CALCULUS
The Mean Value Theorem
Let \( y = L(x) \) be the line through \((3, 1)\) and \((9, 13)\).

2 units rise per unit run

\[
\text{slope} = \frac{13 - 1}{9 - 3} = 2
\]

Problem: Find \( L(8) \):

\[
L(8) = \left[ \frac{13 - 1}{9 - 3} \right][8 - 3] + 1 = 11
\]

8 \( \rightarrow \) \( x \)

\[
L(x) = \left[ \frac{13 - 1}{9 - 3} \right][x - 3] + 1
\]

\[
L'(x) = \frac{13 - 1}{9 - 3} = 2
\]

Question: What is the slope of the tangent line to \( y = L(x) \) at \((8, 11)\)?

\[
L'(8) = \frac{13 - 1}{9 - 3} = 2
\]

Note: Any tangent line to a line is just the line itself. The slope of a tangent line to a line is just the slope of the line.
velocity = rate of change in position w.r.t. time
speed = |velocity|

0 hrs, leave Minneapolis
4 hrs 20 mins, start break
5 hrs, notice problem.

head back
5 hrs 40 mins, arrive Minneapolis

6 hrs, leave Minneapolis
8 hrs, arrive Chicago
velocity = rate of change in position w.r.t. time
speed = |velocity|

THERE ARE CHOICES...

Average velocity over the eight hrs:
\[
\frac{[f(8)] - [f(0)]}{8 - 0} = \frac{400 - 0}{8} = 50
\]

Average velocity is 50 mph from 0 hrs to 8 hrs.

Instantaneous velocity is 50 mph at some time (45 min) (4 hrs 19 min 30 sec) (6 hrs 5 sec)
velocity = rate of change in position w.r.t. time
speed = |velocity|

Expect: Every avg. velocity is an instantaneous velocity.
Expect: Every sec. slope is a tangent slope.
Average velocity is 50 mph from 0 hrs to 8 hrs.

instantaneous velocity is 50 mph at some time (45 min) (4 hrs 19 min 30 sec)
(6 hrs 5 sec)
Let \( a, b \in \mathbb{R} \) and assume that \( a < b \).

Fact: Assume that \( g \) is continuous on \([a, b]\), that \( g \) is differentiable on \((a, b)\),
that \( g(a) = g(b) = 0 \)
and that \( \exists s \in (a, b) \) s.t. \( g(s) > 0 \).

Then \( \exists c \in (a, b) \) s.t. \( g'(c) = 0 \).

Proof:

\[ g(c) \geq g(s) > 0 \]
\[ g(a) = g(b) = 0 \]
\[ c \neq a \text{ and } c \neq b, \text{ so } g \text{ is differentiable at } c. \]
Fermat: \( g'(c) = 0 \). QED

Every sec. slope is a tangent slope.
Let \( a, b \in \mathbb{R} \) and assume that \( a < b \).

**Fact:** Assume that \( g \) is continuous on \([a, b]\), that \( g \) is differentiable on \((a, b)\), that \( g(a) = g(b) = 0 \), and that \( \exists s \in (a, b) \) s.t. \( g(s) \geq 0 \).

Then \( \exists c \in (a, b) \) s.t. \( g'(c) = 0 \).

**Proof:**

**ExtrValThm:** Choose \( c \in [a, b] \) s.t. \( g \) attains a global maximum at \( c \).

\[ g(c) \geq g(s) > 0 \quad g(a) = g(b) = 0 \]

\( c \neq a \) and \( c \neq b \), so \( g \) is differentiable at \( c \). \( g'(c) \) exists.

**Fermat:** \( g'(c) = 0 \). QED

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**Differentiability needed:**

\( g \) is differentiable at both endpoints.

\( a \) and \( b \) are differentiable.

---

**Continuity needed:**

\( g \) is continuous on \([a, b]\).

\( g \) is continuous at both endpoints.
Let \( a, b \in \mathbb{R} \) and assume that \( a < b \).

Fact: Assume that \( g \) is continuous on \([a, b]\), that \( g \) is differentiable on \((a, b)\), that \( g(a) = g(b) = 0 \) and that \( \exists s \in (a, b) \) s.t. \( g(s) \preceq 0 \).

Then \( \exists c \in (a, b) \) s.t. \( g'(c) = 0 \).

Proof: ExtrValThm: Choose \( c \in [a, b] \) s.t. \( g \) attains a global minimum at \( c \).

\[
g(c) \leq g(s) < 0 \quad g(a) = g(b) = 0
\]

\( c \neq a \) and \( c \neq b \), so \( g \) is differentiable at \( c \).

Fermat: \( g'(c) = 0 \). QED (\( g'(c) \) exists.)

Differentiability needed:

Continuity needed:
Let \( a, b \in \mathbb{R} \) and assume that \( a < b \).

**Fact:** Assume that \( g \) is continuous on \([a, b]\), that \( g \) is differentiable on \((a, b)\), that \( g(a) = g(b) = 0 \) and that \( \exists s \in (a, b) \text{ s.t. } g(s) < 0 \).

