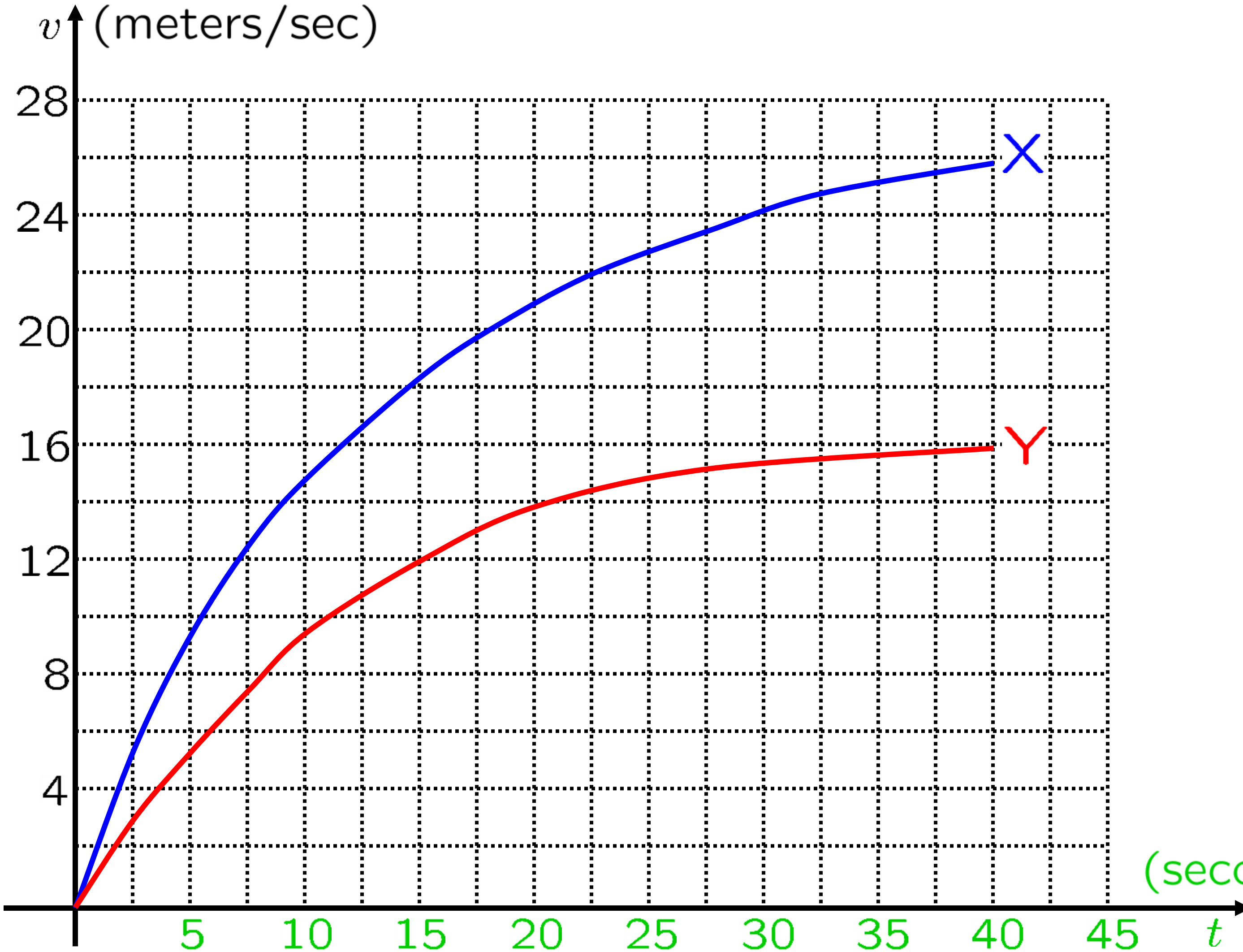
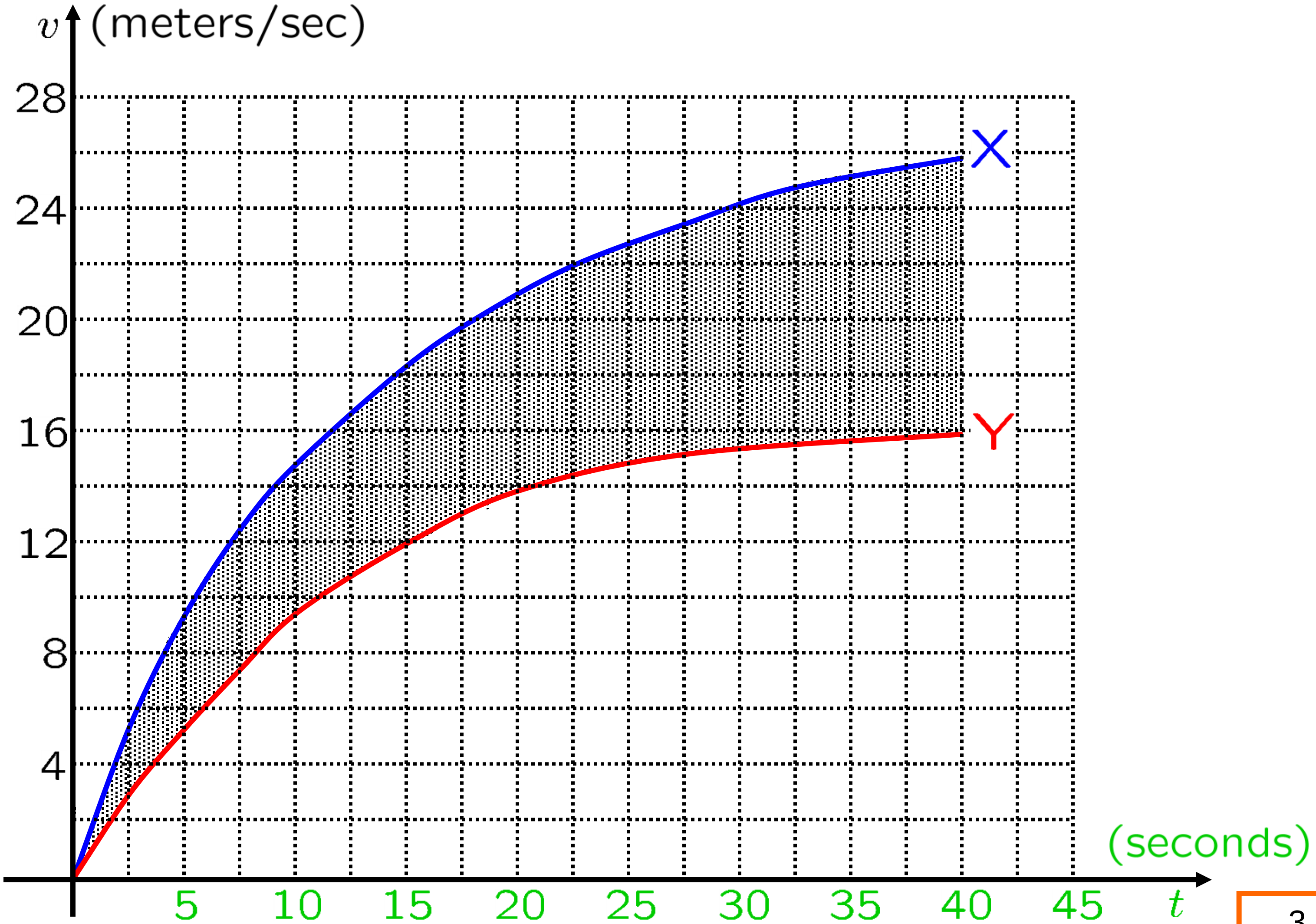
CALCULUS
Area between curves, problems

EXAMPLE: The graph below shows the velocity curves for two trains, X and Y, that start side by side and move along parallel tracks.



EXAMPLE: The graph below shows the velocity curves for two trains, X and Y, that start side by side and move along parallel tracks. What does the area between the the curves represent? Estimate the area using midpoints.



