CALCULUS

Integration by substitution

$u := 1 + x^2$

$$\frac{du}{dx} = 2x$$

antiderivative?

$$\int \left[\sqrt{1 + x^2}\right] [2x] \, dx = \int [\sqrt{u}] \left[\frac{du}{dx}\right] dx$$

Goal:

CHAIN RULE: $\frac{d}{dx}[F(u)] = [F'(u)] \left[\frac{du}{dx}\right]$

antiderivative

Goal: F s.t. $F' = \sqrt{\bullet}$

$$u := 1 + x^2$$

$$\frac{du}{dx} = 2x$$

$$\int \left[\sqrt{1+x^2}\right] [2x]\, dx = \int [\sqrt{u}] \left[\frac{du}{dx}\right] dx = \frac{u^{3/2}}{3/2} + C$$

$$\| $$

$$\frac{(1+x^2)^{3/2}}{3/2} + C$$

$$\frac{d}{dx}\left[\frac{u^{3/2}}{3/2}\right] = [\sqrt{u}] \left[\frac{du}{dx}\right]$$

CHAIN RULE: $\frac{d}{dx}[F(u)] = [F'(u)] \left[\frac{du}{dx}\right]$

Goal: $F(x)$ s.t. $F'(x) = \sqrt{x}$

Goal: F s.t. $F' = \sqrt{\bullet}$

$$F(x) := \frac{x^{3/2}}{3/2}$$

$$F := \frac{(\bullet)^{3/2}}{3/2}$$

$$F(u) = \frac{u^{3/2}}{3/2}$$

Simpler to remember and recognize, as follows. . .

$$u := 1 + x^2$$

$$du := \left[\frac{du}{dx}\right] dx$$

$$\frac{du}{dx} = 2x$$

$$\int \left[\sqrt{1+x^2}\right] [2x]\, dx = \int [\sqrt{u}] \left[\frac{du}{dx}\right] dx = \frac{u^{3/2}}{3/2} + C$$

$$\frac{d}{dx}\left[\frac{u^{3/2}}{3/2}\right] = [\sqrt{u}] \left[\frac{du}{dx}\right]$$

$$\| \qquad \frac{(1+x^2)^{3/2}}{3/2} + C \quad \blacksquare$$

CHAIN RULE: $\frac{d}{dx}[F(u)] = [F'(u)] \left[\frac{du}{dx}\right]$

Goal: F s.t. $F' = \sqrt{\bullet}$

$$F := \frac{(\bullet)^{3/2}}{3/2}$$

Simpler to remember and recognize, as follows. . .

$$u := 1 + x^2$$

$$\frac{du}{dx} = 2x$$

$$du := \left[\frac{du}{dx}\right] dx = [2x]\, dx$$

$$\int \left[\sqrt{1+x^2}\right] [2x]\, dx = \int \left[\sqrt{u}\right] \boxed{du} = \frac{u^{3/2}}{3/2} + C$$

$$\| $$

$$\frac{(1+x^2)^{3/2}}{3/2} + C \quad ■$$

$$\frac{d}{dx}\left[\frac{u^{3/2}}{3/2}\right] = \left[\sqrt{u}\right]\left[\frac{du}{dx}\right]$$

CHAIN RULE: $\frac{d}{dx}[F(u)] = [F'(u)]\left[\frac{du}{dx}\right]$

Goal: F s.t. $F' = \sqrt{\bullet}$

$$F := \frac{(\bullet)^{3/2}}{3/2}$$

Simpler to remember and recognize, as follows. . .

$u := 1 + x^2$

$du = [2x]\,dx$

$$\begin{aligned} du &\\ &= [2x]\,dx \end{aligned}$$

THE KEY STEP

$$\int \left[\sqrt{1+x^2}\right] [2x]\,dx = \int \left[\sqrt{u}\right] du = \frac{u^{3/2}}{3/2} + C$$

$$\| $$

$$\frac{(1+x^2)^{3/2}}{3/2} + C \qquad \blacksquare$$

$$\int \left[\sqrt{u}\right] du = \frac{u^{3/2}}{3/2} + C$$

True even when u is a *DEPENDENT* variable!

