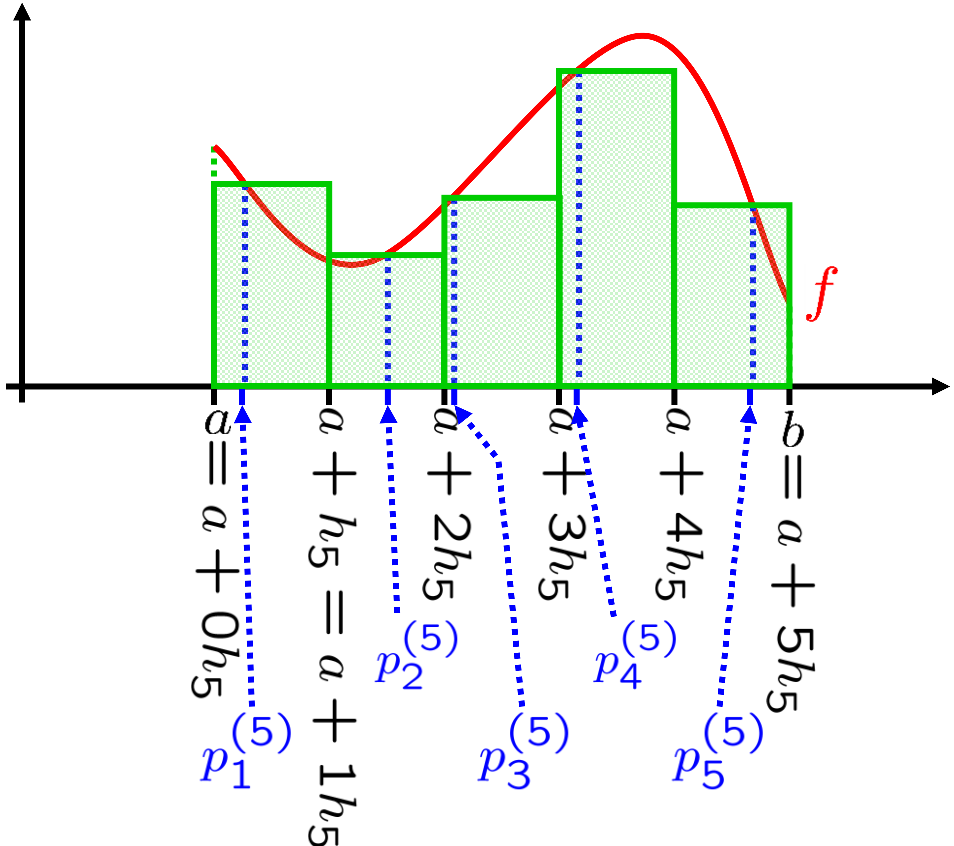
CALCULUS
Variations on the definition of the definite integral

Written form. . .

$$h_5 = \frac{b-a}{5}$$



$$\sum_{j=1}^{5} [h_5][f(p_5^{(j)})]$$

||

total shaded area

Variant: For each partition, pick any point in each subinterval.

e.g.: The fifth partition. . .

DEFINITION OF A DEFINITE INTEGRAL:

$$\boxed{\int_a^b f(x)\,dx} := \lim_{n\to\infty} L_n S_a^b f = \lim_{n\to\infty} M_n S_a^b f = \lim_{n\to\infty} R_n S_a^b f$$

Written form...

THEOREM: Let $a, b \in \mathbb{R}$ satisfy $a < b$.

Let f be a function. Assume that f is continuous on $[a, b]$.

$\forall$integers $n \geq 1$, let $h_n := (b - a)/n$,

let $p_n^{(1)} \in [a, a + h_n]$, $p_n^{(2)} \in [a + h_n, a + 2h_n]$,

$p_n^{(3)} \in [a + 2h_n, a + 3h_n], \ldots, p_n^{(n)} \in [a + (n-1)h_n, b]$

and let $\text{RS}_n := \sum_{j=1}^{n} [h_n][f(p_n^{(j)})]$.

if desired

DOES NOT DEPEND ON j

Then

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \text{RS}_n.$$

REMARK: This kind of sum is a **Riemann sum of** f.

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Other limits yield the area...

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and let $\mathrm{RS}_n := \sum_{j=1}^{n} [h_n][f(p_n^{(j)})]$.

Then
$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \mathrm{RS}_n.$$

Note: True even if f has a finite number of jump discontinuities.

NOTE: Some functions have infinitely many jump discontinuities, and some have discontinuities that are not jump discontinuities. Such functions, in some cases, are not integrable, *i.e.*, the limit of the Riemann sums might not exist, or it might depend on the choices of the points $p_j^{(k)}$.

See STEWART §5.2, p. 378
Exercises 67-68.

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Buzz phrase:
"The definite integral is the limit of the Riemann sums".

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NEXT:
The general theory even allows for subintervals of varying lengths,
but requires that the **mesh of the partition** tends to zero.

Fishing nets with small (fine) mesh catch more fish.

i.e., the length of the longest subinterval

Subintervals can have varying lengths. . .

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Let f be a function. Assume that f is continuous on $[a, b]$.

$\forall$integers $n \geq 1$, let $k_n \geq 1$ be an integer,

"nth partition"

let $a = x_n^{(0)} < \cdots < x_n^{(k_n)} = b$,

nth partition has k_n subintervals.

