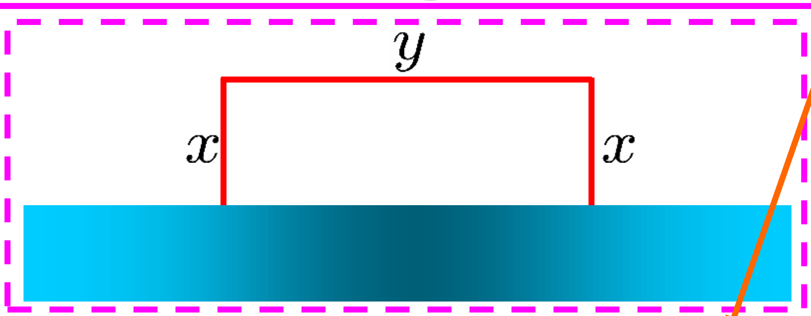
CALCULUS
Optimization

STEPS IN SOLVING OPTIMIZATION PROBLEMS:

1. Understand the problem.
   (Identify the important quantities.)
2. Draw a diagram.
3. Introduce notation.
4. Express the quantity to be extremized as a function (of possibly more than one variable).
   Express the constraints.
5. Express the quantity to be extremized as a function of one variable.
6. Use methods of Chapter 5 to maximize or minimize the function.

EXAMPLE: A farmer has 1,600 ft of fencing and wants to fence a rectangular field next to a straight river. He requires no fence along the river. What are the dimensions of the field that with largest area?



Maximize $A := xy = x(1600 - 2x)$

Constraint: $2x + y = 1600$

$y = 1600 - 2x$

$$A = 1600x - 2x^2,\ x > 0$$

$200 \mapsto 800$

no sign change on $0 < x < 400$

no sign change on $x > 400$

$500 \mapsto -400$

$$dA/dx = 1600 - 4x,\ x > 0$$

pos on $0 < x < 400$
zero only at $x = 400$
neg on $x > 400$

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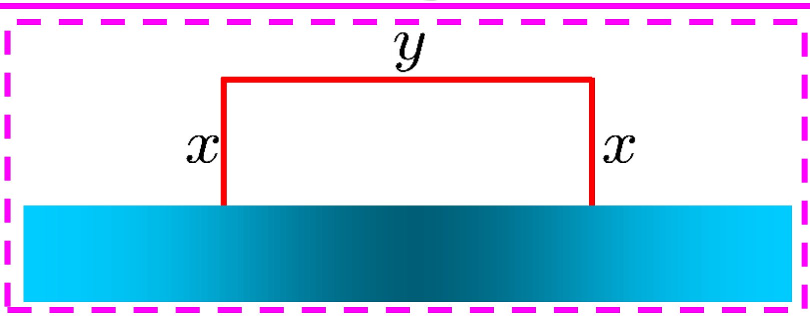

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incr on $0 < x \leq 400$

global max at $x = 400$

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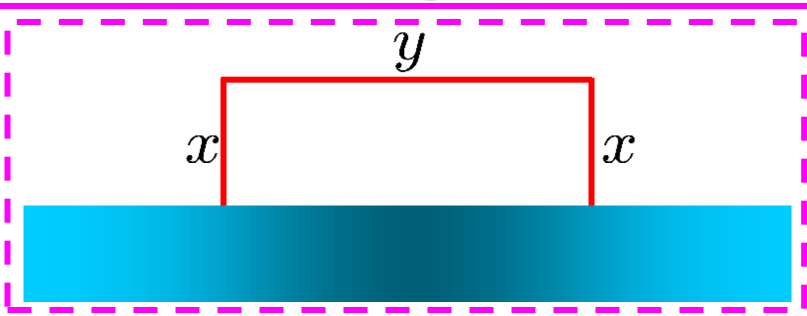

Maximize $A := xy = x(1600 - 2x)$

global max at $x = 400$

$[y]_{x:\to 400} = 800$ ■

$[A]_{x:\to 400} = 320000$

SKILL max-min

Constraint: $2x + y = 1600$

$y = 1600 - 2x$

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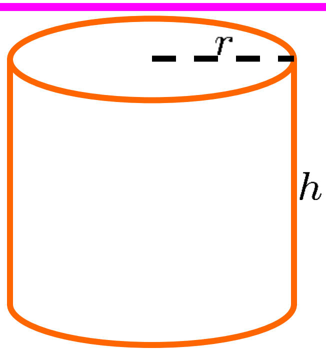

Minimize $A = \pi r^2 + \pi r^2 + 2\pi r h$

TOP BOTTOM SIDE

Constraint: $\pi r^2 h = 5000$, *i.e.*, $h = 5000/(\pi r^2)$

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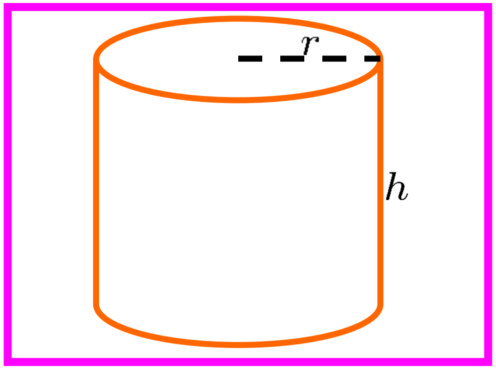

Minimize $A = \pi r^2 + \pi r^2 + 2\pi r h$

$= 2\pi r^2 + 2\pi r(5000/(\pi r^2))$

$= 2\pi r^2 + (10000/r)$ on $r > 0$

Constraint: $\pi r^2 h = 5000$, *i.e.*, $h = 5000/(\pi r^2)$

$$A = 2\pi r^2 + \frac{10000}{r}, \quad r > 0$$

decr on $0 < r \leq \sqrt[3]{2500/\pi}$

incr on $r \geq \sqrt[3]{2500/\pi}$

$r \to 0^+$: $-\infty$

$$\frac{dA}{dr} = 4\pi r - \frac{10000}{r^2}, \quad r > 0$$

neg on $0 < r < \sqrt[3]{2500/\pi}$

pos on $r > \sqrt[3]{2500/\pi}$

$r \to \infty$: ∞

no sign change on $0 < r < \sqrt[3]{2500/\pi}$

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$$\frac{dA}{dr} = 0 \text{ only at } r = \sqrt[3]{2500/\pi}$$

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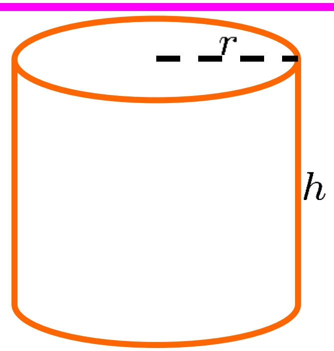

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decr on $0 < r \le \sqrt[3]{2500/\pi}$

incr on $r \ge \sqrt[3]{2500/\pi}$

A takes its global min at $r = \boxed{\sqrt[3]{2500/\pi}}$

$$[h]_{r:\to \sqrt[3]{2500/\pi}} = \frac{5000}{\pi\left(\sqrt[3]{2500/\pi}\right)^2}$$

For you: $[A]_{r:\to \boxed{\sqrt[3]{2500/\pi}}} = ??$ SKILL max-min

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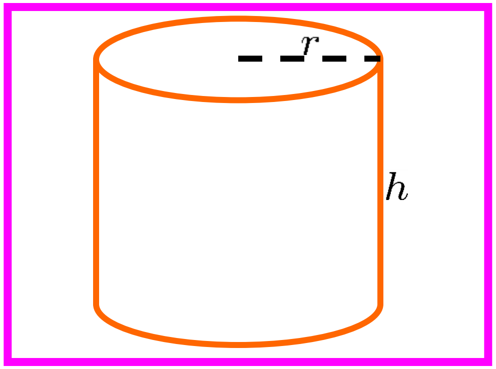