Then \( \exists c \in (a, b) \text{ s.t. } g'(c) = 0 \).

**Proof:**

**ExtrValThm:** Choose \( c \in [a, b] \) s.t. \( g \) attains a global minimum at \( c \).

\[
g(c) \leq g(s) < 0 \quad \text{and} \quad g(a) = g(b) = 0
\]

\( c \neq a \) and \( c \neq b \), so \( g \) is differentiable at \( c \).

Fermat: \( g'(c) = 0 \). QED (\( g'(c) \) exists.)

---

**Differentiability needed:**

---

**Continuity needed:**
Let $a, b \in \mathbb{R}$ and assume that $a < b$.

Fact: Assume that $g$ is continuous on $[a, b]$, that $g$ is differentiable on $(a, b)$, that $g(a) = g(b) = 0$ and that $\exists s \in (a, b)$ s.t. $g(s) < 0$. Then $\exists c \in (a, b)$ s.t. $g'(c) = 0$.

Proof:

ExtrValThm: Choose $c \in [a, b]$ s.t. $g$ attains a global minimum at $c$.

$g(c) \leq g(s) < 0 \quad g(a) = g(b) = 0$

$c \neq a$ and $c \neq b$, so $g$ is differentiable at $c$.

Fermat: $g'(c) = 0$. QED

Fact: Assume that $g$ is continuous on $[a, b]$, that $g$ is differentiable on $(a, b)$, and that $g(a) = g(b) = 0$.

Then $\exists c \in (a, b)$ s.t. $g'(c) = 0$.

Proof: Easy if $g = 0$ on $(a, b)$, so we may assume $\exists s \in (a, b)$ s.t. $g(s) \neq 0$. Done if $g(s) < 0$. 

§6.5
Let \( a, b \in \mathbb{R} \) and assume that \( a < b \).

Fact: Assume that \( g \) is continuous on \([a, b]\),
that \( g \) is differentiable on \((a, b)\),
that \( g(a) = g(b) = 0 \)
and that \( \exists s \in (a, b) \) s.t. \( g(s) > 0 \).
Then \( \exists c \in (a, b) \) s.t. \( g'(c) = 0 \).

Proof:

ExtrValThm: Choose \( c \in [a, b] \)
s.t. \( g \) attains a global maximum at \( c \).

\[
\begin{align*}
g(c) & \geq g(s) > 0 \\
g(a) & = g(b) = 0
\end{align*}
\]

\( c \neq a \) and \( c \neq b \), so \( g \) is differentiable at \( c \).

Fermat: \( g'(c) = 0 \). QED

Fact: Assume that \( g \) is continuous on \([a, b]\),
that \( g \) is differentiable on \((a, b)\),
and that \( g(a) = g(b) = 0 \).
Then \( \exists c \in (a, b) \) s.t. \( g'(c) = 0 \).

Proof: Easy if \( g = 0 \) on \((a, b)\),
so we may assume \( \exists s \in (a, b) \) s.t. \( g(s) \neq 0 \).
Done if \( g(s) < 0 \). Done if \( g(s) > 0 \). QED

\( \S6.5 \)
Let $a, b \in \mathbb{R}$ and assume that $a < b$.

cf. §6.5, p. 134 (TH’M 6.26) MEAN VALUE THEOREM:
Assume that $f$ is continuous on $[a, b]$, and that $f$ is differentiable on $(a, b)$.

"TAME" HYPOTHESES

Fact: Assume that $g$ is continuous on $[a, b]$, that $g$ is differentiable on $(a, b)$, and that $g(a) = g(b) = 0$.

Fact: Then $\exists c \in (a, b)$ s.t. $g'(c) = 0$. on $[a, b]$, that $g$ is differentiable on $(a, b)$, and that $g(a) = g(b) = 0$.

Then $\exists c \in (a, b)$ s.t. $g'(c) = 0$.
Let \( a, b \in \mathbb{R} \) and assume that \( a < b \).

**c. \( \S 6.5, \) p. 134 (TH’M 6.26) MEAN VALUE THEOREM:**

Assume that \( f \) is continuous on \([a, b]\),
and that \( f \) is differentiable on \((a, b)\).

Then \( \exists c \in (a, b) \) such that

\[
  f'(c) = \frac{(f(b)) - (f(a))}{b - a}.
\]

---

**Fact:** Assume that \( g \) is continuous on \([a, b]\),
that \( g \) is differentiable on \((a, b)\),
and that \( g(a) = g(b) = 0 \).

Then \( \exists c \in (a, b) \) s.t. \( g'(c) = 0 \).
Let $a, b \in \mathbb{R}$ and assume that $a < b$.

cf. §6.5, p. 134 (TH’M 6.26) **MEAN VALUE THEOREM:**
Assume that $f$ is continuous on $[a, b]$,
and that $f$ is differentiable on $(a, b)$.

Then $\exists c \in (a, b)$ such that $f'(c) = \frac{(f(b)) - (f(a))}{b - a}$.

pf in a special case...

**SOMETIMES THERE ARE CHOICES...**

Every sec. slope is a tangent slope.

