2-1)^2}{(x^2+1)^2}$$

$$\sqrt{1 - \frac{(x^2-1)^2}{(x^2+1)^2}} \overset{?}{=} \frac{2x}{x^2+1}$$

EXAMPLE: Using calculus, prove, $\forall x \geq 0$, that

$$\operatorname{arccot} x = \frac{1}{2} \arccos\left(\frac{x^2-1}{x^2+1}\right).$$

$$\sqrt{1 - \frac{(x^2-1)^2}{(x^2+1)^2}} \overset{?}{=} \frac{2x}{x^2+1}$$

$$\boxed{1} - \frac{(x^2-1)^2}{(x^2+1)^2} = \boxed{\frac{(x^2+1)^2}{(x^2+1)^2}} - \frac{(x^2-1)^2}{(x^2+1)^2}$$

$$= \frac{(x^2+1)^2 - (x^2-1)^2}{(x^2+1)^2}$$

$$= \frac{(\cancel{x^4} + 2x^2 + \cancel{1}) + (\cancel{x^4} + 2x^2 + \cancel{1})}{(x^2+1)^2}$$

$$= \frac{4x^2}{(x^2+1)^2}$$

EXAMPLE: Using calculus, prove, $\forall x \geq 0$, that

$$\operatorname{arccot} x = \frac{1}{2} \arccos\left(\frac{x^2 - 1}{x^2 + 1}\right).$$

$$\sqrt{1 - \frac{(x^2-1)^2}{(x^2+1)^2}} \overset{?}{=} \frac{2x}{x^2+1}$$

$$\sqrt{} \longrightarrow 1 - \frac{(x^2-1)^2}{(x^2+1)^2} = \frac{4x^2}{(x^2+1)^2}$$

$$\sqrt{1 - \frac{(x^2-1)^2}{(x^2+1)^2}}$$

$$= \frac{4x^2}{(x^2+1)^2}$$

EXAMPLE: Using calculus, prove, $\forall x \geq 0$, that

$$\operatorname{arccot} x = \frac{1}{2} \arccos\left(\frac{x^2-1}{x^2+1}\right).$$

$$\sqrt{1 - \frac{(x^2-1)^2}{(x^2+1)^2}} \overset{?}{=} \frac{2x}{x^2+1}$$ ☺ ■

$$\sqrt{} \longrightarrow 1 - \frac{(x^2-1)^2}{(x^2+1)^2} = \frac{4x^2}{(x^2+1)^2}$$

$$\sqrt{4x^2} = 2x$$

$$\sqrt{(x^2+1)^2} = x^2+1$$

$$\sqrt{1 - \frac{(x^2-1)^2}{(x^2+1)^2}} = \frac{2x}{x^2+1}$$

SKILL
calculus proves
algebra/trig identity

SKILL

Mean Value Theorem

Whitman problems

§6.5, p. 136–137, #1-13