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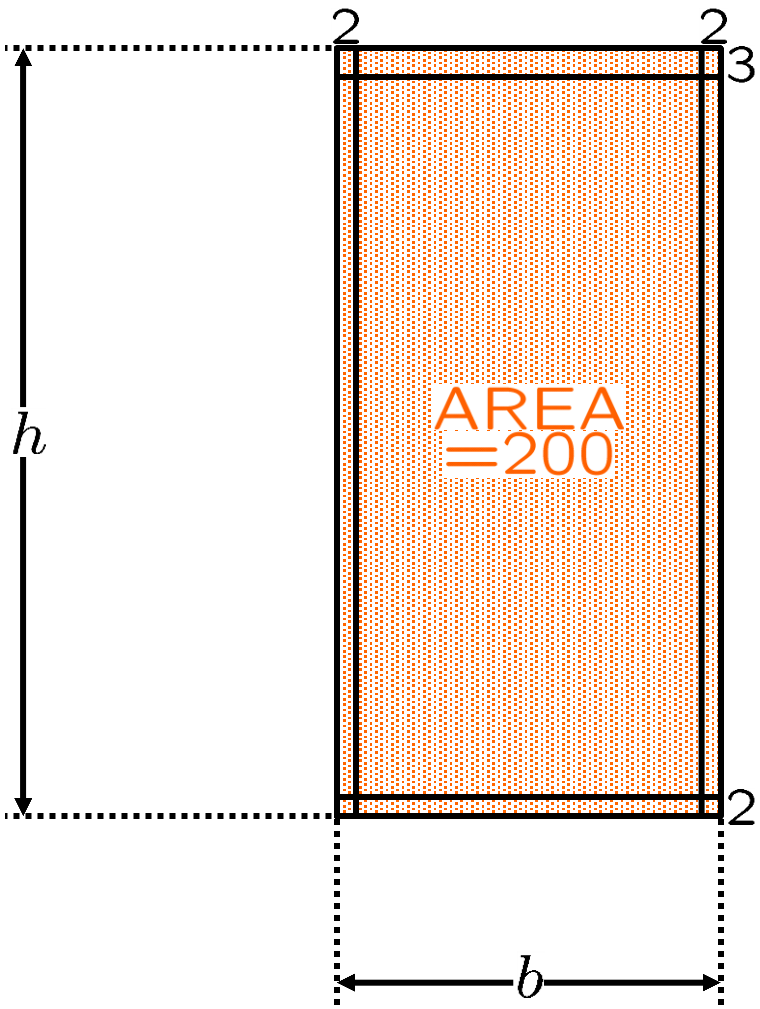
EXERCISE: A poster is to have an area of 200 in^2 with 2-inch margins at the bottom and sides and a 3-inch margin at the top. What dimensions will give the largest printed area?



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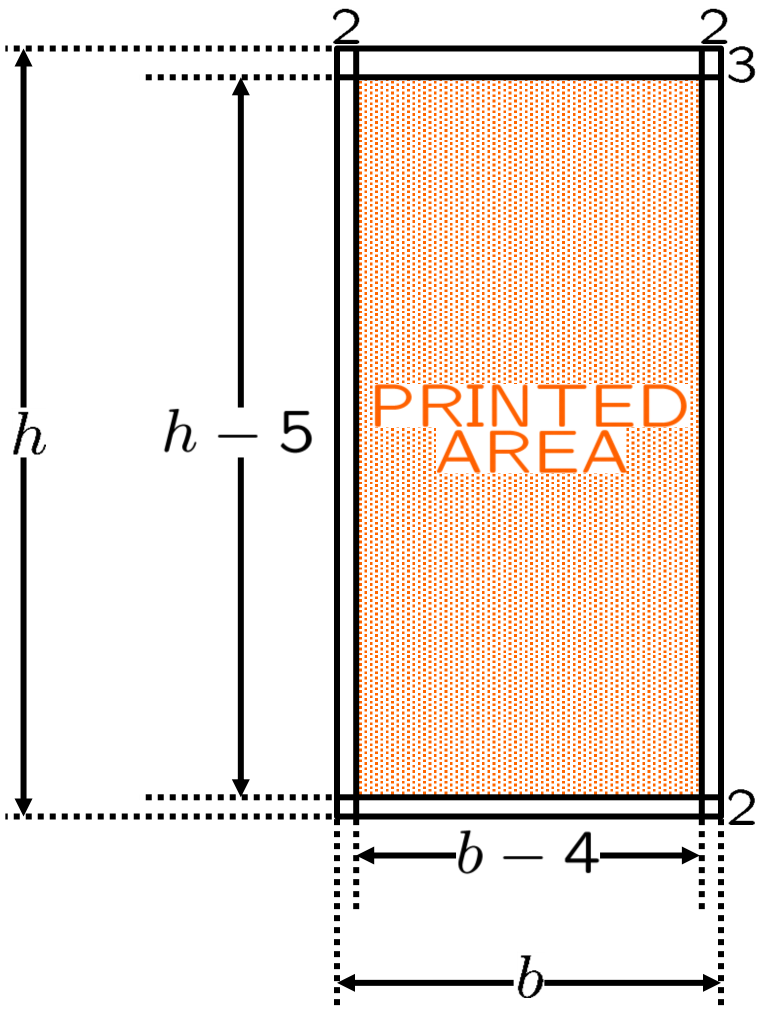

Maximize: $P := (h-5)(b-4)$

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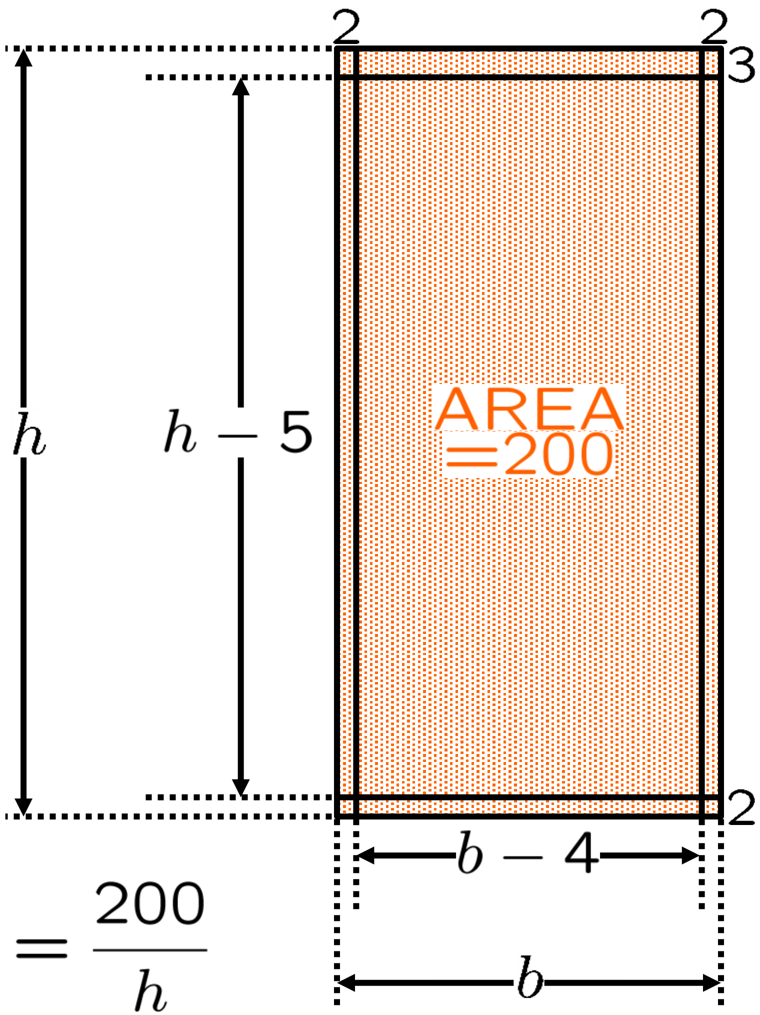

Maximize: $P := (h-5)(b-4)$

$$b = \frac{200}{h}$$

Constraint: $hb = 200$

EXERCISE: A poster is to have an area of 200 in^2 with 2-inch margins at the bottom and sides and a 3-inch margin at the top. What dimensions will give the largest printed area?

Maximize: $P := (h-5)(b-4)$

Max $= (h-5)\left(\frac{200}{h}-4\right)$

$$b = \frac{200}{h}$$

Constraint: $hb = 200$

EXERCISE: A poster is to have an area of 200 in^2 with 2-inch margins at the bottom and sides and a 3-inch margin at the top. What dimensions will give the largest printed area?