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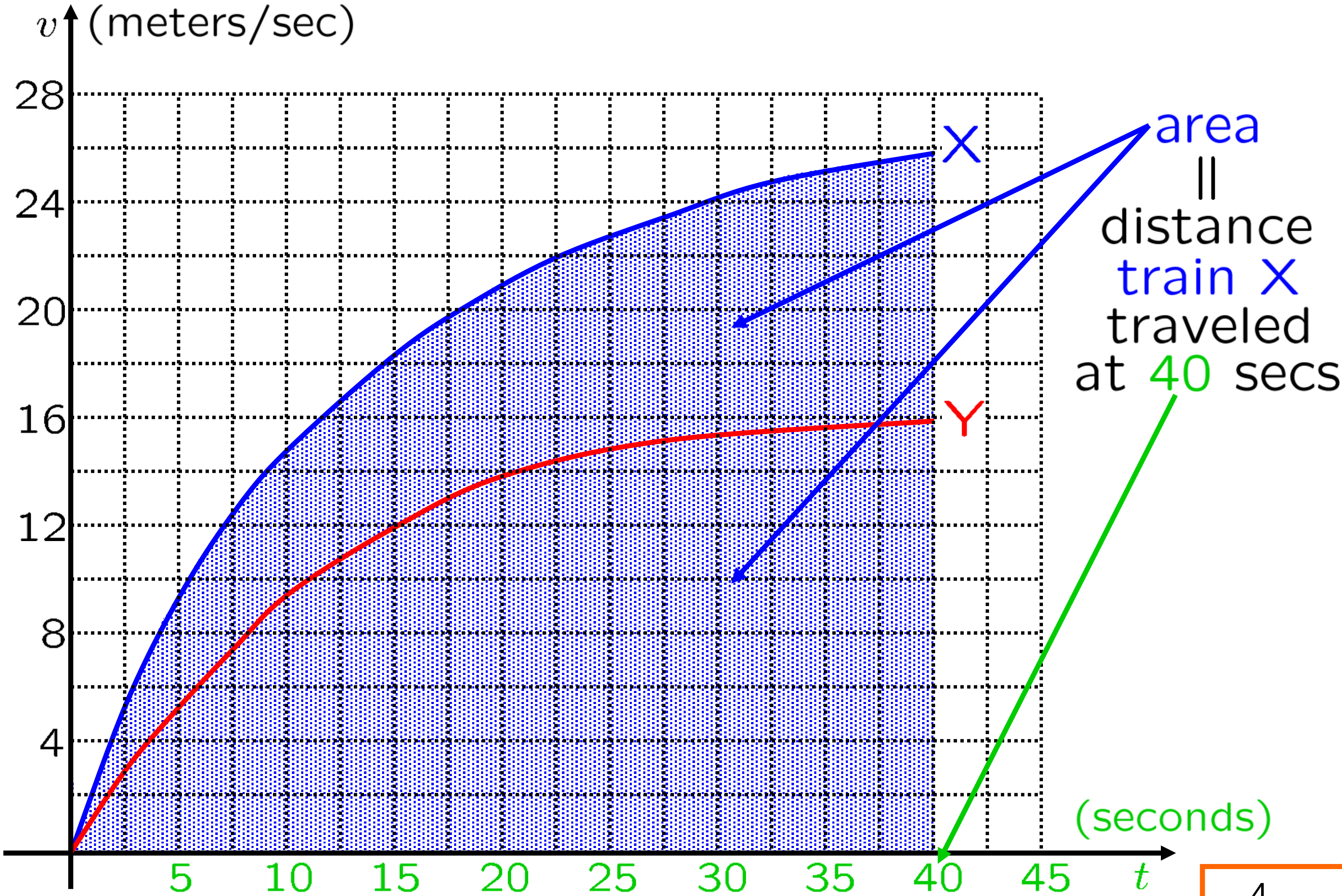

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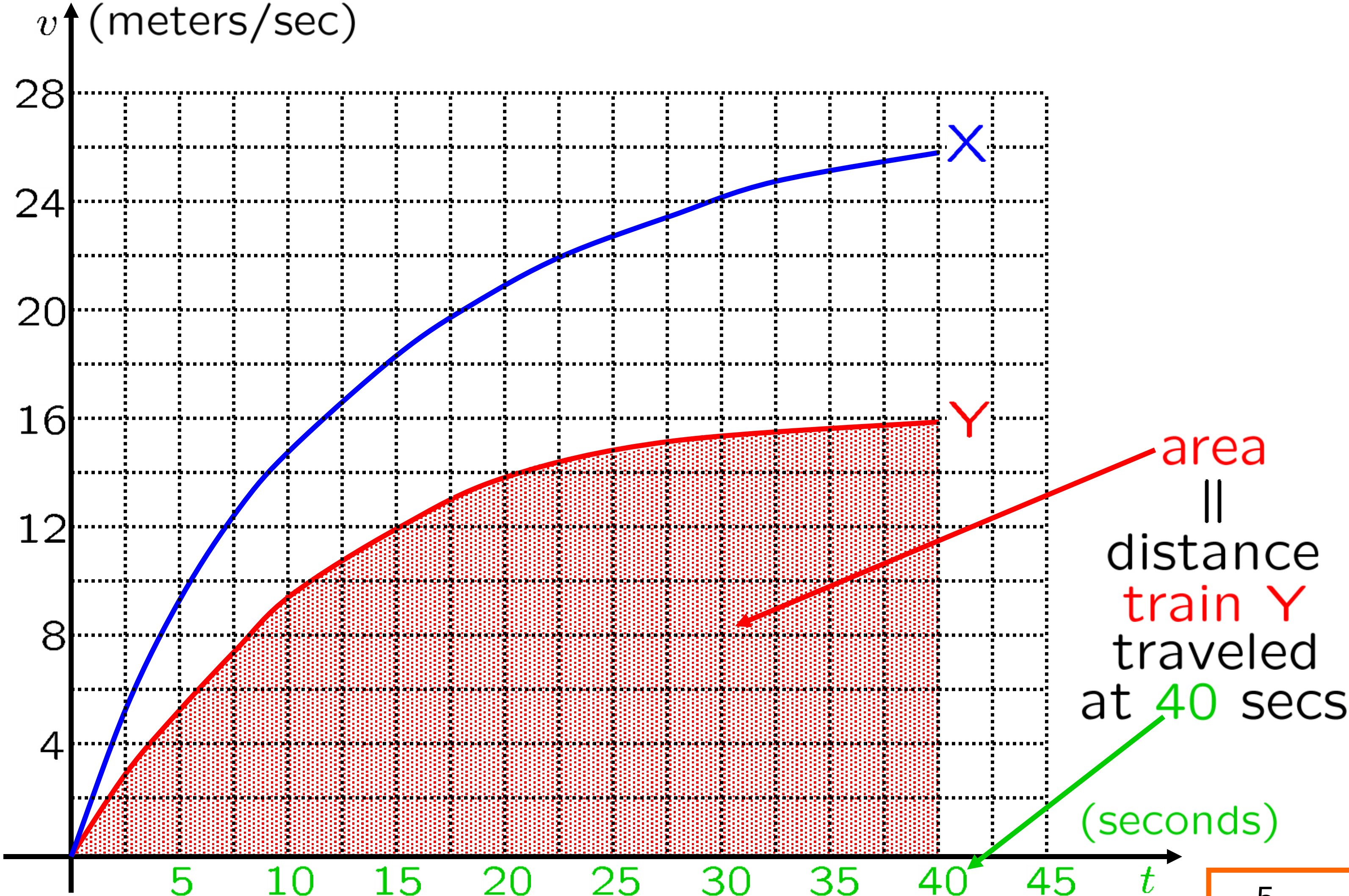

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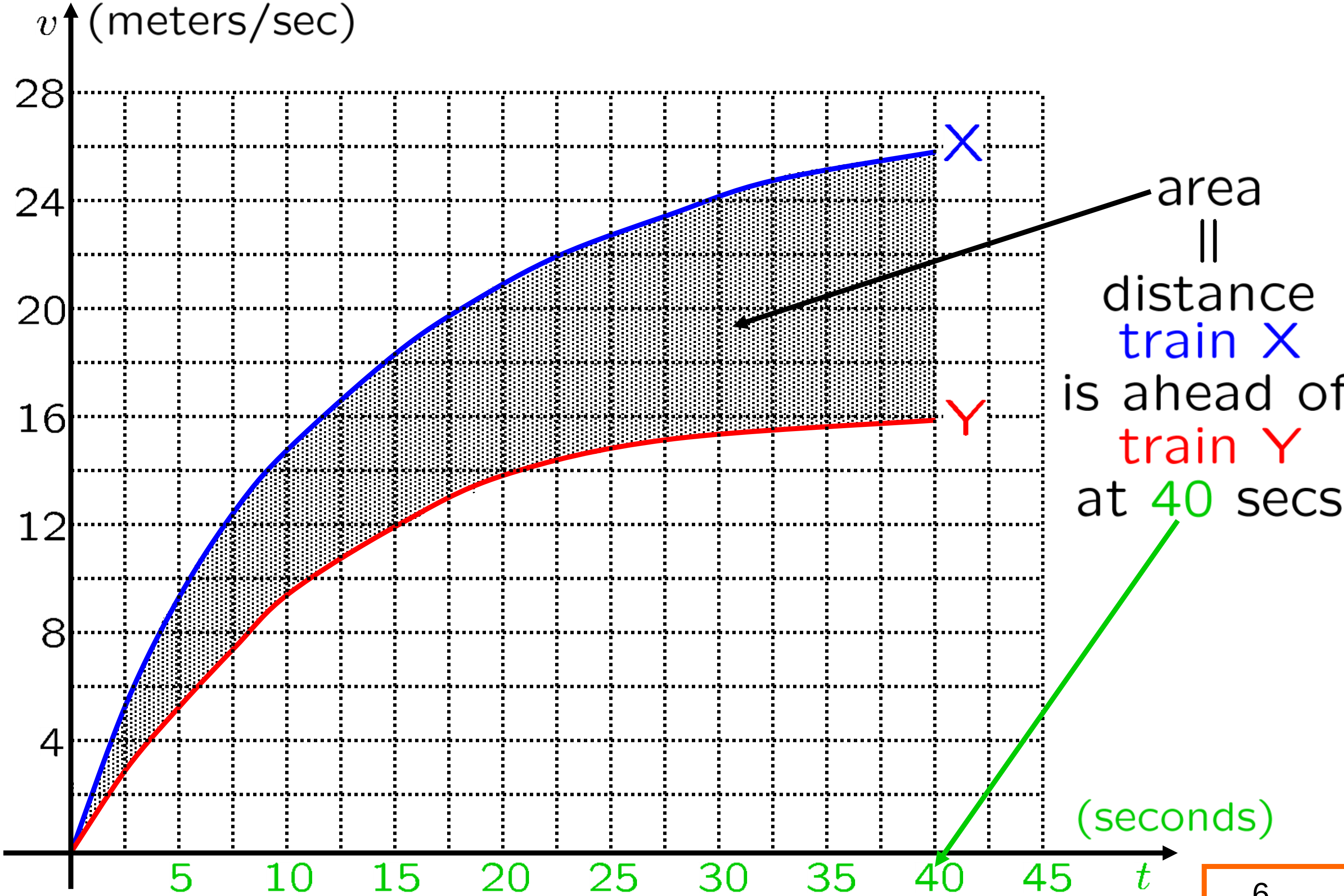

t	0	5	10	15	20	25	30	35	40
v_X	0	9.2 meters/sec							
v_Y	0	5.1							
$v_X - v_Y$	0	4.1							

Estimate the area using midpoints.