INDEFINITE INTEGRATION BY SUBSTITUTION

For any function F, for any expression u of x,

domain of $F(u)$ might not be an interval...

$$\int F'(u)\,du = (F(u)) + C.$$

sometimes sloppy

Proof:

du ∴ $\left[\frac{du}{dx}\right] dx$

Want: $\int [F'(u)] \left[\frac{du}{dx}\right] dx = (F(u)) + C$

d/dx

Chain Rule implies: $\frac{d}{dx}[F(u)] = [F'(u)] \left[\frac{du}{dx}\right]$

QED

$$F := \frac{(\bullet)^{3/2}}{3/2} \qquad u := 1 + x^2$$

$$\int \left[\sqrt{u}\right] du = \frac{u^{3/2}}{3/2} + C$$

True even when u is a *DEPENDENT* variable!

INDEFINITE INTEGRATION BY SUBSTITUTION

For any function F, for any expression u of x,

$$\int F'(u)\,du = (F(u)) + C.$$

Proof:

$$du \;\ddot{=}\; \left[\frac{du}{dx}\right] dx$$

Want: $\displaystyle\int [F'(u)]\left[\frac{du}{dx}\right] dx = (F(u)) + C$

Chain Rule implies: $\displaystyle\frac{d}{dx}[F(u)] = [F'(u)]\left[\frac{du}{dx}\right]$

QED

EXAMPLE: Find $\displaystyle\int [x^4][\cos(x^5-6)]\,dx$.

$$\int [x^4][\cos(x^5-6)]\,dx = \frac{1}{5}\int [\cos(u)]\,du = \frac{1}{5}\sin(u) + C$$

$$u := x^5 - 6$$

$$= \frac{1}{5}\sin(x^5-6) + C$$

$$\frac{1}{5}du = x^4\,dx \qquad du = 5x^4\,dx$$

SKILL
Integration by substitution

EXAMPLE: Calculate $\int \tan x \, dx$.

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\int \frac{+du}{u} = -[\ln(|u|)] + C = -[\ln(|\cos x|)] + C \quad \blacksquare$$

$du = -\sin x \, dx$

$u := \cos x$

LINEARITY OF $\int$

SKILL
Integration by substitution

INDEFINITE INTEGRATION BY SUBSTITUTION

For any function F, for any expression u of x,

$$\int F'(u) \, du \overset{\text{sometimes sloppy}}{=} (F(u)) + C.$$

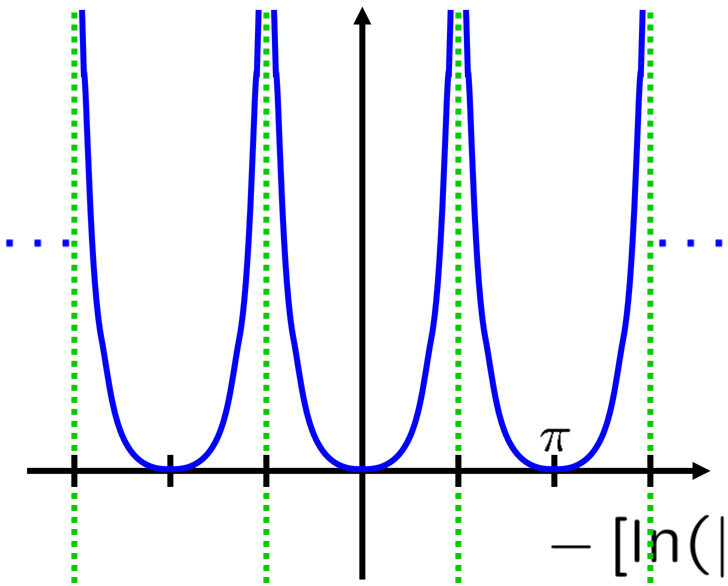
domain of $F(u)$ might not be an interval. . .

$$\int \tan x \, dx$$

$$\|$$

$$-[\ln(|\cos x|)] + C$$

$$-[\ln(|\cos x|)]$$



$$\|$$

$$-[\ln(|\cos x|)] + C$$

INDEFINITE INTEGRATION BY SUBSTITUTION

For any function F, for any expression u of x,

domain of $F(u)$ might not be an interval. . .