Maximize: $P := (h-5)(b-4) = (h-5)\left(\frac{200}{h} - 4\right)$ on $h > 0$

$$0 = \frac{dP}{dh} = \left(\frac{200}{h} - 4\right) + (h-5)\left(-\frac{200}{h^2}\right) \text{ on } h > 0$$

$h^2 \quad \times \quad \left[\left(\frac{200}{h} - 4\right) = (h-5)\left(\frac{200}{h^2}\right)\right]$

$(1/4) \quad \times \quad \left[200h + 4h^2 = 200h + 1000\right]$

$$h^2 = 250$$

$b = \frac{200}{h} \qquad \frac{dP}{dh} = 0$ only at $h = \sqrt{250}$

Constraint: $hb = 200$

§6.1

EXERCISE: A poster is to have an area of 200 in^2 with 2-inch margins at the bottom and sides and a 3-inch margin at the top. What dimensions will give the largest printed area?

Maximize: $P := (h-5)(b-4) = (h-5)\left(\frac{200}{h} - 4\right)$ on $h > 0$

$$\frac{dP}{dh} = \boxed{\left(\frac{200}{h} - 4\right) + (h-5)\left(-\frac{200}{h^2}\right)} \text{ on } h > 0$$

$\to \infty$, as $h \to 0^+$

$\to -4$, as $h \to \infty$

$$\frac{dP}{dh} = 0 \text{ only at } h = \sqrt{250}$$

$$b = \frac{200}{h} \qquad \frac{dP}{dh} = 0 \text{ only at } h = \sqrt{250}$$

Constraint: $hb = 200$

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Max value: $\left(\sqrt{250}-5\right)\left(\frac{200}{\sqrt{250}}-4\right)$

Maximize: $P := (h-5)(b-4) = (h-5)\left(\frac{200}{h}-4\right)$ on $h>0$

$$\frac{dP}{dh} = \left(\frac{200}{h}-4\right) + (h-5)\left(-\frac{200}{h^2}\right) \text{ on } h>0$$

$\rightarrow \infty$, as $h \rightarrow 0^+$

$\rightarrow -4$, as $h \rightarrow \infty$

$\frac{dP}{dh} = 0$ only at $h = \sqrt{250}$

pos on $0 < h < \sqrt{250}$

neg on $\sqrt{250} < h$

P incr on $0 < h \leq \sqrt{250}$

P decr on $\sqrt{250} \leq h$

P takes its global max at $h = \sqrt{250}$

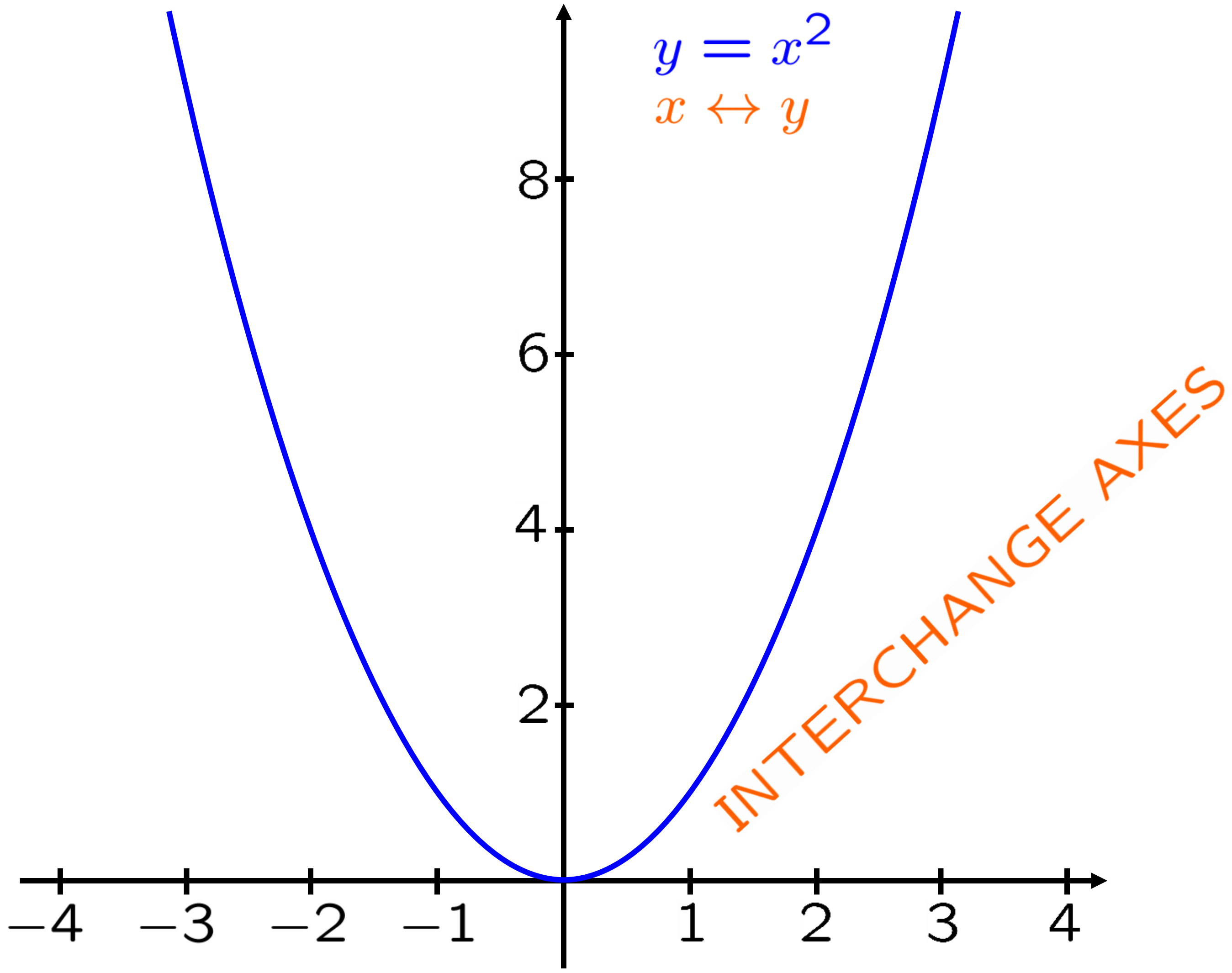
SKILL max-min

$b = \frac{200}{h}$

Constraint: $hb = 200$

$[b]_{h:\rightarrow\sqrt{250}} = \frac{200}{\sqrt{250}}$ ■

EXAMPLE: Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.