**Fact:** Assume that $g$ is continuous on $[a, b]$,
that $g$ is differentiable on $(a, b)$,
and that $g(a) = g(b) = 0$.

Then $\exists c \in (a, b)$ s.t. $g'(c) = 0$. 

§6.5
Let \( a, b \in \mathbb{R} \) and assume that \( a < b \).

cf. §6.5, p. 134 (TH’M 6.26) MEAN VALUE THEOREM:
Assume that \( f \) is continuous on \([a, b]\),
and that \( f \) is differentiable on \((a, b)\).

Then \( \exists c \in (a, b) \) such that
\[
f'(c) = \frac{f(b) - f(a)}{b - a}.
\]

pf in a special case... e.g.: Suppose \( a = 3, \ b = 9, \ f(3) = 1 \) and \( f(9) = 13 \).

Want: \( \exists c \in (a, b) \) such that
\[
f'(c) = \frac{13 - 1}{9 - 3}.
\]

The curve \( y = f(x) \) goes through \((3, 1)\) and \((9, 13)\).
Let \( y = L(x) \) be the line through \((3, 1)\) and \((9, 13)\).
Then \( L(3) = 1 \) and \( L(9) = 13 \).

Define \( g : [a, b] \to \mathbb{R} \) by \( g(x) = [f(x)] - [L(x)] \).
\[
g(3) = [f(3)] - [L(3)] = 1 - 1 = 0
\]
\[
g(9) = [f(9)] - [L(9)] = 13 - 13 = 0
\]

DIFFERENTIATE

Choose \( c \in (a, b) \) s.t. \( g'(c) = 0 \).

Fact: Assume that \( g \) is continuous on \([a, b]\),
that \( g \) is differentiable on \((a, b)\),
and that \( g(a) = g(b) = 0 \).
Then \( \exists c \in (a, b) \) s.t. \( g'(c) = 0 \).
Let \( a, b \in \mathbb{R} \) and assume that \( a < b \).

cf. §6.5, p. 134 (TH’M 6.26) **MEAN VALUE THEOREM:**
Assume that \( f \) is continuous on \([a, b]\),
and that \( f \) is differentiable on \((a, b)\).

Then \( \exists c \in (a, b) \) such that \( f'(c) = \frac{(f(b)) − (f(a))}{b − a} \).

**pf in a special case...**

**e.g.:** Suppose \( a = 3, b = 9, f(3) = 1 \) and \( f(9) = 13 \).

**Want:** \( \exists c \in (a, b) \) such that \( f'(c) = \frac{13 − 1}{9 − 3} \).

The curve \( y = f(x) \) goes through \((3, 1)\) and \((9, 13)\).

Let \( y = L(x) \) be the line through \((3, 1)\) and \((9, 13)\).

Then \( L(3) = 1 \) and \( L(9) = 13 \).

**Define** \( g : [a, b] \to \mathbb{R} \) by \( g(x) = [f(x)] − [L(x)] \).

**DIFFERENTIATE**

\[
\begin{align*}
g(3) &= [f(3)] − [L(3)] = 1 − 1 = 0 \\
g(9) &= [f(9)] − [L(9)] = 13 − 13 = 0
\end{align*}
\]

Choose \( c \in (a, b) \) s.t. \( g'(c) = 0 \).

\[
\begin{align*}
g'(x) &\quad \downarrow \quad g'(c) \\
[f'(c)] − [L'(c)] &\quad \downarrow \\
[f'(x)] − [L'(x)]
\end{align*}
\]

\( f'(c) = L'(c) \)
Let \( a, b \in \mathbb{R} \) and assume that \( a < b \).

cf. §6.5, p. 134 (TH’M 6.26) MEAN VALUE THEOREM:

Assume that \( f \) is continuous on \([a, b]\),
and that \( f \) is differentiable on \((a, b)\).

Then \( \exists c \in (a, b) \) such that

\[
    f'(c) = \frac{f(b) - f(a)}{b - a}.
\]

pf in a special case...

e.g.: Suppose \( a = 3, b = 9, f(3) = 1 \) and \( f(9) = 13 \).

Want: \( \exists c \in (a, b) \) such that

\[
    f'(c) = \frac{13 - 1}{9 - 3}.
\]

The curve \( y = f(x) \) goes through \((3, 1)\) and \((9, 13)\).

Let \( y = L(x) \) be the line through \((3, 1)\) and \((9, 13)\).

Then \( L(3) = 1 \) and \( L(9) = 13 \).

Define \( g : [a, b] \to \mathbb{R} \) by

\[
    g(x) = [f(x)] - [L(x)].
\]

\[
    g(3) = [f(3)] - [L(3)] = 1 - 1 = 0
\]

\[
    g(9) = [f(9)] - [L(9)] = 13 - 13 = 0
\]

Choose \( c \in (a, b) \) s.t. \( g'(c) = 0 \).