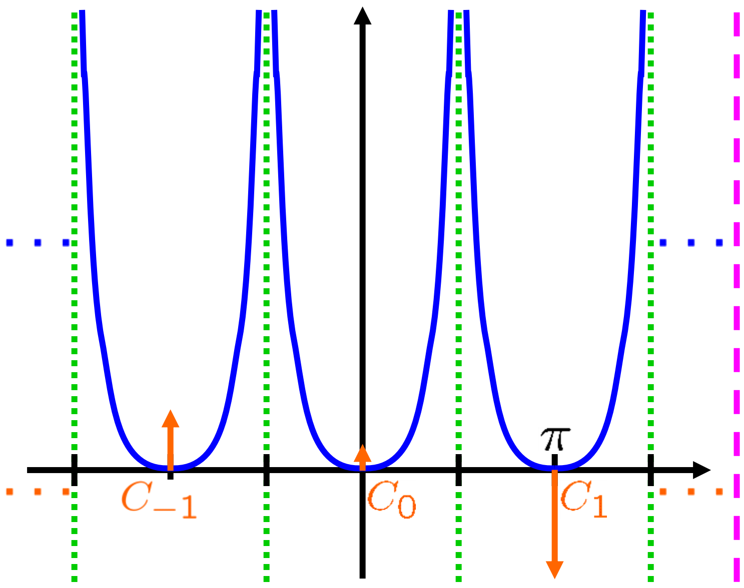
$$\int F'(u)\, du \overset{\text{sometimes sloppy}}{=} (F(u)) + C.$$

$$\int \tan x \, dx$$

sloppy $=$

$-[\ln(|\cos x|)] + C$

$-[\ln(|\cos x|)]$



INDEFINITE INTEGRATION BY SUBSTITUTION

For any function F, for any expression u of x,

domain of $F(u)$ might not be an interval...

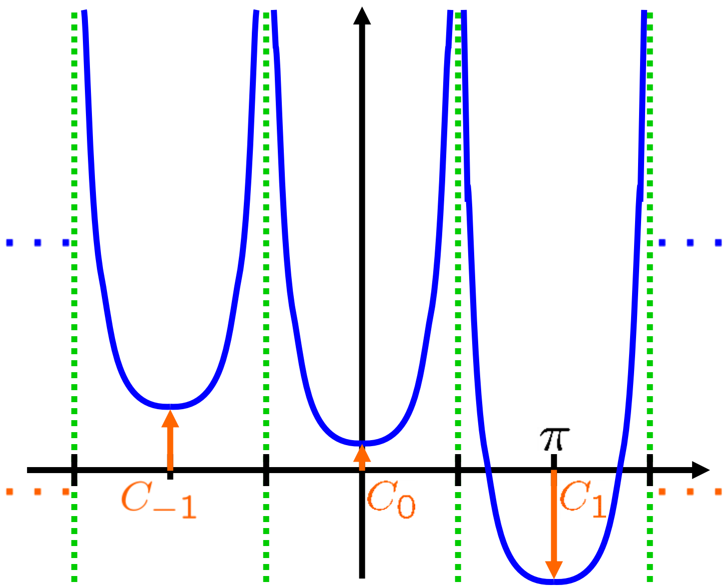
$$\int F'(u)\, du \overset{\text{sometimes sloppy}}{=} (F(u)) + C.$$

$$\int \tan x\, dx$$

sloppy $=$ correct $=$

$$-[\ln(|\cos x|)] + C$$

This has same deriv.