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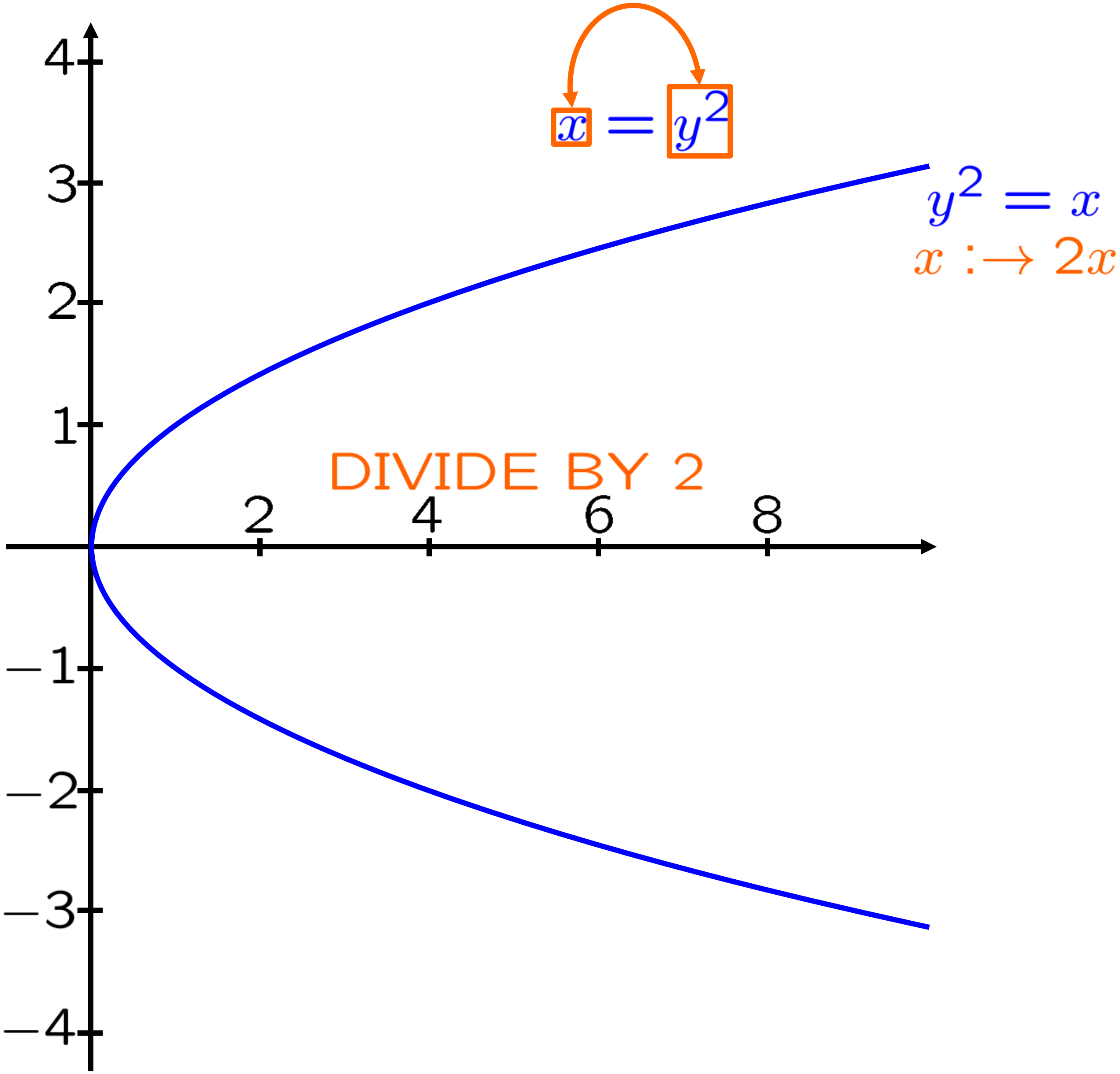

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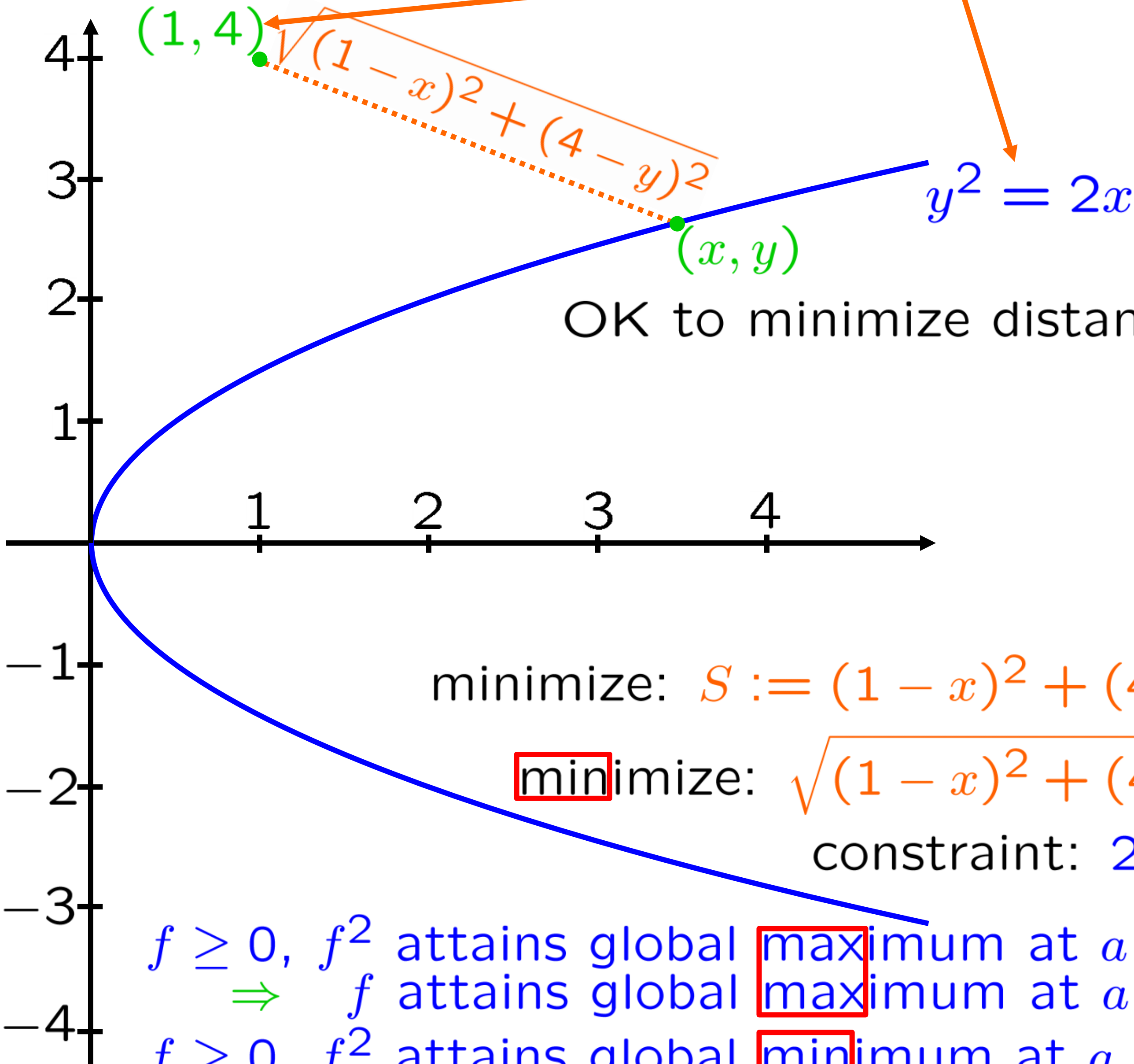

OK to minimize distance^2

minimize: $S := (1-x)^2 + (4-y)^2$

minimize: $\sqrt{(1-x)^2+(4-y)^2}$

constraint: $2x = y^2$

$f \geq 0$, f^2 attains global maximum at a
$\Rightarrow$ f attains global maximum at a

$f \geq 0$, f^2 attains global minimum at a
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EXAMPLE: Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

$$x = y^2/2$$

minimize: $S := (1 - x)^2 + (4 - y)^2 = (1 - (y^2/2))^2 + (4 - y)^2$

constraint: $2x = y^2$

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minimize: $S := (1-x)^2 + (4-y)^2 = (1-(y^2/2))^2 + (4-y)^2$