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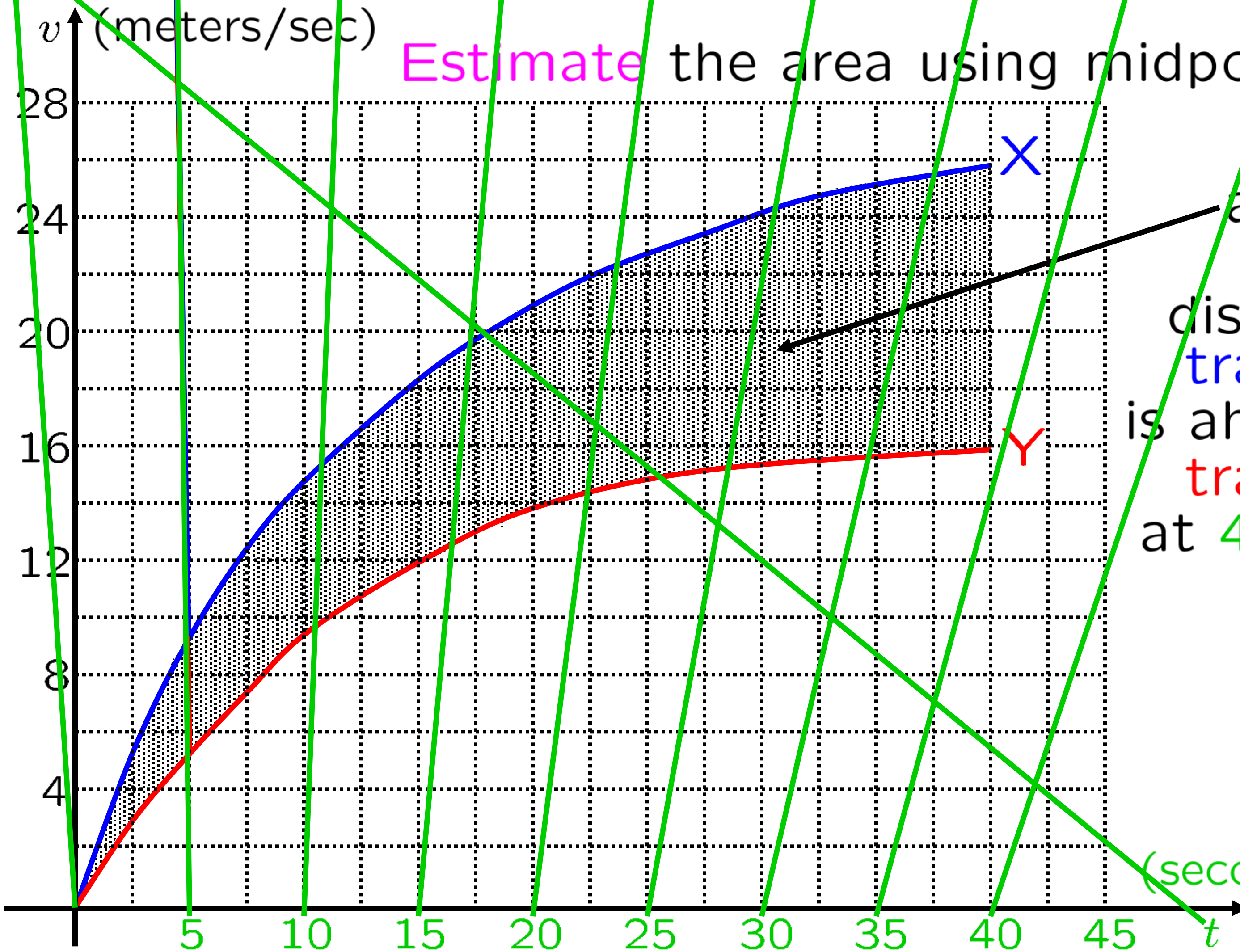
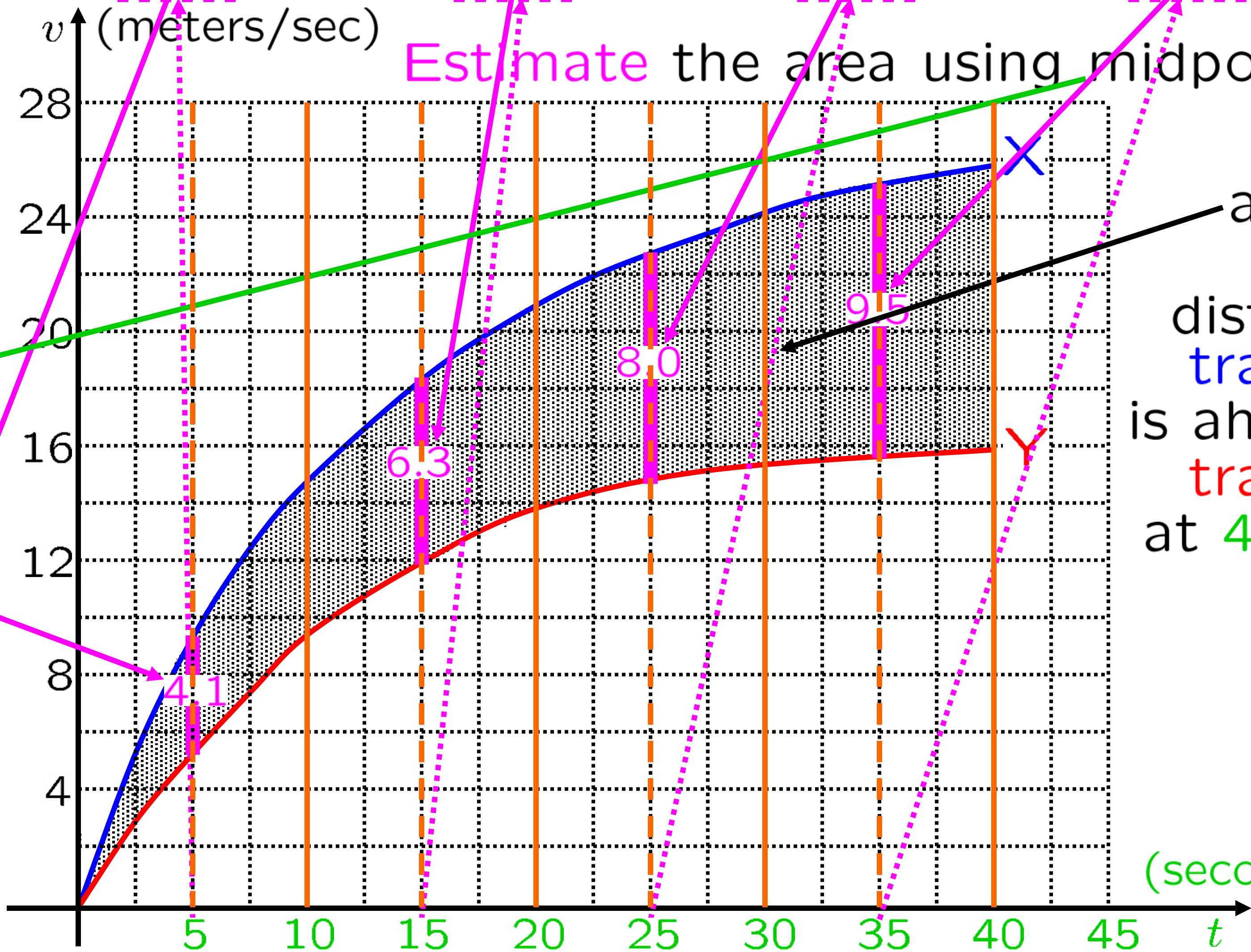


t	0	5	10	15	20	25	30	35	40
v_X	0	9.2	14.7	18.2	20.9	22.7	24.1	25.1	25.8
v_Y	0	5.1	9.3	11.9	13.7	14.7	15.3	15.6	15.9
$v_X - v_Y$	0	4.1	5.4	6.3	7.2	8.0	8.8	9.5	9.9

Let's choose FOUR subintervals
Mark midpoints.