$$\begin{pmatrix} \vdots \\ -[\ln(-\cos x)] + C_{-1}, \text{ if } -\frac{3\pi}{2} < x < -\frac{\pi}{2} \\ -[\ln(\cos x)] + C_0, \text{ if } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ -[\ln(-\cos x)] + C_1, \text{ if } \frac{\pi}{2} < x < \frac{3\pi}{2} \\ \vdots \end{pmatrix}$$

INFINITELY MANY
"DEGREES OF FREEDOM"

INDEFINITE INTEGRATION BY SUBSTITUTION

For any function F, for any expression u of x,

domain of $F(u)$ might not be an interval. . .

$$\int F'(u)\, du \overset{\text{sometimes sloppy}}{=} (F(u)) + C.$$

Next: DEFINITE integration via substitution. . .

INDEFINITE INTEGRATION BY SUBSTITUTION

For any function F, for any expression u of x,

$$\int F'(u)\,du = (F(u)) + C.$$

$$\int_{4}^{7} \cos(x^3)\,[3x^2]\,dx$$

INDEFINITE INTEGRATION BY SUBSTITUTION

For any function F, for any expression u of x,

$$\int F'(u)\,du = (F(u)) + C.$$

Next: DEFINITE integration via substitution. . .

INDEFINITE INTEGRATION BY SUBSTITUTION

For any function F, for any expression u of x,

$$\int F'(u)\,du = (F(u)) + C.$$

$$\sin' = \cos$$

$$\int [\cos(x^3)]\,[3x^2]\,dx = \int [\cos(u)]\,du = \sin(u) + C$$

$$u = x^3$$

$$du = [3x^2]\,dx$$

$$= \sin(x^3) + C$$

$$\int_{4}^{7} \cos(x^3)\,[3x^2]\,dx = [\sin(x^3)]_{x:\to 4}^{x:\to 7}$$

$$= [\sin(7^3)] - [\sin(4^3)]$$ note. . .

$$= [\sin(w)]_{w:\to 4^3}^{w:\to 7^3}$$

$$= \int_{4^3}^{7^3} \cos(w)\,dw$$

INDEFINITE INTEGRATION BY SUBSTITUTION

For any function F, for any expression u of x,

$$\int F'(u)\,du = (F(u)) + C.$$

$$\int_4^7 \cos(x^3)\,[3x^2]\,dx$$

$$\int_4^{\cdot} \cos(\underbrace{x^3}_{w})\,\underbrace{[3x^2]\,dx}_{dw} = \int_{4^3}^{7^3} \cos(w)\,dw$$

w is both independent and dependent.

$$= \int_{4^3}^{7^3} \cos(w)\,dw$$

INDEFINITE INTEGRATION BY SUBSTITUTION

For any function F, for any expression u of x,

$$\int F'(u)\,du = (F(u)) + C.$$

DEFINITE INTEGRATION BY SUBSTITUTION

For any functions f, g, for any $a, b \in \mathbb{R}$,

$[w]_{x:\to b} = g(b)$

$$\int_a^b f(\underbrace{g(x)}_{w})\,\underbrace{[g'(x)]\,dx}_{dw} = \int_{g(a)}^{g(b)} f(w)\,dw$$

$[w]_{x:\to a} = g(a)$

$[w]_{x:\to 7} = 7^3$

$$\int_4^7 \cos(\underbrace{x^3}_{w})\,\underbrace{[3x^2]\,dx}_{dw} = \int_{4^3}^{7^3} \cos(w)\,dw$$

$[w]_{x:\to 4} = 4^3$

w is independent.

w is both independent and dependent.