The slope of a tangent line to a line is just the slope of the line.

\[
    3 \to a \quad 1 \to f(a) \quad 9 \to b \quad 13 \to f(b)
\]

\[
    f'(c) = L'(c) = \frac{13 - 1}{9 - 3}
\]
Let \( a, b \in \mathbb{R} \) and assume that \( a < b \).

cf. §6.5, p. 134 (TH’M 6.26) MEAN VALUE THEOREM:
Assume that \( f \) is continuous on \([a, b]\),
and that \( f \) is differentiable on \((a, b)\).

Then there exists \( c \in (a, b) \) such that
\[
 f'(c) = \frac{f(b) - f(a)}{b - a}.
\]

**Proof:**

Want: \( c \in (a, b) \) such that
\[
 f'(c) = \frac{f(b) - f(a)}{b - a}.
\]

The curve \( y = f(x) \) goes through \((a, f(a))\) and \((b, f(b))\).

Let \( y = L(x) \) be the line through \((a, f(a))\) and \((b, f(b))\).

Then \( L(a) = f(a) \) and \( L(b) = f(b) \).

Define \( g : [a, b] \to \mathbb{R} \) by
\[
 g(x) = [f(x)] - [L(x)].
\]

- \( g(a) = [f(a)] - [L(a)] = 0 \)
- \( g(b) = [f(b)] - [L(b)] = 0 \)

Choose \( c \in (a, b) \) s.t. \( g'(c) = 0 \).

The slope of a tangent line to a line is just the slope of the line.

\[ L'(x) = \frac{f(b) - f(a)}{b - a} \]

\[ f'(c) = L'(c) = \frac{f(b) - f(a)}{b - a} \]

\( \square \)
Let \( a, b \in \mathbb{R} \) and assume that \( a < b \).

**MEAN VALUE THEOREM:**
Assume that \( f \) is continuous on \([a, b]\), and that \( f \) is differentiable on \((a, b)\).

Then \( \exists c \in (a, b) \) such that \( f'(c) = \frac{(f(b)) - (f(a))}{b - a} \).

**ONE-TO-ONE TEST:**
If \( f'(x) \neq 0 \), for all \( x \) in an interval \( I \), then \( f \) is one-to-one on \( I \).

**ROLLE'S THEOREM:**
Assume that \( f \) is continuous on \([a, b]\), that \( f \) is differentiable on \((a, b)\) and that \( f(a) = f(b) \).

Then \( \exists c \in (a, b) \) such that \( f'(c) = 0 \).

Every sec. slope is a tangent slope.

Idea: If some secant line is horizontal, then some tangent line is horizontal.
Let $a, b \in \mathbb{R}$ and assume that $a < b$.

Cf. §6.5, p. 134 (TH’M 6.26) MEAN VALUE THEOREM:
Assume that $f$ is continuous on $[a, b]$, and that $f$ is differentiable on $(a, b)$.

Then $\exists c \in (a, b)$ such that
\[
f'(c) = \frac{(f(b)) - (f(a))}{b - a}.
\]

**THEOREM (ONE-TO-ONE TEST):**

If $f'(x) \not\equiv 0$, for all $x$ in an interval $I$,

then $f$ is one-to-one on $I$.

Proof: Let $a, b \in I$.

Want: $f(a) \not\equiv f(b)$.

Assume $a < b$.

Assume $f(a) = f(b)$.

Want: Contradiction.

Choose $c \in (a, b)$ such that
\[
f'(c) = \frac{(f(b)) - (f(a))}{b - a}.
\]

Every sec. slope is a tangent slope.

Idea: If no tangent line is horizontal, then no secant line is horizontal.

Contradiction. QED
Let \( a, b \in \mathbb{R} \) and assume that \( a < b \).

Cf. §6.5, p. 134 (TH’M 6.26) **MEAN VALUE THEOREM:**

Assume that \( f \) is continuous on \([a, b]\),
and that \( f \) is differentiable on \((a, b)\).

Then \( \exists c \in (a, b) \) such that

\[
f'(c) = \frac{(f(b)) - (f(a))}{b - a}
\]

**THEOREM (CONSTANT TEST):**

If \( f'(x) = 0 \), for all \( x \) in an interval \( I \),
then \( f \) is constant on \( I \).

**Proof:** Let \( a, b \in I \).

Want: \( f(a) = f(b) \).

Assume \( a < b \).

Assume \( f(a) \neq f(b) \).

Want: Contradiction.

Choose \( c \in (a, b) \) such that

\[
f'(c) = \frac{(f(b)) - (f(a))}{b - a}
\]

\( \equiv 0 \)

Contradiction. **QED**

Every sec. slope is a tangent slope.

Idea: If every tangent line is horizontal,
then every secant line is horizontal.
Let \( a, b \in \mathbb{R} \) and assume that \( a < b \).

**THEOREM (CONSTANT TEST):**

If \( f'(x) \equiv 0 \) for all \( x \) in an interval \( I \), then \( f \) is constant on \( I \).

**Proof:** Let \( f := g - h \).

Then \( \forall x \in I, \quad f'(x) = (g'(x)) - (h'(x)) = 0 \).

So \( f \) is constant on \( I \). Choose \( k \in \mathbb{R} \) s.t. \( f = k \) on \( I \).