Estimate the area using midpoints.



area
||
distance
train X
is ahead of
train Y
at 40 secs

t	0	5	10	15	20	25	30	35	40
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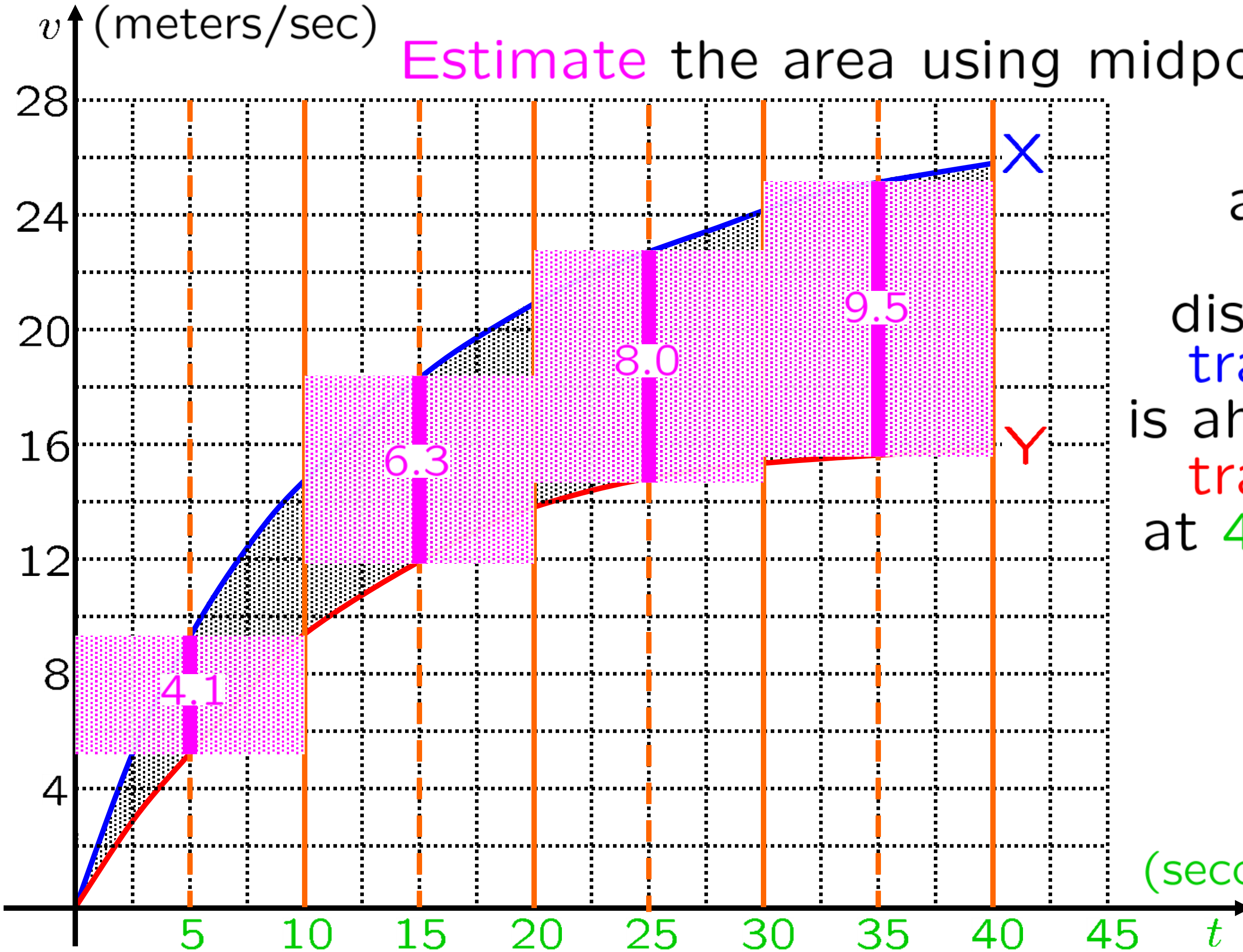

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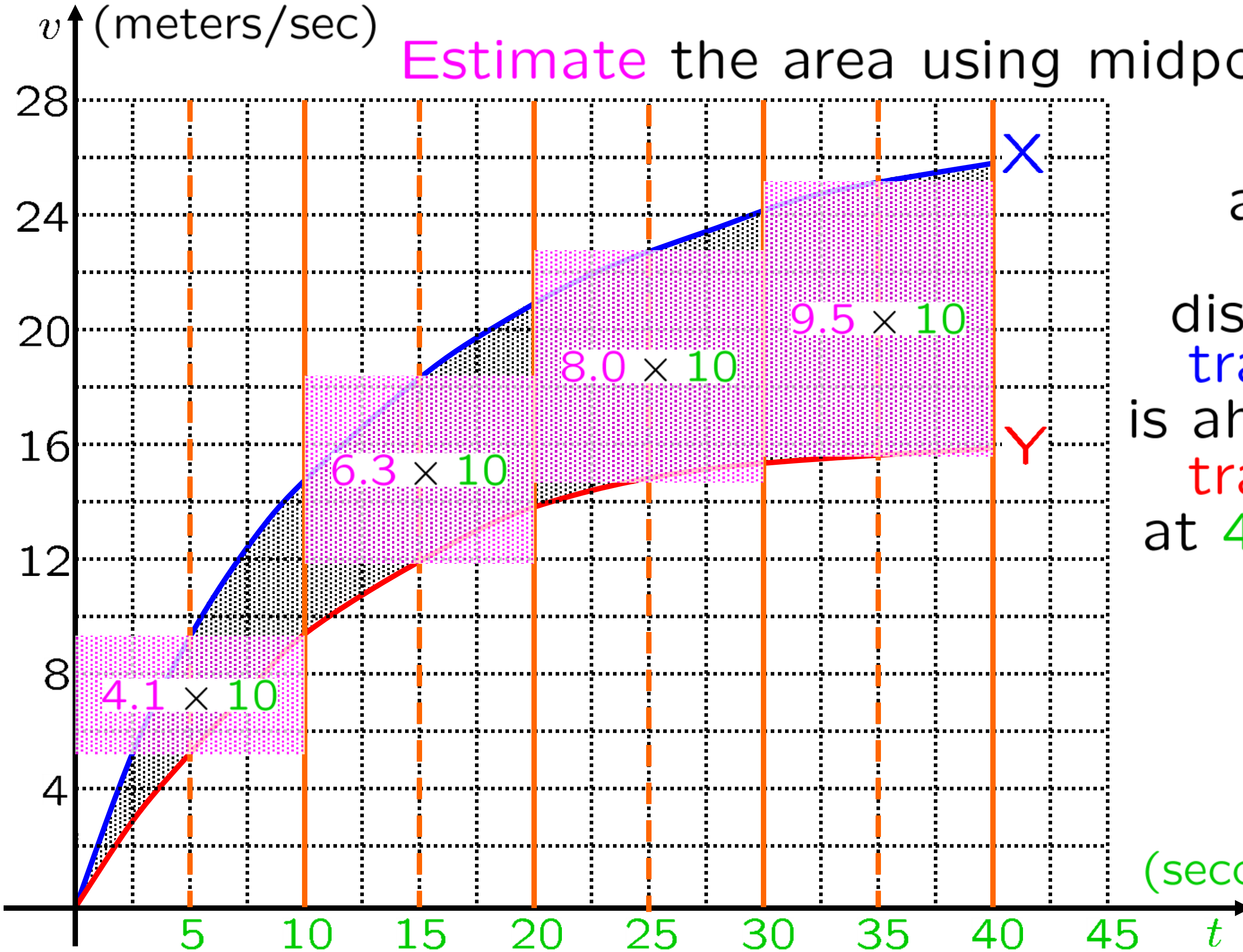