INDEFINITE INTEGRATION BY SUBSTITUTION

For any function F, for any expression u of x,

$$\int F'(u)\,du = (F(u)) + C.$$

DEFINITE INTEGRATION BY SUBSTITUTION

For any functions f, g, for any $a, b \in \mathbb{R}$,

$[w]_{x:\to b} = g(b)$

$$\int_a^b f(\underbrace{g(x)}_{w})\,\underbrace{[g'(x)]\,dx}_{dw} = \int_{g(a)}^{g(b)} f(w)\,dw$$

$[w]_{x:\to a} = g(a)$

w is being used
both as a dependent variable
and as an independent variable.

INDEFINITE INTEGRATION BY SUBSTITUTION

For any function F, for any expression u of x,

$$\int F'(u)\,du = (F(u)) + C.$$

DEFINITE INTEGRATION BY SUBSTITUTION

For any functions f, g, for any $a, b \in \mathbb{R}$,

$[w]_{x:\to b} = g(b)$

$$\int_a^b f(\underbrace{g(x)}_{w}))\underbrace{[g'(x)]\,dx}_{dw} = \int_{g(a)}^{g(b)} f(w)\,dw$$

$[w]_{x:\to a} = g(a)$

EXAMPLE: Evaluate $\displaystyle\int_0^7 \sqrt{3x+2}\,dx$.

w is now an independent variable.

$[w]_{x:\to 7} = 23$

$$\int_0^7 \sqrt{\underbrace{3x+2}_{w}}\,\underbrace{dx}_{\frac{dw}{3}} = \frac{1}{3}\int_2^{23} \sqrt{w}\,dw = \frac{1}{3}\left[\frac{w^{3/2}}{3/2}\right]_{w:\to 2}^{w:\to 23}$$

$[w]_{x:\to 0} = 2$

$$= \frac{1}{3}\left[\frac{23^{3/2}}{3/2} - \frac{2^{3/2}}{3/2}\right] \blacksquare$$

REMARK: $\int_2^{23} \sqrt{w}\,dw = \int_2^{23} \sqrt{t}\,dt = \int_2^{23} \sqrt{x}\,dx$

$w :\to t \qquad t :\to x$

Next: Integrating symmetric functions

Can change to other independent variables . . .

DEFINITE INTEGRATION BY SUBSTITUTION

For any functions f, g, for any $a, b \in \mathbb{R}$,

$[w]_{x:\to b} = g(b)$

$$\int_a^b f(\underbrace{g(x)}_{w})\,\underbrace{[g'(x)]\,dx}_{dw} = \int_{g(a)}^{g(b)} f(w)\,dw$$

$[w]_{x:\to a} = g(a)$

EXAMPLE: Evaluate $\int_0^7 \sqrt{3x+2}\,dx$.

w is now an independent variable.

$$\int_0^7 \sqrt{3x+2}\,dx = \frac{1}{3}\int_2^{23} \sqrt{w}\,dw = \frac{1}{3}\left[\frac{w^{3/2}}{3/2}\right]_{w:\to 2}^{w:\to 23}$$

$$= \frac{1}{3}\left[\frac{23^{3/2}}{3/2} - \frac{2^{3/2}}{3/2}\right] \blacksquare$$

INTEGRATING SYMMETRIC FUNCTIONS

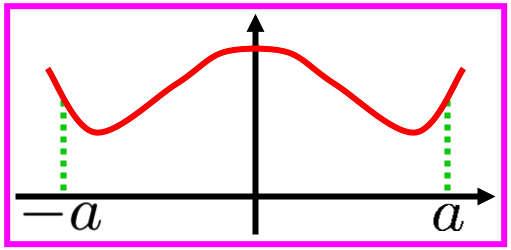
Suppose f is continuous on $[-a, a]$.