constraint: $2x = y^2$

S has global min at $y = 2$

S decr on $y \leq 2$

S incr on $y \geq 2$

$$\frac{dS}{dy} = \frac{d}{dy}[(1-(y^2/2))^2 + (4-y)^2]$$

$$= 2(1-(y^2/2))(-2y/2) + 2(4-y)(-1)$$

$$= -(2-y^2)y - 2(4-y)$$

$$= -(2y - y^3) - (8 - 2y)$$

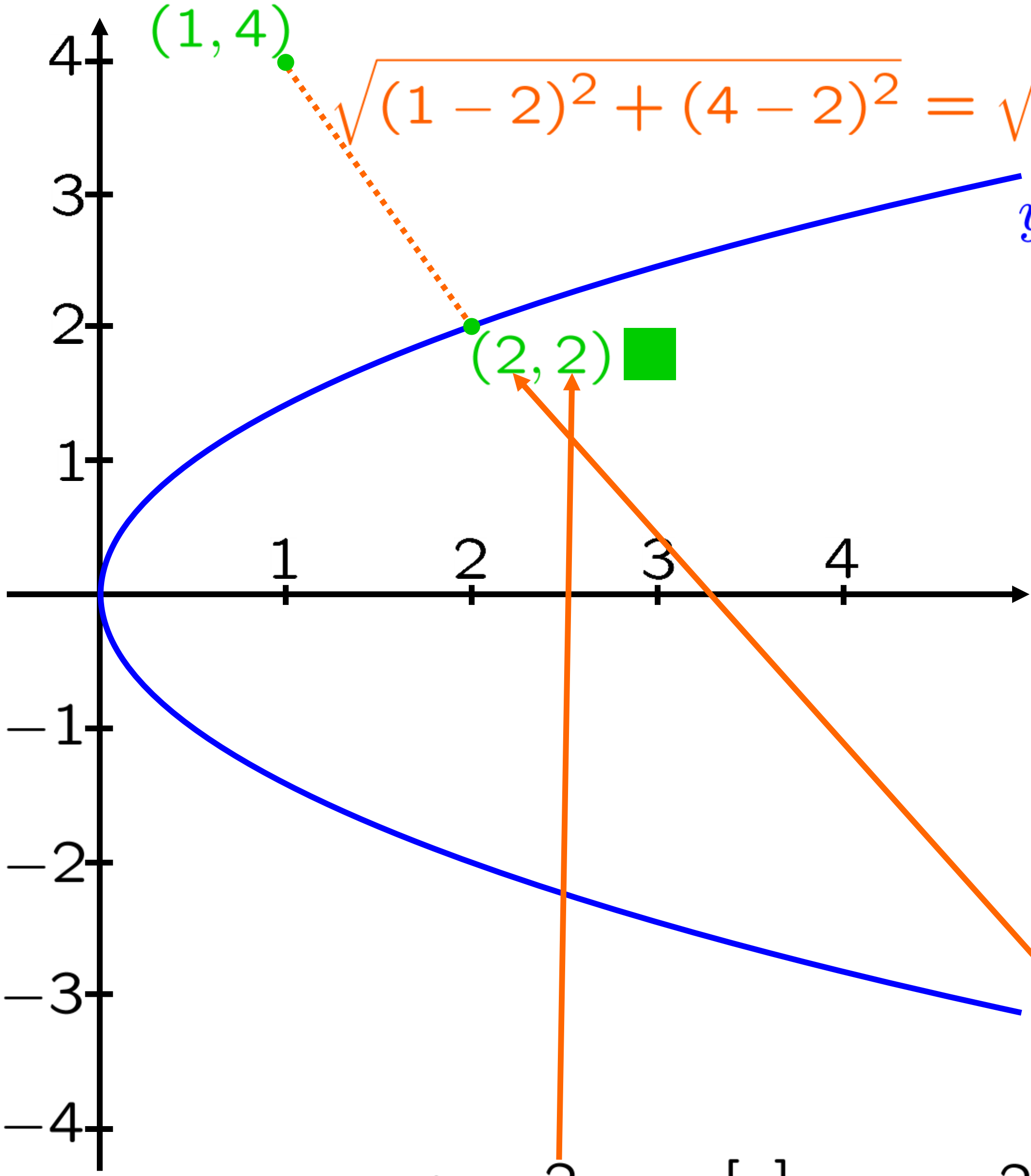
$$= y^3 - 8$$

dS/dy neg on $y < 2$

dS/dy pos on $y > 2$

CRITICAL FOR S: $y^3 = 8$ $\quad y = 2 \quad [x]_{y:\to 2} = 2^2/2 = 2$

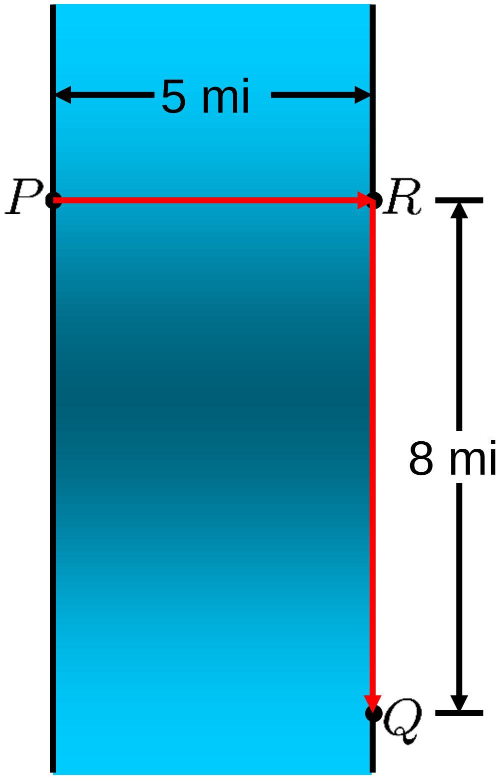
EXAMPLE: Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1,4)$. distance= ??



SKILL
max-min

$y = 2$ $\qquad [x]_{y:\to 2} = 2^2/2 = 2$

EXAMPLE: A man launches a boat from a point P on a bank of a straight river that is 5 miles wide. He wants to get to a point Q that is on the opposite bank and is 8 miles downstream. He could row directly across the river to point R and then run to Q



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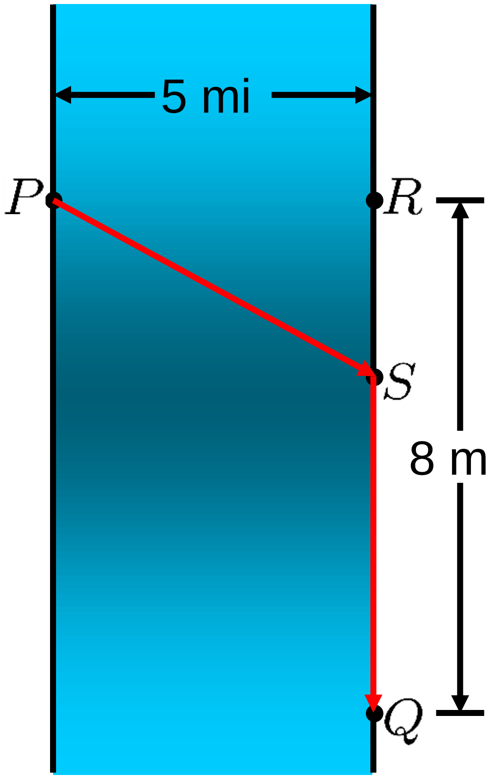

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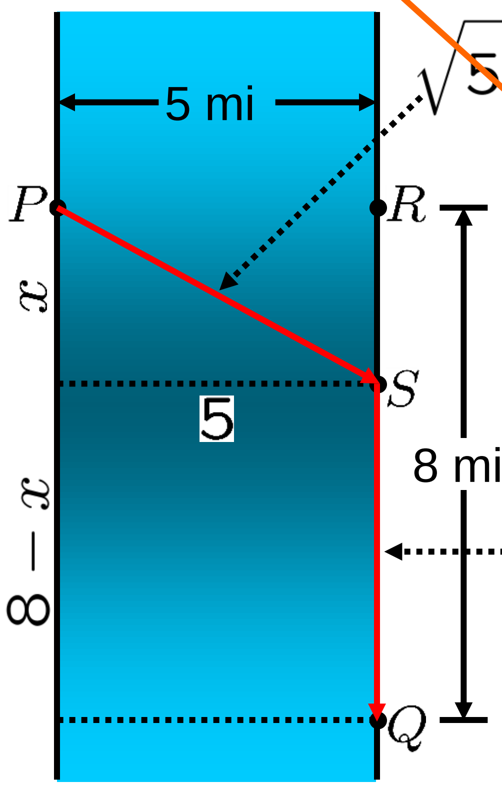

$\sqrt{5^2 + x^2}$ mi at 7 mi/h

minimize: $T :=$ total hours of travel

$$T = \frac{\sqrt{5^2 + x^2}}{7} + \frac{8 - x}{9} \quad \text{on } x \in [0, 8]$$

$8 - x$ mi at 9 mi/h

minimize: $T :=$ total hours of travel

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cf. §6.1, p. 105 TH'M 6.2 (EXTREME VALUE TH'M):

If D is a compact (*i.e.*, closed, bounded) interval,
and if $f : D \to \mathbb{R}$ is continuous on D,
then f has a global max and a global min.