That is, \( \forall x \in I, \quad f(x) = k \).

That is, \( \forall x \in I, \quad (g(x)) - (h(x)) = k \). QED
Let \( a, b \in \mathbb{R} \) and assume that \( a < b \).

Cf. §6.5, p. 134 (TH’M 6.26) MEAN VALUE THEOREM:
Assume that \( f \) is continuous on \([a, b]\),
and that \( f \) is differentiable on \((a, b)\).

Then \( \exists c \in (a, b) \) such that
\[
    f'(c) = \frac{(f(b)) - (f(a))}{b - a}.
\]

INCREASING TEST:
If \( f'(x) \geq 0 \), for all \( x \) in an interval \( I \),
then \( f \) is increasing on \( I \).

Proof:
Let \( a, b \in I \).
Want: \( f(a) < f(b) \).
Assume \( a < b \).
Assume \( f(a) \geq f(b) \).
Want: Contradiction.

Choose \( c \in (a, b) \) such that
\[
    f'(c) = \frac{(f(b)) - (f(a))}{b - a}.
\]

Every sec. slope is a tangent slope.

Idea: If every tangent line runs uphill,
then every secant line runs uphill.

QED
Let \( a, b \in \mathbb{R} \) and assume that \( a < b \).

Cf. §6.5, p. 134 (TH’M 6.26) **MEAN VALUE THEOREM:**

Assume that \( f \) is continuous on \([a, b]\),
and that \( f \) is differentiable on \((a, b)\).

Then \( \exists c \in (a, b) \) such that
\[
\frac{(f(b)) - (f(a))}{b - a} = f'(c).
\]

**DECREASING TEST:**
If \( f'(x) < 0 \), for all \( x \) in an interval \( I \),
then \( f \) is decreasing on \( I \).

**Proof:** Let \( a, b \in I \).

Want: \( f(a) > f(b) \).

Assume \( a < b \).

Assume \( f(a) \leq f(b) \).

Want: Contradiction.

Choose \( c \in (a, b) \) such that
\[
\frac{(f(b)) - (f(a))}{b - a} = f'(c).
\]

\[ f'(c) < 0 \]

Contradiction. QED

Every sec. slope is a tangent slope.

Idea: If every tangent line runs downhill,
then every secant line runs downhill.
NONINCREASING TEST:
If $f'(x) \leq 0$, for all $x$ in an interval $I$, then $f$ is nonincreasing on $I$.

- semi-decreasing
- no secant line runs uphill

DECREASING TEST:
If $f'(x) < 0$, for all $x$ in an interval $I$, then $f$ is decreasing on $I$.

Proof: Let $a, b \in I$.
Assume $a < b$.
Assume $f(a) \leq f(b)$.
Want: Contradiction.

Choose $c \in (a, b)$ such that $f'(c) = \frac{(f(b)) - (f(a))}{b - a}$.

Every sec. slope is a tangent slope.

Idea: If every tangent line runs downhill, then every secant line runs downhill.

Pf is similar.
works for any kind of interval (open, closed, half-open) (bdd, unbdd)
NONINCREASING TEST:
If \( f'(x) \leq 0 \), for all \( x \) in an interval \( I \), then \( f \) is **nonincreasing** on \( I \).
- semi-decreasing
- no secant line runs uphill

Converse for the NONINCREASING TEST...

NONDECREASING TEST:
If \( f'(x) \geq 0 \), for all \( x \) in an interval \( I \), then \( f \) is **nondecreasing** on \( I \).
- semi-increasing
- no secant line runs downhill

Converse...

THEOREM:
If \( f \) is nondecreasing and differentiable on an open interval \( I \) then \( f'(x) \geq 0 \), for all \( x \in I \).
- limit of slopes of secant lines
- slopes of secant lines \( \geq 0 \) QED
NONINCREASING TEST:
If \( f'(x) \leq 0 \), for all \( x \) in an interval \( I \),
then \( f \) is nonincreasing on \( I \).

Converse for the NONINCREASING TEST . . .

THEOREM:
If \( f \) is nonincreasing and differentiable
on an open interval \( I \)
then \( f'(x) \leq 0 \), for all \( x \in I \).

limit of slopes of secant lines
slopes of secant lines \( \leq 0 \) QED

THEOREM:
If \( f \) is nondecreasing and differentiable
on an open interval \( I \)
then \( f'(x) \geq 0 \), for all \( x \in I \).

limit of slopes of secant lines
slopes of secant lines \( \geq 0 \) QED

NO perfect converse for the INCREASING TEST . . .
NONINCREASING TEST:
If \( f'(x) \leq 0 \), for all \( x \) in an interval \( I \),
then \( f \) is nonincreasing on \( I \).

THEOREM:
If \( f \) is nonincreasing and differentiable on an open interval \( I \),
then \( f'(x) \leq 0 \), for all \( x \in I \).

INCREASING TEST:
If \( f'(x) > 0 \), for all \( x \) in an interval \( I \),
then \( f \) is increasing on \( I \).

NO perfect converse for the INCREASING TEST...
WARNING: $f$ is increasing and differentiable on an open interval $I$ if and only if $f'(x) > 0$, for all $x \in I$.