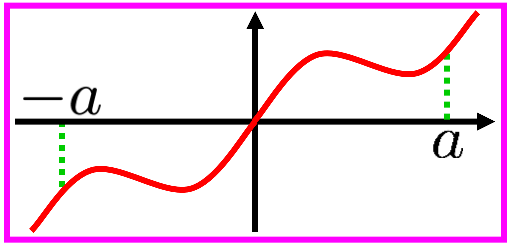
(a) If f is even, *i.e.*, $f(-x) = f(x)$,

then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(b) If f is odd, *i.e.*, $f(-x) = -(f(x))$,

then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.





$dw = -dx$

$w := -x$

Proof of (a): $\displaystyle\int_{-a}^{a} f(x)\,dx = \left[\int_{-a}^{0} f(x)\,dx\right] + \left[\int_{0}^{a} f(x)\,dx\right]$

$f(-w)$

$x = -w$

INTEGRATING SYMMETRIC FUNCTIONS

Suppose f is continuous on $[-a, a]$.

(a) If f is even, *i.e.*, $f(-x) = f(x)$,

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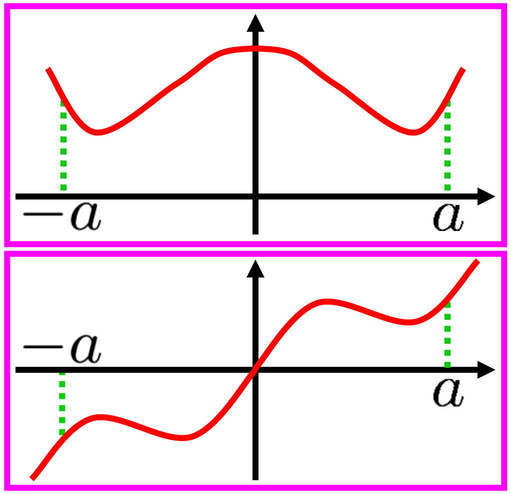

$dw = -dx$

$w := -x$

Proof of (a): $\displaystyle\int_{-a}^{a} f(x)\,dx = \left[\int_{-a}^{0} f(x)\,dx\right] + \left[\int_{0}^{a} f(x)\,dx\right]$

$$\int_{-a}^{0} f(x)\,dx = \int_{a}^{0} f(-w)\,[-dw]$$

$[w]_{x:\to 0} = 0$

$[w]_{x:\to -a} = a$

$dx = -dw$

w is now an independent variable . . .

Suppose f is continuous on $[-a, a]$.

(a) If f is even, *i.e.*, $f(-x) = f(x)$,

then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

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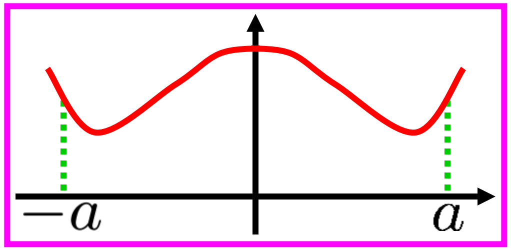
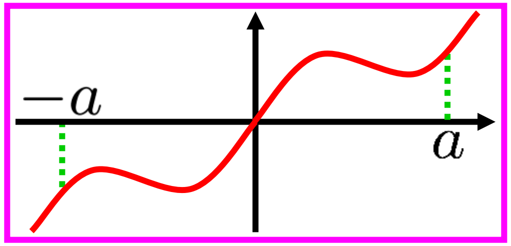




Proof of (a): $\int_{-a}^{a} f(x)\,dx = \left[\int_{-a}^{0} f(x)\,dx\right] + \left[\int_0^a f(x)\,dx\right]$

$$\int_{-a}^{0} f(x)\,dx = \int_a^0 f(-w)\,[-dw] = \int_0^a f(-w)\,dw$$

delete

$$\int_0^a f(x)\,dx = \int_0^a f(-x)\,dx = \int_0^a f(-w)\,dw$$

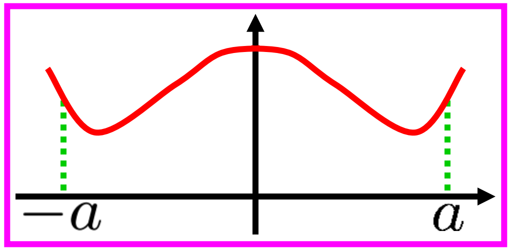
$w :\to x$

INTEGRATING SYMMETRIC FUNCTIONS

Suppose f is continuous on $[-a, a]$.