FINDING GLOBAL EXTREMA:

If you know a function has a global max (resp. min),
then you can find it:

compute the values at critical points
and find the largest (resp. the smallest).

minimize: $T :=$ total hours of travel

$$T = \frac{\sqrt{5^2 + x^2}}{7} + \frac{8-x}{9} \text{ on } x \in [0,8]$$

CRITICAL FOR T: $x = 0$, $x = 8$ and . . .

$$0 = \frac{dT}{dx} = \frac{1}{7}\left[\frac{\cancel{2}x}{\cancel{2}\sqrt{5^2+x^2}}\right] + \frac{1}{9}(-1)$$

$$7 \times \left(\frac{1}{7}\left[\frac{x}{\sqrt{5^2+x^2}}\right] = \frac{1}{9}\right)$$

$$\frac{x}{\sqrt{5^2+x^2}} = \frac{7}{9}$$

cf. §6.1, p. [illegible] (EXTREME VALUE TH'M):

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and if $f : D$ [illegible]ous on D,

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$$\frac{1}{7}\left[\frac{x}{\sqrt{5^2 + x^2}}\right] = \frac{1}{9}$$

$$\frac{x}{\sqrt{5^2 + x^2}} = \frac{7}{9}$$

$$\left(\frac{\sqrt{5^2 + x^2}}{x} = \frac{9}{7}\right)^2$$

$$\frac{5^2 + x^2}{x^2} = \frac{81}{49}$$

$$\frac{5^2}{x^2} \boxed{+ 1}$$

$$\frac{5^2}{x^2} = \frac{81}{49} - 1 = \frac{32}{49}$$

$\sqrt{\ }$ →

$$\frac{5}{x} = \frac{\sqrt{32}}{\sqrt{49}} = \frac{4\sqrt{2}}{7}$$

$$\frac{x}{5} = \frac{7}{4\sqrt{2}}$$

§6.1

minimize: $T :=$ total hours of travel

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CRITICAL FOR T: $x = 0$, $x = 8$ and $x = \frac{35}{4\sqrt{2}} \doteq 6.19$

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$$\frac{1}{7}\left[\frac{x}{\sqrt{5^2 + x^2}}\right] = \frac{1}{9}$$

$$\frac{x}{\sqrt{5^2 + x^2}} = \frac{7}{9}$$

$$\frac{\sqrt{5^2 + x^2}}{x} = \frac{9}{7}$$

$$5 \times \left(\frac{x}{5} = \frac{7}{4\sqrt{2}}\right)$$

$$\frac{5}{x} = \frac{\sqrt{32}}{\sqrt{49}} = \frac{4\sqrt{2}}{7}$$

$$\frac{5^2}{x^2} = \frac{81}{49} - 1 = \frac{32}{49}$$

$$\frac{5^2}{x^2} + 1$$

$$\|$$

$$\frac{5^2 + x^2}{x^2} = \frac{81}{49}$$

minimize: $T :=$ total hours of travel

$$T = \frac{\sqrt{5^2 + x^2}}{7} + \frac{8-x}{9} \text{ on } x \in [0,8]$$

CRITICAL FOR T: $x = 0$, $x = 8$ and $x = \dfrac{35}{4\sqrt{2}} \doteq 6.19$

$$[T]_{x:\to 0} = \frac{5}{7} + \frac{8}{9} \doteq 1.60$$

$$[T]_{x:\to 8} = \frac{\sqrt{5^2 + 8^2}}{7} \doteq 1.35$$

$$[T]_{x:\to 35/(4\sqrt{2})} = \frac{\sqrt{5^2 + (35^2/32)}}{7} + \frac{8 - (35/(4\sqrt{2}))}{9}$$

$$\doteq \frac{7.9550}{7} + \frac{1.8128}{9} \doteq 1.34$$

global minimum value

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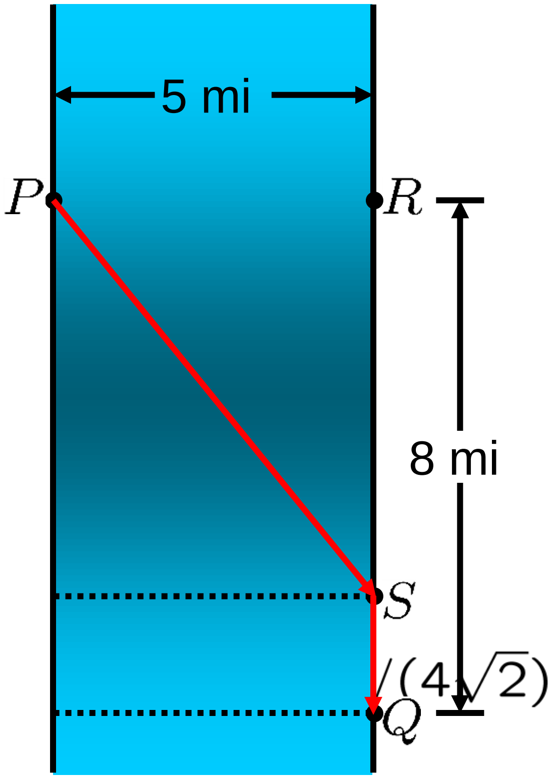

$$x = \frac{35}{4\sqrt{2}} \doteq 6.19$$

$$[T]_{x:\to 35/(4\sqrt{2})} \doteq 1.34$$

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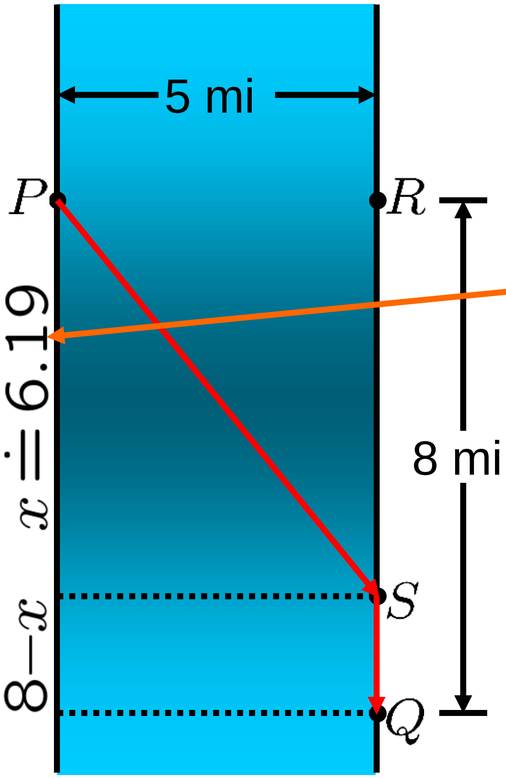
Total time $\doteq$ 1.34 hrs

SKILL
max-min



$$x = \frac{35}{4\sqrt{2}} \doteq 6.19$$

$$[T]_{x:\to 35/(4\sqrt{2})} \doteq 1.34$$

EXAMPLE: A store has been selling 200 laptops per week at \$800 each. A market survey indicates that, for each \$1 reduction in the price, the average number of laptops sold per week will increase by $\frac{1}{2}$.

How large a reduction should the store offer to maximize its revenue?

$x :=$ the price reduction

$800 - x =$ the price of a laptop, after reduction

$R :=$ revenue per week

$= $ (number laptops sold)(price per laptop)

$= (200 + \frac{1}{2}x)(800 - x),\ x \geq 0$

incr on $0 \leq x \leq 200$
global max at $x = 200$ ■
decr on $x \geq 200$

$$\frac{dR}{dx} = \frac{1}{2}(800 - x) + (200 + \frac{1}{2}x)(-1)$$

$$= 200 - x,\ x > 0$$

pos on $0 < x < 200$
zero only at $x = 200$
neg on $x > 200$

SKILL
max-min

For you:
$[R]_{x:\to 200} = ??$

x and y, $x \le y$

EXERCISE: Find two numbers whose difference is 10 and whose product is a minimum.

SKILL
max-min

Minimum value?

Minimum value: $(-5)(5) = -25$

P takes its global min at $x = -5$

$[y]_{x:\to -5} = [10 + x]_{x:\to -5} = 10 - 5 = 5$ ■

P decreasing on $x \le -5$

P increasing on $x \ge -5$

negative on $x < -5$
zero only at $x = -5$
positive on $x > -5$

$$\frac{dP}{dx} = 10 + 2x$$

Minimize: $P := xy = x(10 + x) = 10x + x^2$

Constraint: $y - x = 10$, *i.e.*, $y = 10 + x$

EXERCISE: Find the dimensions of a rectangle with area 2,000 ft^2 whose perimeter is minimal.