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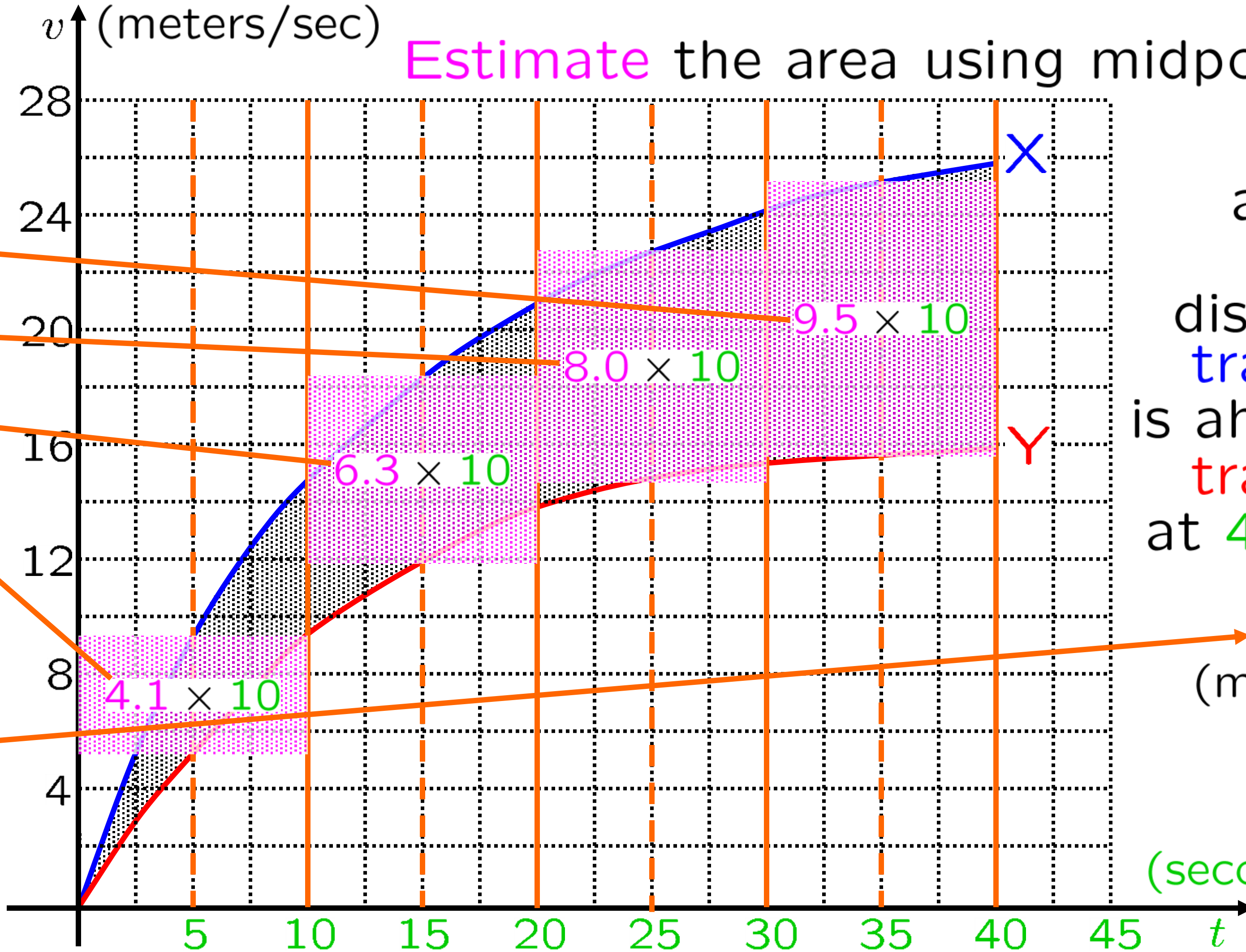
Estimate the area using midpoints.

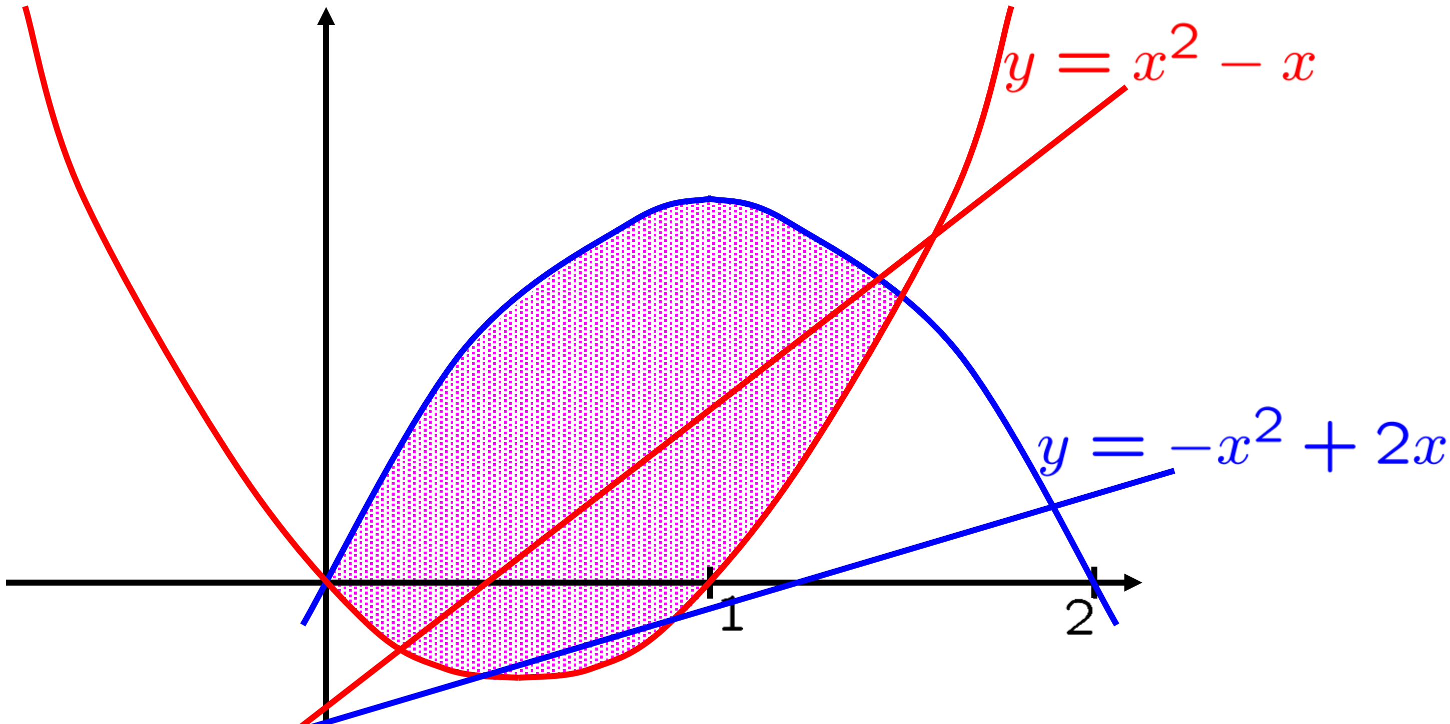
9.5×10
$+$
8.0×10
$+$
6.3×10
$+$
4.1×10
$\|$
27.9×10
$\|$
279 (meters)



area
$\|$
distance
train X
is ahead of
train Y
at 40 secs
$\approx$
279
(meters)
■

SKILL interpret & estimate area between curves

EXAMPLE: Find the shaded area, shown below.



$$-x^2 + 2x = x^2 - x \quad \Leftrightarrow \quad 0 = 2x^2 - 3x = 2x\left(x - \frac{3}{2}\right)$$

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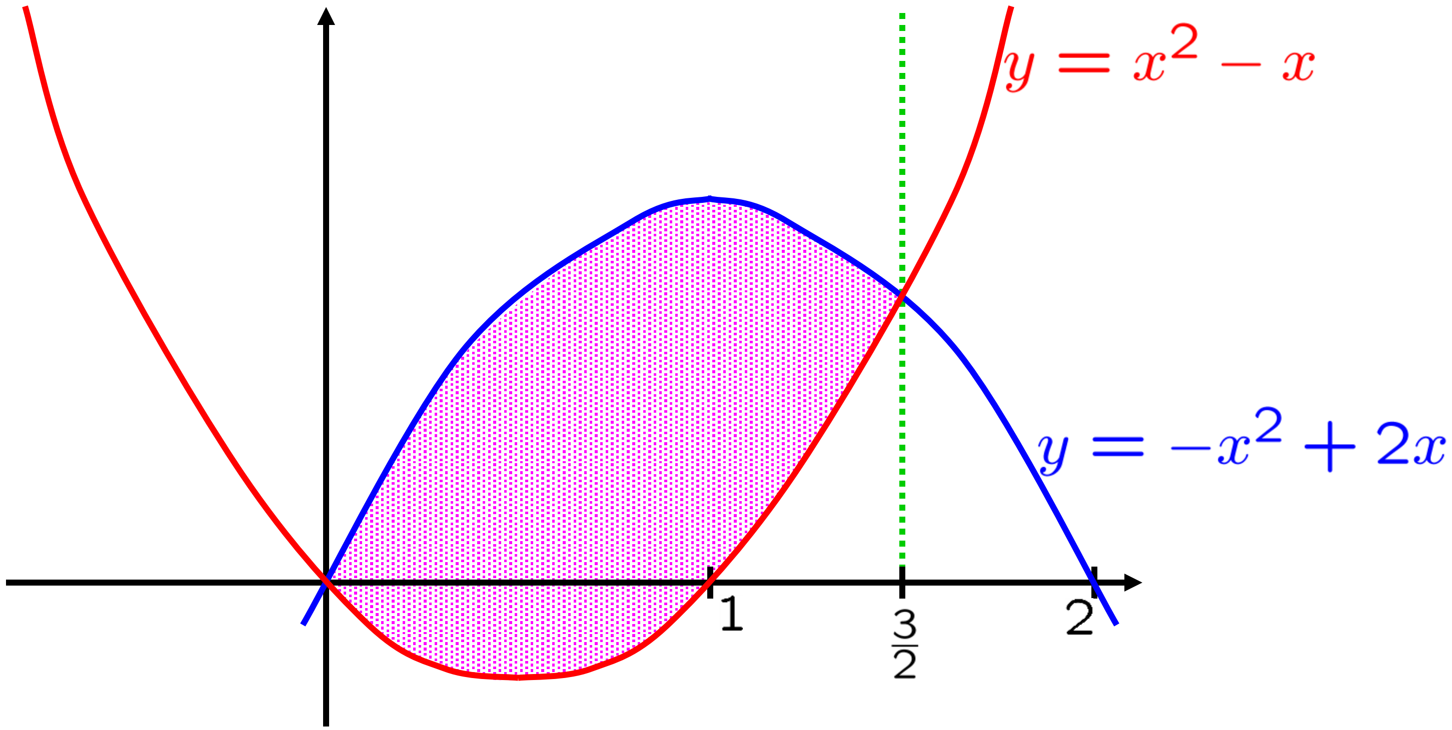