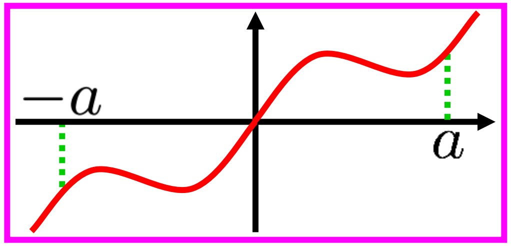
(a) If f is even, *i.e.*, $f(-x) = f(x)$,

then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.



(b) If f is odd, *i.e.*, $f(-x) = -(f(x))$,

then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.



Proof of (a): $\displaystyle\int_{-a}^{a} f(x)\,dx = \left[\int_0^a f(x)\,dx\right] + \left[\int_0^a f(x)\,dx\right]$

$\displaystyle\int_0^a f(x)\,dx$ $\displaystyle = 2\int_0^a f(x)\,dx$ QED

INTEGRATING SYMMETRIC FUNCTIONS

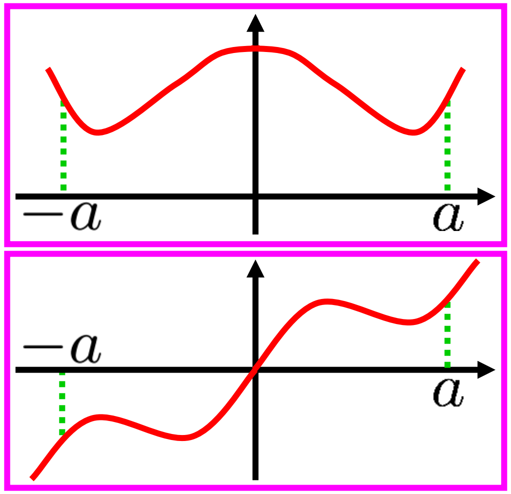
Suppose f is continuous on $[-a,a]$.

(a) If f is even, *i.e.*, $f(-x) = f(x)$,

then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

(b) If f is odd, *i.e.*, $f(-x) = -(f(x))$,

then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.



Proof of (b): $$\int_{-a}^{a} f(x)\,dx = \underbrace{\left[\int_{-a}^{0} f(x)\,dx\right]}_{\displaystyle \| } + \left[\int_0^a f(x)\,dx\right]$$

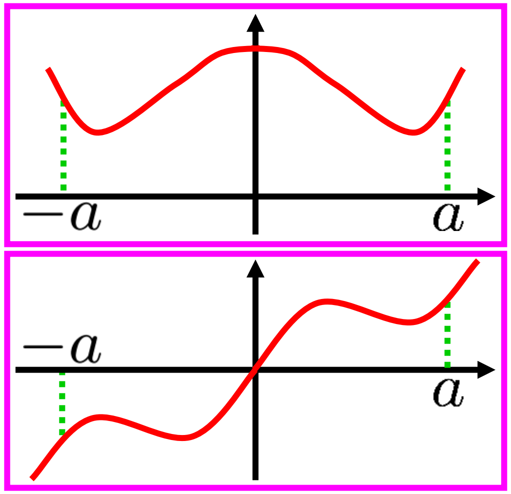
$$\int_a^0 f(-w)\,[-dw]$$

$$\|$$

$$\int_0^a -(f(x))\,dx \quad \| \quad \int_0^a f(-x)\,dx = \int_0^a f(-w)\,dw$$

INTEGRATING SYMMETRIC FUNCTIONS

Suppose f is continuous on $[-a, a]$.