Minimum value?

Minimum value: $2\sqrt{2000} + 2\sqrt{2000} = 4\sqrt{2000}$

P takes its global min at $x = \sqrt{2000}$

SKILL max-min

$[y]_{x:\to\sqrt{2000}} = [2000/x]_{x:\to\sqrt{2000}} = \sqrt{2000}$ ■

P decreasing on $0 < x \leq \sqrt{2000}$

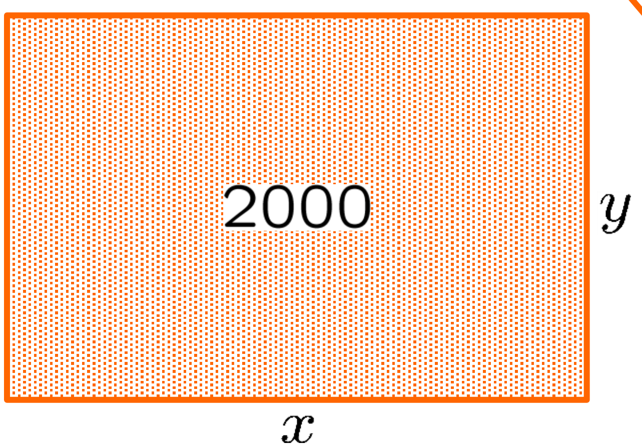
P increasing on $x \geq \sqrt{2000}$

$\frac{dP}{dx} = 0$ only at $x = \sqrt{2000}$

$\frac{dP}{dx} = \boxed{2 - \frac{4000}{x^2}}$ on $x > 0$

neg on $0 < x < \sqrt{2000}$

pos on $x > \sqrt{2000}$



$2x + \frac{4000}{x}$ on $x > 0$

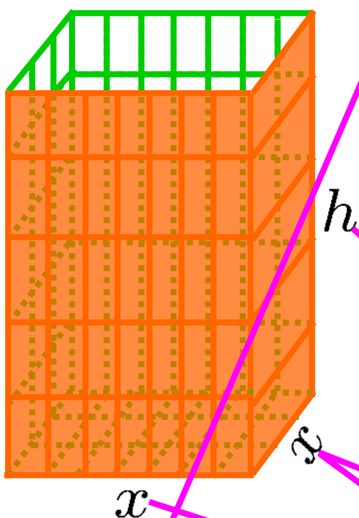
||

Minimize: $P := 2x + 2y = 2x + 2(2000/x)$

Constraint: $xy = 2000$, *i.e.*, $y = 2000/x$

EXERCISE: A box with a square base and open top must have a volume of $16,000\ \text{ft}^3$. Find the dimensions of the box that minimizes the amount of material used.

vol $= 16000$



ALL FOUR SIDES

Minimize: $M := x^2 + 4xh$

BOTTOM OF BOX A SIDE OF BOX

EXERCISE: A box with a square base and open top must have a volume of 16,000 ft^3. Find the dimensions of the box that minimizes the amount of material used.

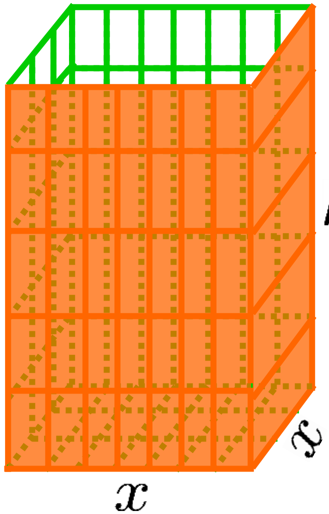
Minimum value: $(\sqrt[3]{32000})^2 + (64000/\sqrt[3]{32000})$

SKILL
max-min

M takes its global min at $x = \sqrt[3]{32000}$

$[h]_{x:\to\sqrt[3]{32000}} = 16000/(\sqrt[3]{32000})^2$ ■

vol = 16000



M decreasing on $0 < x \leq \sqrt[3]{32000}$

M increasing on $x \geq \sqrt[3]{32000}$

$\frac{dM}{dx} = 0$ only at $x = \sqrt[3]{32000}$

neg on $0 < x < \sqrt[3]{32000}$

pos on $x > \sqrt[3]{32000}$

$\frac{dM}{dx} = 2x - \frac{64000}{x^2}$ on $x > 0$

$x^2 + (64000/x)$ on $x > 0$

||

Minimize: $M := x^2 + 4xh = x^2 + 4x(16000/x^2)$

$h = 16000/x^2$

§6.1 Constraint: $x^2 h = 16000$

EXERCISE: Find the point on the line $5x + y = 3$ that that is closest to the point $(-4, 4)$.

For you: Graph $5x + y = 3$ and plot $(-4, 4)$.

Minimize: $\text{dist}((x, y), (-4, 4)) = \sqrt{(x - (-4))^2 + (y - 4)^2}$

Constraint: $5x + y = 3$, *i.e.*, $y = 3 - 5x$

Minimize: $s = (x - (-4))^2 + (y - 4)^2$

$f \geq 0$, f^2 attains global minimum at a
$\Rightarrow$ f attains global minimum at a

EXERCISE: Find the point on the line $5x + y = 3$ that that is closest to the point $(-4, 4)$. min distance?

For you: Graph $5x + y = 3$ and plot $(-4, 4)$.

Minimize: $\text{dist}((x,y),(-4,4)) = \sqrt{(x-(-4))^2 + (y-4)^2}$

Constraint: $5x + y = 3$, *i.e.*, $y = 3 - 5x$

Minimize: $s = (x-(-4))^2 + (y-4)^2$

$= (x+(+4))^2 + (3 - 5x - 4)^2$

$= (x+4)^2 + (-5x - 1)^2$

decr on $x \le -9/26$
incr on $x \ge -9/26$

$ds/dx = 2(x+4)(1) + 2(-5x-1)(-5)$

neg on $x < -9/26$
pos on $x > -9/26$

$= (2x+8) + 2(25x+5) = 52x + 18$

$ds/dx = 0$ only at $x = -18/52 = -9/26$

s takes its global min at $x = -9/26$

$[y]_{x:\to -9/26} = 3 - 5(-9/26) = 4 + (19/26)$

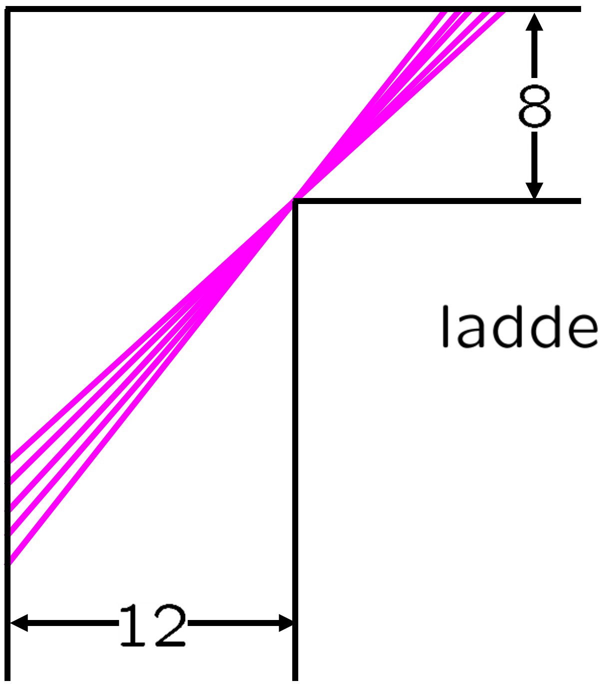
Closest point: $(-9/26, 4 + (19/26))$ ■

For you: dist($(-9/26, 4 + (19/26))$, $(-4, 4)$)

SKILL max-min

EXERCISE: A ladder is being carried down a hallway 12 ft wide. At the end of the hall there is a right-angle turn into a narrower hallway 8 ft wide. What is the length of the longest ladder that can be carried horizontally around the corner?

Focus on one purple line segment . . .