Slopes of secant lines $> 0$ $\Rightarrow$ limit of slopes of secant lines $> 0$

$f(x) = x^3$ is increasing on $I = (-\infty, \infty)$, but $f'(0) = 0$.

An increasing function can "level off for an instant".

$f$ is increasing and differentiable on an open interval $I$ if and only if $f'(x) \geq 0$, for all $x \in I$.

INCREASING TEST:
If $f'(x) > 0$, for all $x$ in an interval $I$, then $f$ is increasing on $I$.

NO perfect converse for the DECREASING TEST...
WARNING: $f$ is decreasing and differentiable on an open interval $I$
\[ f'(x) < 0, \text{ for all } x \in I. \]

Next: problems...

$f(x) = -x^3$ is decreasing on $I = (-\infty, \infty)$, but $f'(0) = 0$.

A decreasing function can “level off for an instant”.

$f$ is decreasing and differentiable on an open interval $I$
\[ f'(x) \leq 0, \text{ for all } x \in I. \]

DECREASING TEST:
If $f'(x) < 0$, for all $x$ in an interval $I$,
then $f$ is decreasing on $I$.

NO perfect converse for the DECREASING TEST...
EXAMPLE: Suppose that \( f(0) = -3 \) and that \( f'(t) \leq 5, \forall t \in [0, 2] \). How large can \( f(2) \) possibly be?

By MVT, choose \( c \in (0, 2) \) s.t.

\[
\frac{(f(2)) - (-3)}{2} = \frac{(f(2)) - (f(0))}{2 - 0} = f'(c) \leq 5 \times 2
\]

\[
(f(2)) - (-3) \leq 10
\]

\[
(f(2)) = 10 + (-3) = 7
\]

Note: If \( f(t) = -3 + 5t \), then \( f'(t) = 5 \), and \( f(2) = 7 \).
EXAMPLE: Assume \( \frac{d}{dx}[(\arctan x) + (\arccot x)] = 0. \)

Prove the identity \( (\arctan x) + (\arccot x) = \frac{\pi}{2}. \)

\[(\arctan + \arccot)'(x) = 0\]

\(\arctan + \arccot\) is constant on \( \mathbb{R} \).

\[(\arctan + \arccot)(1) = (\frac{\pi}{4}) + (\frac{\pi}{4}) = \frac{\pi}{2}\]

\(\arctan + \arccot = \frac{\pi}{2}\) on \( \mathbb{R} \)

\[(\arctan x) + (\arccot x) = \frac{\pi}{2}\]

\(\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} = \cos(\frac{\pi}{4})\)

\(\tan(\frac{\pi}{4}) = 1 = \cot(\frac{\pi}{4})\)

\(\arctan(1) = \frac{\pi}{4} = \arccot(1)\)

**SKILL**
calculus proves algebra/trig identity

**THEOREM (CONSTANT TEST):**
If \( f'(x) = 0 \), for all \( x \) in an interval \( I \),
then \( f \) is constant on \( I \).
EXAMPLE: Verify that the fn \( f(x) = x^4 - 23x^2 + 42x + 5 \) satisfies the three hypotheses of Rolle’s Theorem on \([0, 3]\).

Find all \( c \) that satisfy the conclusion of Rolle’s Theorem.

\[
f(x) = x^4 - 23x^2 + 42x + 5 \text{ is contin. on } [0, 3],
\]
because polynomials are continuous.

\[
f(0) = 5 \quad f(3) = 81 - (23 \cdot 9) + 126 + 5
\]
\[= 212 - 207 = 5 = f(0)
\]

\[
f'(x) = 4x^3 - 46x + 42 \text{ is defined on } (0, 3),
\]
because polynomials defined everywhere.

\[
f'(1) = 4 - 46 + 42 = 0
\]

\[
f'(x) = 4x^3 - 46x + 42 \quad \text{divisible by } x - 1
\]
\[= (x - 1)(4x^2 + 4x - 42) \quad \text{factor out } 4
\]
\[= 4(x - 1)(x^2 + x - \frac{21}{2}) \quad \text{use the quadratic formula}
\]
\[
f'(x) = 0 \quad \text{iff } x \in \left\{ 1, \frac{-1 \pm \sqrt{1 + 4 \cdot \frac{21}{2}}}{2} \right\}
\]
\[= \frac{-1 \pm \sqrt{43}}{2} \approx -0.5 \pm 3.2787
\]

\section*{SKILL}

Rolle’s Theorem
EXAMPLE: Verify that the fn \( f(x) = x^4 - 23x^2 + 42x + 5 \) satisfies the three hypotheses of Rolle’s Theorem on \([0, 3]\).