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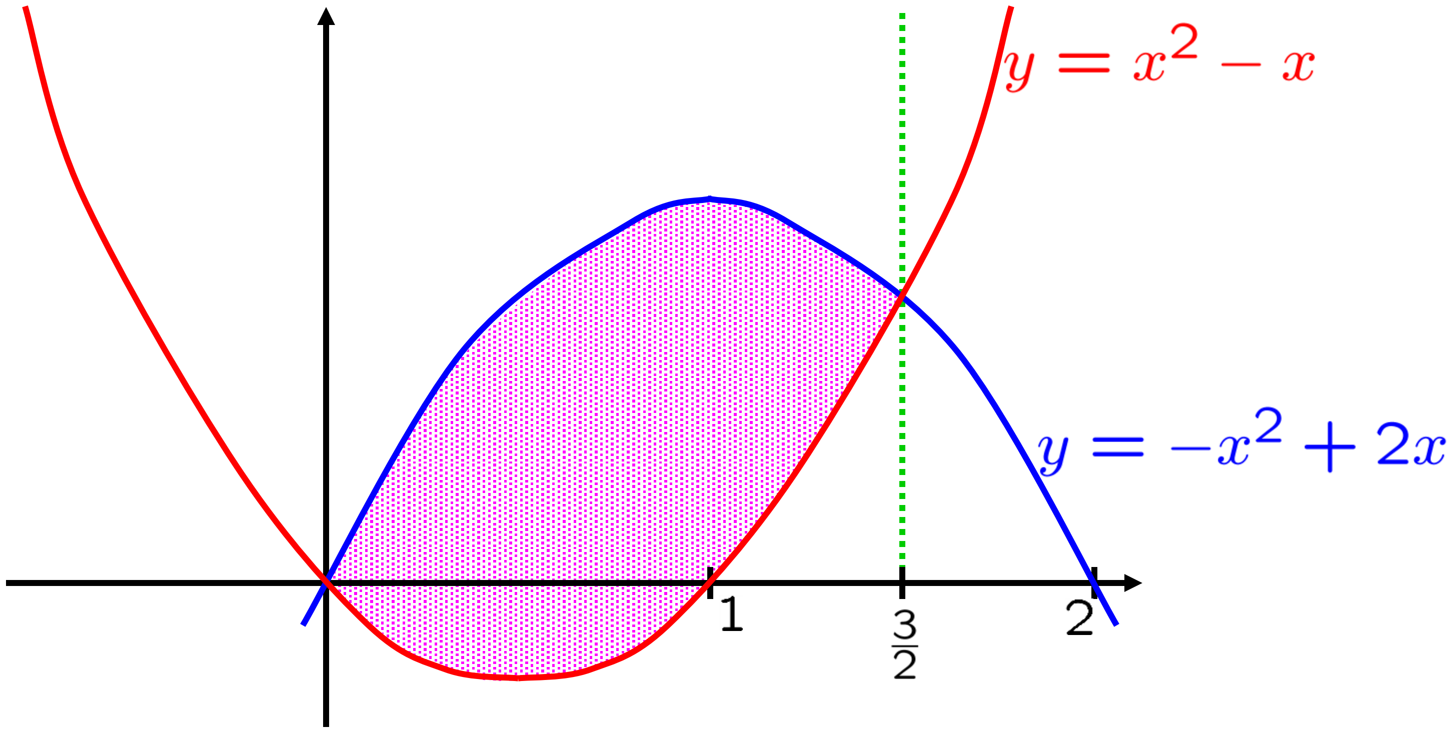


$$\int_0^{3/2} \left|(-x^2 + 2x) - (x^2 - x)\right| dx = \int_0^{3/2} (-2x^2 + 3x)\, dx$$

$$= \left[-2\frac{x^3}{3} + 3\frac{x^2}{2}\right]_{x:\to 0}^{x:\to 3/2} = \left[-2\frac{(3/2)^3}{3} + 3\frac{(3/2)^2}{2}\right] - [0]$$

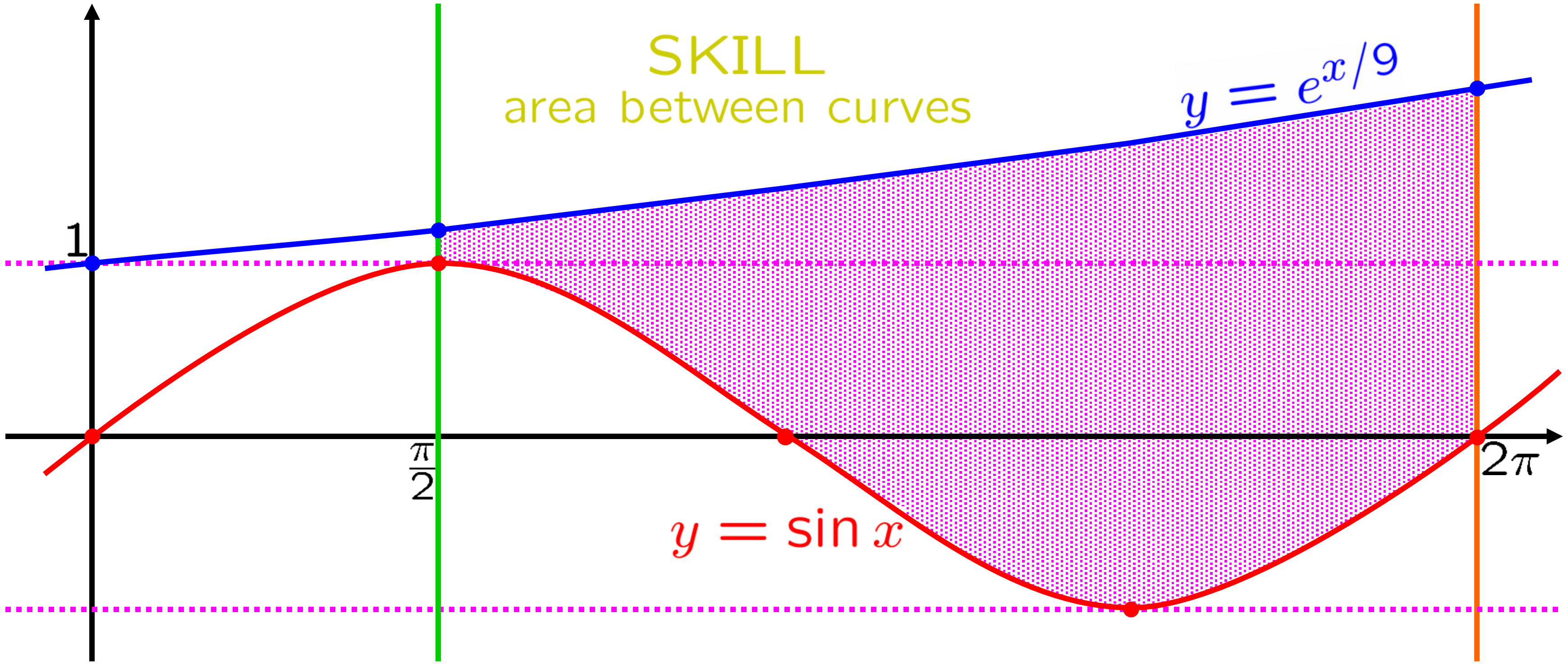
$$= -\frac{18}{8} + \frac{27}{8} = \frac{9}{8} \quad \blacksquare$$

SKILL
area between curves

EXAMPLE: Sketch the region enclosed by

$y = \sin x$, $y = e^{x/9}$, $x = \pi/2$ and $x = 2\pi$.

Find the area of that region.



$$\int_{\pi/2}^{2\pi} \left(e^{x/9} - \sin x\right)\, dx = \left[\frac{e^{x/9}}{1/9} - (-\cos x)\right]_{x:\to\pi/2}^{x:\to 2\pi} = \left[\frac{e^{x/9}}{1/9} + \cos x\right]_{x:\to\pi/2}^{x:\to 2\pi}$$