(a) If f is even, *i.e.*, $f(-x) = f(x)$,

then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

(b) If f is odd, *i.e.*, $f(-x) = -(f(x))$,

then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.

Proof of (b): $\displaystyle\int_{-a}^{a} f(x)\,dx = \left[\int_0^a -(f(x))\,dx\right] + \left[\int_0^a f(x)\,dx\right]$

LINEARITY OF DEFINITE INTEGRATION

$\displaystyle\int_0^a -(f(x))\,dx = -\left[\int_0^a f(x)\,dx\right] + \left[\int_0^a f(x)\,dx\right]$

$= 0$ QED

INTEGRATING SYMMETRIC FUNCTIONS

Suppose f is continuous on $[-a, a]$.

(a) If f is even, *i.e.*, $f(-x) = f(x)$,

then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(b) If f is odd, *i.e.*, $f(-x) = -(f(x))$,

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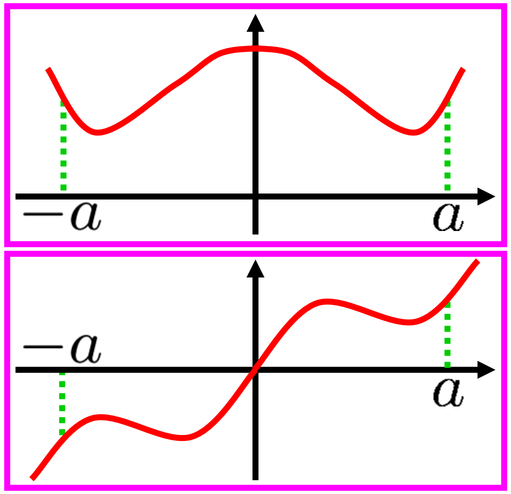

EXAMPLE: Evaluate $\displaystyle\int_{-7}^{7} (3x^4 + x^2 - 4)\,dx$.

even in x

$$\| $$

$$2\int_{0}^{7} (3x^4 + x^2 - 4)\,dx$$

$$\| $$

SKILL
Integration using symmetry

$$2\left[\frac{3\cdot 7^5}{5} + \frac{7^3}{3} - 4\cdot 7\right] = 2\left[\frac{3x^5}{5} + \frac{x^3}{3} - 4x\right]_{x:\to 0}^{x:\to 7}$$

INTEGRATING SYMMETRIC FUNCTIONS

Suppose f is continuous on $[-a, a]$.

(a) If f is even, *i.e.*, $f(-x) = f(x)$,

then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

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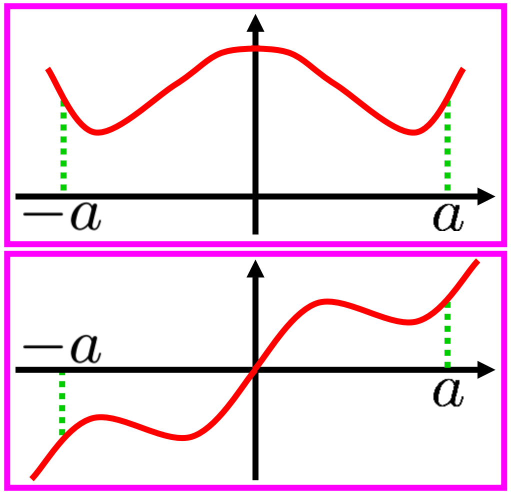

EXAMPLE: Evaluate $\displaystyle\int_{-1}^{1} \frac{\tan x}{3 + 2x^4 + 5x^8}\,dx$.

$\tan x$: odd in x

$3 + 2x^4 + 5x^8$: even in x

The integrand is odd in x.

$$= 0$$

$\tan = \dfrac{\sin}{\cos}$ is $\dfrac{\text{odd}}{\text{even}}$, which is odd.

■

SKILL
Integration using symmetry

STOP