Find all \( c \) that satisfy the conclusion of Rolle’s Theorem.

\[ f(x) = x^4 - 23x^2 + 42x + 5 \text{ is contin. on } [0, 3], \]

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\[ f(0) = 5 \]

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\[ = 212 - 207 = 5 = f(0) \]

\[ f'(x) = 4x^3 - 46x + 42 \text{ is defined on } (0, 3), \]

because polynomials defined everywhere.

\[ f'(x) = 0 \text{ iff } x \in \left\{ 1, \frac{-1 \pm \sqrt{1 + 4 \cdot \frac{21}{2}}}{2} \right\} \]

\[ \frac{-1 \pm \sqrt{43}}{2} \approx -0.5 \pm 3.2787 \]

\[ f'(c) = 0 \text{ c \in [0, 3]} \]

\[ f'(x) = 0 \text{ iff } x \in \left\{ 1, \frac{-1 \pm \sqrt{1 + 4 \cdot \frac{21}{2}}}{2} \right\} \]

\[ \frac{-1 \pm \sqrt{43}}{2} \approx -0.5 \pm 3.2787 \]
EXAMPLE: Verify that the fn \( f(x) = x^4 - 23x^2 + 42x + 5 \) satisfies the three hypotheses of Rolle’s Theorem on \([0, 3]\).

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\[ f(x) = x^4 - 23x^2 + 42x + 5 \text{ is contin. on } [0, 3], \]

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\[ f'(x) = 4x^3 - 46x + 42 \text{ is defined on } (0, 3), \]

*because* polynomials defined everywhere.

\[ f'(x) = 0 \quad \text{iff} \quad x \in \left\{ 1, \frac{-1 \pm \sqrt{1 + 4 \cdot 21}}{2} \right\} \]

\[ \frac{-1 \pm \sqrt{43}}{2} \approx -0.5 \pm 3.2787 \]

\[ \frac{-1 - \sqrt{43}}{2} < 0 \quad 1, \frac{-1 + \sqrt{43}}{2} \in [0, 3] \]

**SKILL**

Rolle’s Theorem

\[ c = 1 \quad \text{or} \quad c = \frac{-1 + \sqrt{43}}{2} \]
EXAMPLE: Let \( f(x) = 3 + \tan x \). Show that \( f(0) = f(\pi) \), but that there is no number \( c \in (0, \pi) \) such that \( f'(c) = 0 \).

Why does this not contradict Rolle’s Theorem?

\[
\tan \text{ is } \pi\text{-periodic, so } \tan 0 = \tan \pi.
\]

\[
f(0) = 3 + \tan 0 = 3 + \tan \pi = f(\pi)
\]

\[
f' = 0 + \tan' = \sec^2 = \frac{1}{\cos^2} \text{ is never equal to 0.}
\]

\[
f = 3 + \tan = 3 + \frac{\sin}{\cos} \text{ is not continuous at } \pi/2,
\]

so \( f \) is not continuous on \([0, \pi]\),

and \( f \) is not differentiable on \((0, \pi)\).

SKILL: Rolle’s Theorem

Tame hypotheses are important.
EXAMPLE: Let \( f(x) = (x - 5)^{-8} \). Show that there is no \( c \in (4, 6) \) s.t. \( f'(c) = \frac{(f(6)) - (f(4))}{6 - 4} \).

Why does this not contradict the Mean Value Theorem?

\[
\begin{align*}
f(6) &= (6 - 5)^{-8} = 1^{-8} = 1 \\
f(4) &= (4 - 5)^{-8} = (-1)^{-8} = 1 \\
(f(6)) - (f(4)) &= 0 \\
\frac{(f(6)) - (f(4))}{6 - 4} &= 0
\end{align*}
\]

\[
f'(x) = (-8)(x - 5)^{-9}(1) = \frac{-8}{(x - 5)^9} \text{ is never 0.}
\]

\[
f(x) = (x - 5)^{-8} = \frac{1}{(x - 5)^8} \text{ is not defined at } x = 5,
\]

so \( f \) is not continuous on \([4, 6]\),

and \( f \) is not differentiable on \((4, 6)\).

SKILL

Mean Value Theorem

\[\text{§6.5}\]

Tame hypotheses are important.
EXAMPLE: Show, \( \forall k \in \mathbb{R} \), that the equation \( x^4 - 5x + k = 0 \) has at most one root in the interval \([-1, 1]\).

Let \( g(x) = x^4 - 5x + k \).

Want: \( g \) has at most one root on \([-1, 1]\).

We’ll show: \( g \) is decreasing on \([-1, 1]\).

\[
-1 \leq x \leq 1 \quad \Rightarrow \quad \left\{ -1 \leq x^3 \leq 1 \right\} \times 4 \\
\Rightarrow \quad \left\{ -4 \leq 4x^3 \leq 4 \right\} - 5 \\
\Rightarrow \quad -9 \leq 4x^3 - 5 \leq -1 \\
\Rightarrow \quad 4x^3 - 5 < 0
\]

\( g'(x) = 4x^3 - 5 \) is < 0 on \([-1, 1]\).

By the **DECREASING TEST**, \( g \) is decreasing on \([-1, 1]\).
EXAMPLE: Using calculus, prove, $\forall x \geq 0$, that

First, check when $x \rightarrow 0$:

\[
\frac{x^2 - 1}{x^2 + 1} \bigg|_{x \rightarrow 0} = -1
\]

\[
\arccot 0 = \frac{1}{2} \arccos (-1) = \frac{\pi}{2}
\]

$0 = \cot (\pi/2)$, so $\arccot 0 = \pi/2$.