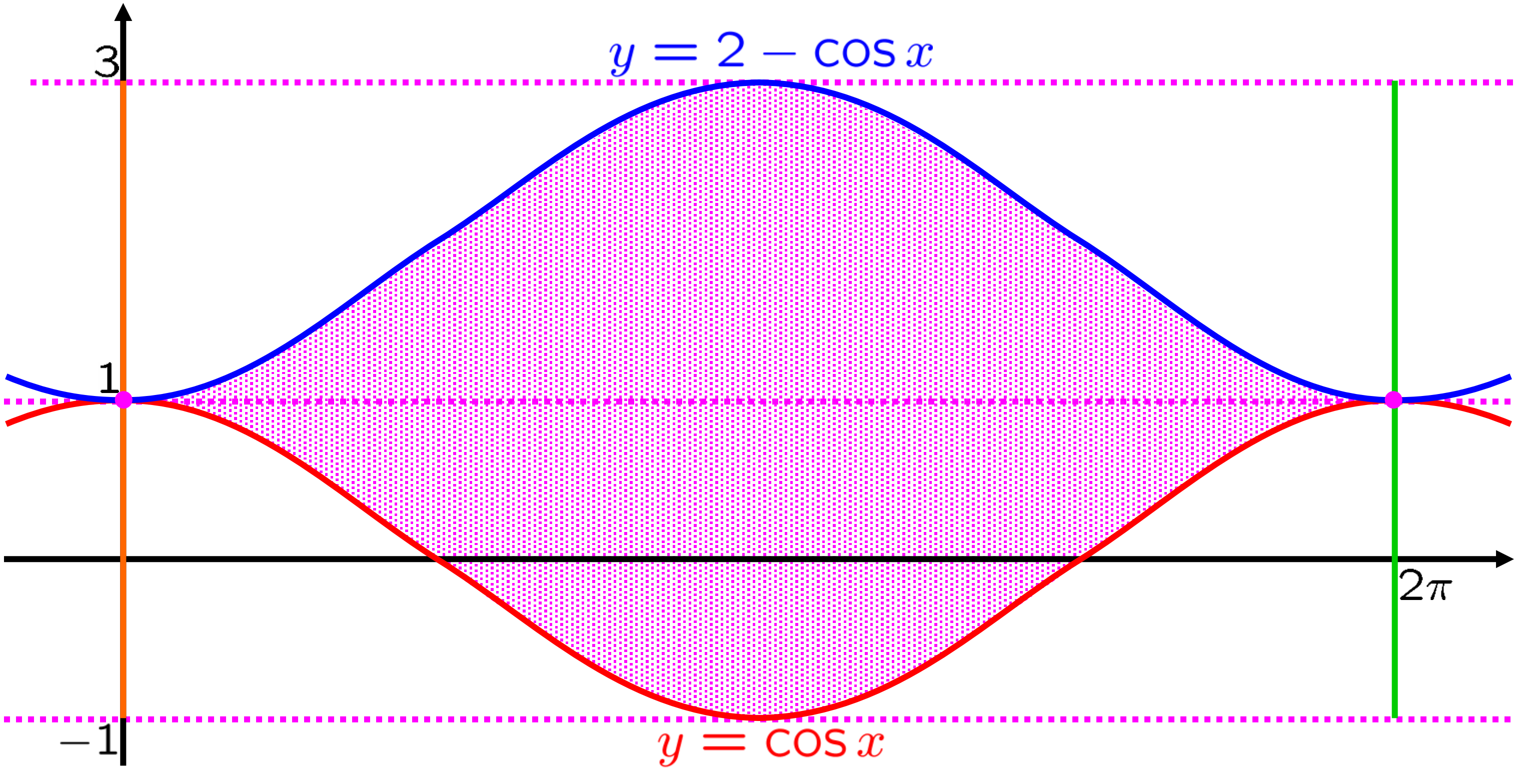
$$= \left[\frac{e^{2\pi/9} - e^{\pi/18}}{1/9}\right] + [(\cos(2\pi)) - (\cos(\pi/2))]$$

$$= 9\left[e^{2\pi/9} - e^{\pi/18}\right] + [1 - 0] = 9\left[e^{2\pi/9} - e^{\pi/18}\right] + 1 \blacksquare$$

EXAMPLE: Sketch the region enclosed by

$$y = 2 - (\cos x), \quad y = \cos x, \quad x = 0 \quad \text{and} \quad x = 2\pi.$$

Find the area of that region.



$$\int_0^{2\pi} [(2 - \cos x) - (\cos x)]\, dx = \int_0^{2\pi} [2 - 2\cos x]\, dx$$

SKILL
area between curves

$$= [2x - 2\sin x]_{x:\to 0}^{x:\to 2\pi}$$

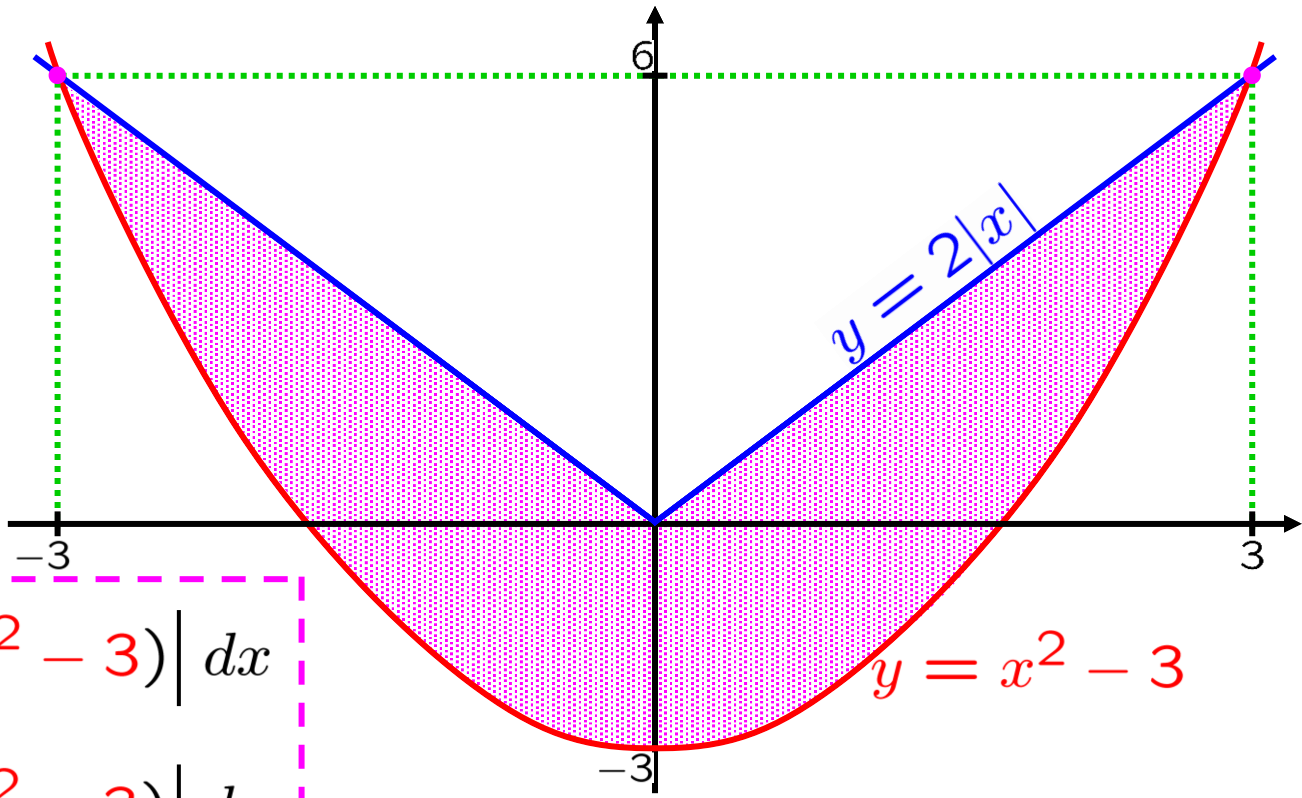
$$= 2[2\pi - 0] - 2[(\sin(2\pi)) - (\sin(0))]$$

$$= 2[2\pi - 0] - 2[0 - 0] = 4\pi \blacksquare$$

EXAMPLE: Sketch the region enclosed by

$$y = 2|x| \quad \text{and} \quad y = x^2 - 3.$$

Find the area of that region.



$$\int_{-3}^{3} \left| (2|x|) - (x^2 - 3) \right| dx$$

$$= 2\int_{0}^{3} \left| (2|x|) - (x^2 - 3) \right| dx$$

$$= 2\int_{0}^{3} (-x^2 + 2x + 3)\, dx$$

$$= 2\left[-\frac{x^3}{3} + x^2 + 3x \right]_{x:\to 0}^{x:\to 3} = 2\left[-\frac{3^3}{3} + 3^2 + 3\cdot 3 \right] = 18$$