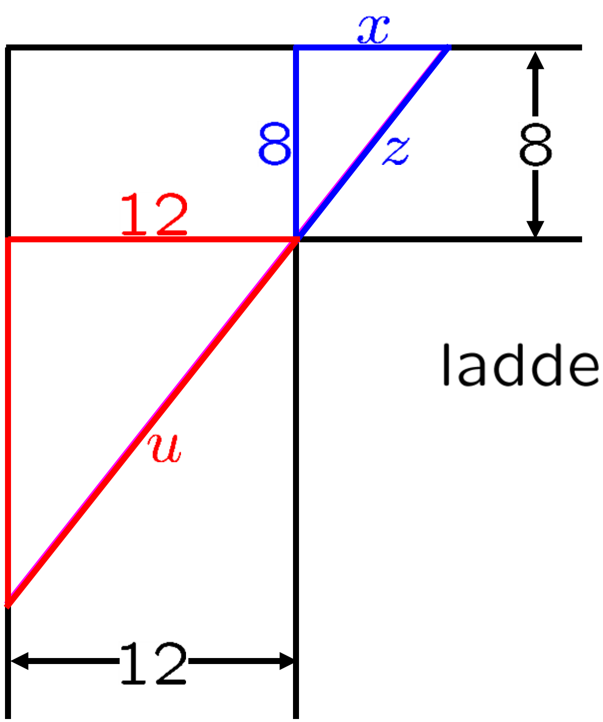


ladder length $\leq$ length purple line segments

max ladder length = min length of purple line segments

EXERCISE: A ladder is being carried down a hallway 12 ft wide. At the end of the hall there is a right-angle turn into a narrower hallway 8 ft wide. What is the length of the longest ladder that can be carried horizontally around the corner?

$x > 0$



Constraint: $z = \sqrt{x^2 + 8^2}$

$$\frac{z}{x} = \frac{u}{12}$$

ladder length $\leq$ length purple line segments

max ladder length $=$ min length of purple line segments

Minimize: $L = z + u$

Minimize: $L = z + u$

$x > 0$

$x > 0$

Constraint: $z = \sqrt{x^2 + 8^2}$

$$12 \times \left(\frac{z}{x} = \frac{u}{12} \right)$$

$$\frac{12z}{x} = u$$

Minimize: $L = z + u$

Minimize: $L = z + u$

$x > 0$

$$= z + \frac{12z}{x}$$

$$= \left[1 + \frac{12}{x}\right] z$$

Constraint: $z = \sqrt{x^2 + 8^2}$

$$\frac{z}{x} = \frac{u}{12}$$

$$\frac{12z}{x} = u$$

Minimize: $S := L^2$

$$S = \left[1 + \frac{12}{x}\right]^2 z^2$$

$$= \left[1 + \frac{12}{x}\right]^2 \left[x^2 + 8^2\right]$$

$f \geq 0$, f^2 attains global minimum at a

$\Rightarrow$ f attains global minimum at a

Minimize: L

Minimize: L

Minimize: $S := L^2$

$x > 0$

$$S = \left[1 + \frac{12}{x}\right]^2 \left[x^2 + 8^2\right]$$

$$x > 0$$

$\forall x > 0,$

Minimize: $S := L^2$

$$\frac{dS}{dx} = \left[1 + \frac{12}{x}\right]^2 [2x] + 2\left[1 + \frac{12}{x}\right]\left[-\frac{12}{x^2}\right]\left[x^2 + 8^2\right]$$

S

$$= \left[1 + \frac{12}{x}\right]^2 \left[x^2 + 8^2\right]$$

Minimize: L

Minimize: $S := L^2$

$$S = \left[1 + \frac{12}{x}\right]^2 \left[x^2 + 8^2\right]$$

$$x > 0$$

decr on $0 < x \le \sqrt[3]{12 \cdot 8^2}$

global min at $x = \sqrt[3]{12 \cdot 8^2}$

incr on $x \ge \sqrt[3]{12 \cdot 8^2}$

$\forall x > 0,$

$$\frac{dS}{dx} = \left[1 + \frac{12}{x}\right]^2 [2x] + 2\left[1 + \frac{12}{x}\right]\left[-\frac{12}{x^2}\right]\left[x^2 + 8^2\right]$$

expand expand

$$= 2\left[1 + \frac{12}{x}\right]\left(\left[1 + \frac{12}{x}\right]x + \left[-\frac{12}{x^2}\right]\left[x^2 + 8^2\right]\right)$$

$$= 2\left[1 + \frac{12}{x}\right]\left(x + 12 - 12 - \frac{12 \cdot 8^2}{x^2}\right)$$

$$= 2\left[1 + \frac{12}{x}\right]\left(x - \frac{12 \cdot 8^2}{x^2}\right)$$

neg on $0 < x < \sqrt[3]{12 \cdot 8^2}$

zero only at $x = \sqrt[3]{12 \cdot 8^2}$

pos on $x > \sqrt[3]{12 \cdot 8^2}$

Minimize: L ☺

Minimize: $S := L^2$ ☺ $\quad L = S^{1/2}$

$$S = \left[1 + \frac{12}{x}\right]^2 \left[x^2 + 8^2\right]$$

$$x > 0$$

decr on $0 < x \le \sqrt[3]{12 \cdot 8^2}$

global min at $x = \sqrt[3]{12 \cdot 8^2}$

incr on $x \ge \sqrt[3]{12 \cdot 8^2}$

$$[S]_{x :\to \sqrt[3]{12 \cdot 8^2}} = [S]_{x :\to 12^{1/3} \cdot 8^{2/3}}$$

$$= \left[1 + \frac{12}{12^{1/3} \cdot 8^{2/3}}\right]^2 \left[(12^{1/3} \cdot 8^{2/3})^2 + 8^2\right]$$

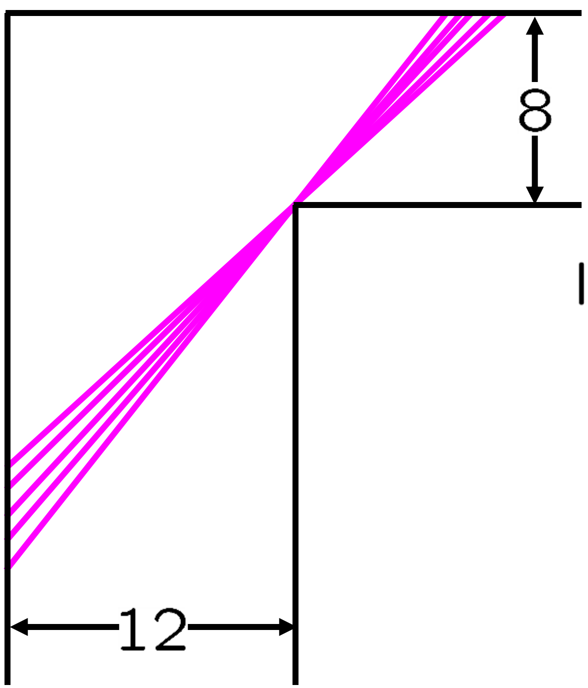
minimum value of S

$$[L]_{x :\to \sqrt[3]{12 \cdot 8^2}} = \left[S^{1/2}\right]_{x :\to \sqrt[3]{12 \cdot 8^2}}$$

$$= \left[1 + \frac{12}{12^{1/3} \cdot 8^{2/3}}\right] \left[(12^{1/3} \cdot 8^{2/3})^2 + 8^2\right]^{1/2}$$

minimum value of L

EXERCISE: A ladder is being carried down a hallway 12 ft wide. At the end of the hall there is a right-angle turn into a narrower hallway 8 ft wide. What is the length of the longest ladder that can be carried horizontally around the corner?



SKILL
max-min

longest ladder: $\left[8^{2/3} + 12^{2/3}\right]^{3/2}$ ft ■

$$\left[8^{2/3} + 12^{2/3}\right]^{3/2}$$

$\|$ exercise

$$\left[1 + \frac{12}{12^{1/3} \cdot 8^{2/3}}\right] \left[(12^{1/3} \cdot 8^{2/3})^2 + 8^2\right]^{1/2}$$

minimum length of a purple line segment

SKILL

max-min

Whitman problems

§6.1, p. 115–118, #2-34