"points in subintervals in the nth partition"

let $p_n^{(1)} \in [x_n^{(0)}, x_n^{(1)}]$, . . . , $p_n^{(k_n)} \in [x_n^{(k_n-1)}, x_n^{(k_n)}]$,

"mesh of the nth partition"

let $\mu_n :=$ max$\{ x_n^{(1)} - x_n^{(0)}$, . . . , $x_n^{(k_n)} - x_n^{(k_n-1)} \}$

and let $\text{RS}_n := \sum_{j=1}^{k_n} [x_n^{(j)} - x_n^{(j-1)}][f(p_n^{(j)})]$.

REMARK: This kind of sum is a **Riemann sum of** f.

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let $\mu_n := \max\{\, x_n^{(1)} - x_n^{(0)} \,, \ldots, \, x_n^{(k_n)} - x_n^{(k_n-1)} \,\}$

and let $\mathrm{RS}_n := \sum_{j=1}^{k_n} [x_n^{(j)} - x_n^{(j-1)}][f(p_n^{(j)})]$.

Assume $\lim_{n \to \infty} \mu_n = 0$.

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$$\text{let } \mu_n := \max\{\, x_n^{(1)} - x_n^{(0)}, \ldots, x_n^{(k_n)} - x_n^{(k_n-1)} \,\}$$

DEPENDS ON j

$$\text{and let } \mathrm{RS}_n := \sum_{j=1}^{k_n} \boxed{[x_n^{(j)} - x_n^{(j-1)}]}[f(p_n^{(j)})].$$

CANNOT BE FACTORED OUT!

Assume $\lim_{n\to\infty} \mu_n = 0$. Then $\int_a^b f(x)\,dx = \lim_{n\to\infty} \mathrm{RS}_n$.

Note: True even if f has a finite number of jump discontinuities.

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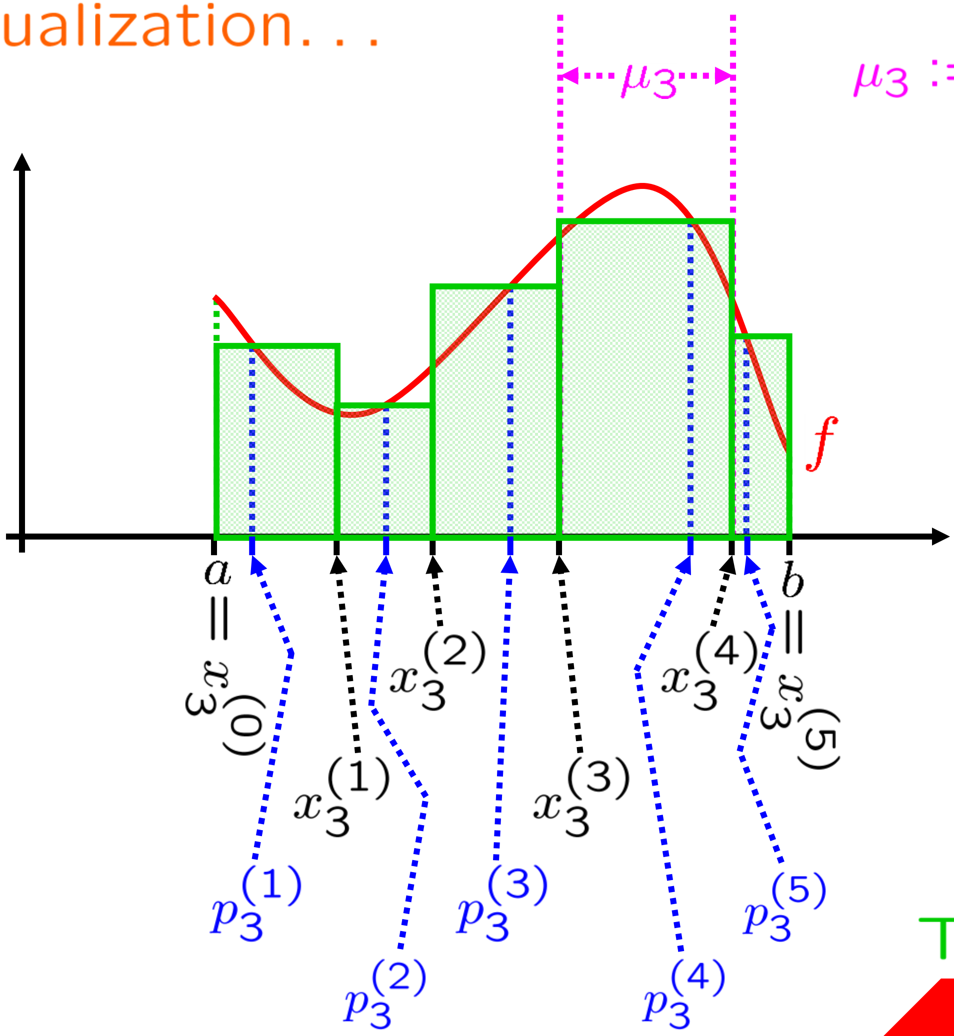
Visualization. . .



$\mu_3 :=$ the mesh of the third partition

$$\sum_{j=1}^{5} [x_3^{(j)} - x_3^{(j-1)}][f(p_3^{(j)})]$$

$$\|$$

RS$_3$ = total shaded area

Need a sequence RS$_1$, RS$_2$, RS$_3$, RS$_4$, . . .
with $\mu_1, \mu_2, \mu_3, \mu_4, \ldots \to 0$.

Then RS$_1$, RS$_2$, RS$_3$, RS$_4$, . . .
$$\to \int_a^b f(x)\,dx.$$

The third partition

$k_3 = 5$ subintervals in that partition
with varying lengths

Subintervals can have varying lengths. . .