$-1 = \cos (\pi)$, so $\arccos (-1) = \pi$.

Want: $\forall x \geq 0$, 
\[
\frac{d}{dx} [\arccot x] = \frac{d}{dx} \left[ \frac{1}{2} \arccos \left( \frac{x^2 - 1}{x^2 + 1} \right) \right]
\]
EXAMPLE: Using calculus, prove, \( \forall x \geq 0 \), that

\[
\frac{d}{dx} \arccot x = \frac{1}{2} \frac{d}{dx} \arccos \left( \frac{x^2 - 1}{x^2 + 1} \right).
\]

\[
\frac{-1}{x^2 + 1} \overset{?}{=} \frac{1}{2} \frac{\frac{-1}{2} \left( x^2 + 1 \right) \left( 2x \right) - \left( x^2 - 1 \right) \left( 2x \right)}{\left( x^2 + 1 \right)^2}
\]

Want: \( \forall x \geq 0 \), \( \frac{d}{dx} [\arccot x] = \frac{d}{dx} \left[ \frac{1}{2} \arccos \left( \frac{x^2 - 1}{x^2 + 1} \right) \right] \)
EXAMPLE: Using calculus, prove, $\forall x \geq 0$, that

$$\text{arccot } x = \frac{1}{2} \text{ arccos} \left( \frac{x^2 - 1}{x^2 + 1} \right).$$

$$\frac{+1}{x^2 + 1} \quad \frac{1}{2} \quad \frac{+1}{\sqrt{1 - \left( \frac{x^2 - 1}{x^2 + 1} \right)^2}}$$

Expand

$$(x^2 + 1)(2x) - (x^2 - 1)(2x)$$

$$(x^2 + 1)^2$$

$$\frac{1}{x^2 + 1} \quad \sqrt{1 - \left( \frac{x^2 - 1}{x^2 + 1} \right)^2}$$

$$\frac{1}{2} \quad \frac{(2x^3 + 2x) + (2x^3 + 2x)}{(x^2 + 1)^2}$$

$$\frac{2}{(x^2 + 1)^2}$$

$$(\frac{a}{b})^2$$

$$\frac{a^2}{b^2}$$

$\S 6.5$
EXAMPLE: Using calculus, prove, \( \forall x \geq 0 \), that

\[
\arccot x = \frac{1}{2} \arccos \left( \frac{x^2 - 1}{x^2 + 1} \right).
\]

\[
\sqrt{1 - \left( \frac{x^2 - 1}{x^2 + 1} \right)^2} \quad \frac{?}{2x} \quad \frac{2x}{x^2 + 1}
\]

\[
1 - \frac{(x^2 - 1)^2}{(x^2 + 1)^2}
\]
EXAMPLE: Using calculus, prove, \( \forall x \geq 0 \), that

\[
\arccot x = \frac{1}{2} \arccos \left( \frac{x^2 - 1}{x^2 + 1} \right).
\]

\[
\sqrt{1 - \frac{(x^2 - 1)^2}{(x^2 + 1)^2}} \overset{?}{=} \frac{2x}{x^2 + 1}
\]

\[
1 - \frac{(x^2 - 1)^2}{(x^2 + 1)^2} = \frac{(x^2 + 1)^2}{(x^2 + 1)^2} - \frac{(x^2 - 1)^2}{(x^2 + 1)^2}
\]

\[
= \frac{(x^2 + 1)^2 - (x^2 - 1)^2}{(x^2 + 1)^2}
\]

\[
= \frac{x^4 + 2x^2 + 1 + x^4 + 2x^2 + 1}{(x^2 + 1)^2}
\]

\[
= \frac{4x^2}{(x^2 + 1)^2}
\]
EXAMPLE: Using calculus, prove, $\forall x \geq 0$, that

$$\arccot x = \frac{1}{2} \arccos \left( \frac{x^2 - 1}{x^2 + 1} \right).$$

\[
\sqrt{1 - \frac{(x^2 - 1)^2}{(x^2 + 1)^2}} \overset{?}{=} \frac{2x}{x^2 + 1}
\]

\[
\sqrt{1 - \frac{(x^2 - 1)^2}{(x^2 + 1)^2}} = \frac{4x^2}{(x^2 + 1)^2}
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EXAMPLE: Using calculus, prove, \( \forall x \geq 0 \), that

\[
\arccot x = \frac{1}{2} \arccos \left( \frac{x^2 - 1}{x^2 + 1} \right).
\]

\[
\sqrt{1 - \frac{(x^2 - 1)^2}{(x^2 + 1)^2}} = \frac{2x}{x^2 + 1}
\]

\[
\sqrt{4x^2} = 2x
\]

\[
\sqrt{(x^2 + 1)^2} = x^2 + 1
\]

SKILL

calculus proves
algebra/trig identity

\( \S 6.5 \)
SKILL
Mean Value Theorem
Whitman problems
§6.5, p. 136–137, #1-13