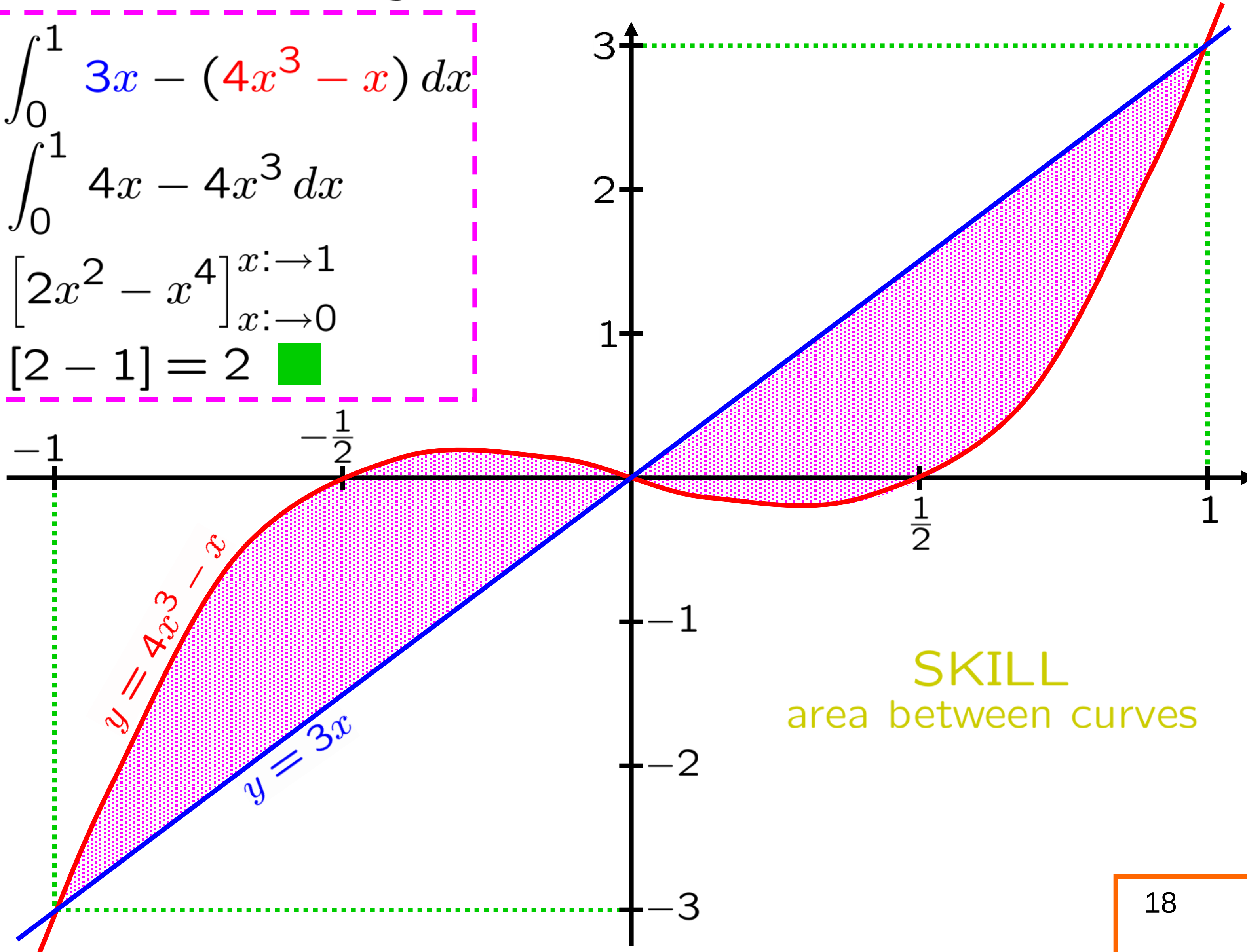
SKILL
area between curves

EXAMPLE: Sketch the region enclosed by

$$y = 3x \quad \text{and} \quad y = 4x^3 - x$$

Find the area of that region.

$$\begin{aligned} \text{Area} &= 2\int_0^1 3x - (4x^3 - x)\,dx \\ &= 2\int_0^1 4x - 4x^3\,dx \\ &= 2\left[2x^2 - x^4\right]_{x:\to 0}^{x:\to 1} \\ &= 2\,[2-1] = 2 \end{aligned}$$

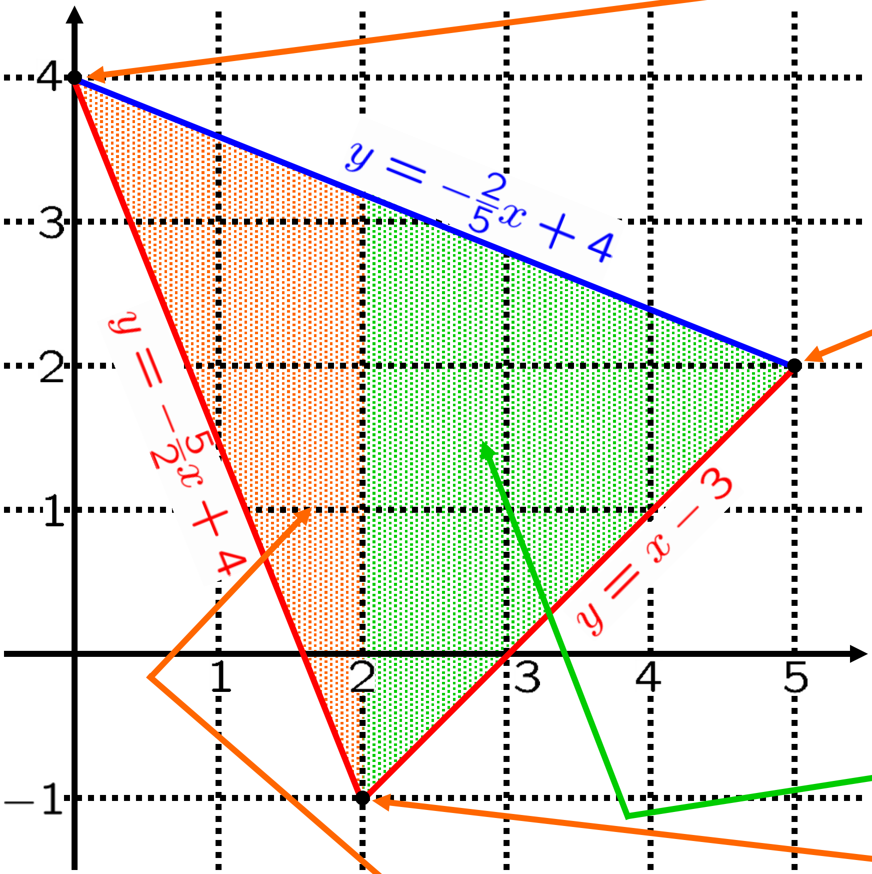


SKILL
area between curves

EXAMPLE: Use calculus to find the area of the triangle whose vertices are: $(0,4)$, $(2,-1)$ and $(5,2)$.

SKILL
area of triangle from vertices

■



$$\left(\frac{21}{10}\right)\frac{2^2}{2} + \left(-\frac{7}{5}\right)\frac{5^2-2^2}{2} + 7(5-2)$$

$$\|$$

$$\int_0^2 \left(\frac{21}{10}\right) x\,dx + \int_2^5 \left(-\frac{7}{5}\right) x + 7\,dx$$

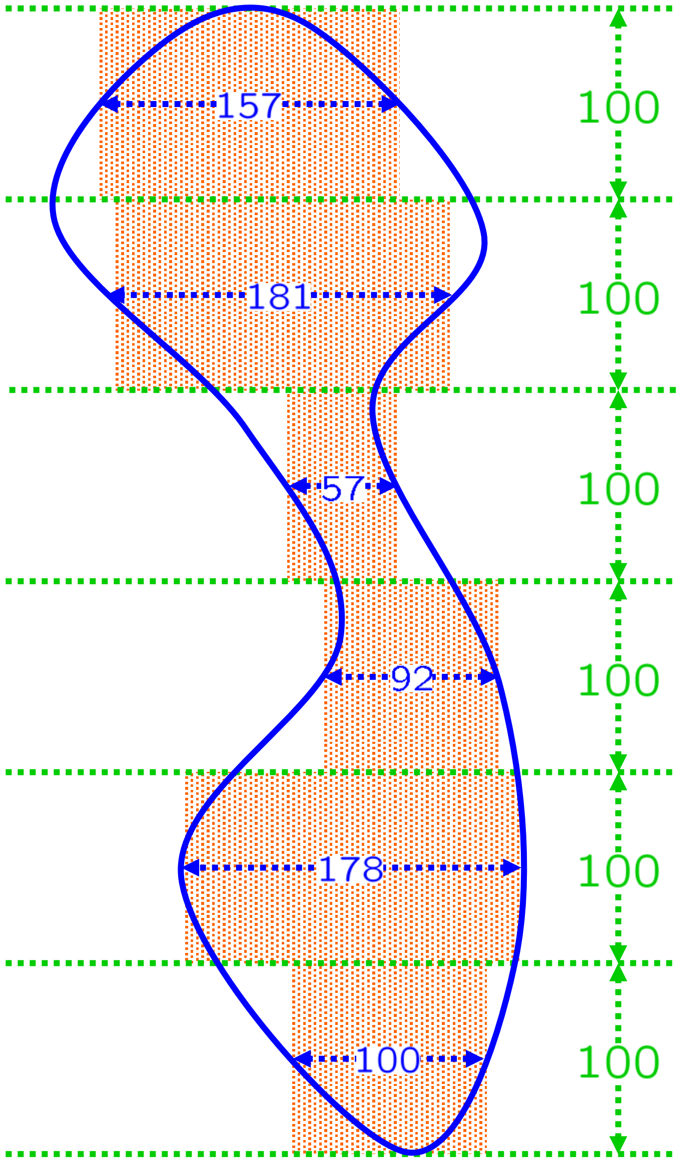
$$\|$$

$$\int_0^2 \left(-\frac{2}{5} + \frac{5}{2}\right) x\,dx + \int_2^5 \left(-\frac{2}{5} - 1\right) x + 7\,dx$$

$$//$$

$$\int_0^2 \left(-\frac{2}{5}x + 4\right) - \left(-\frac{5}{2}x + 4\right)\,dx + \int_2^5 \left(-\frac{2}{5}x + 4\right) - (x - 3)\,dx$$

EXAMPLE: An irregular property has been surveyed and is shown below, with measurements made of some cross-sections. Estimate its area.

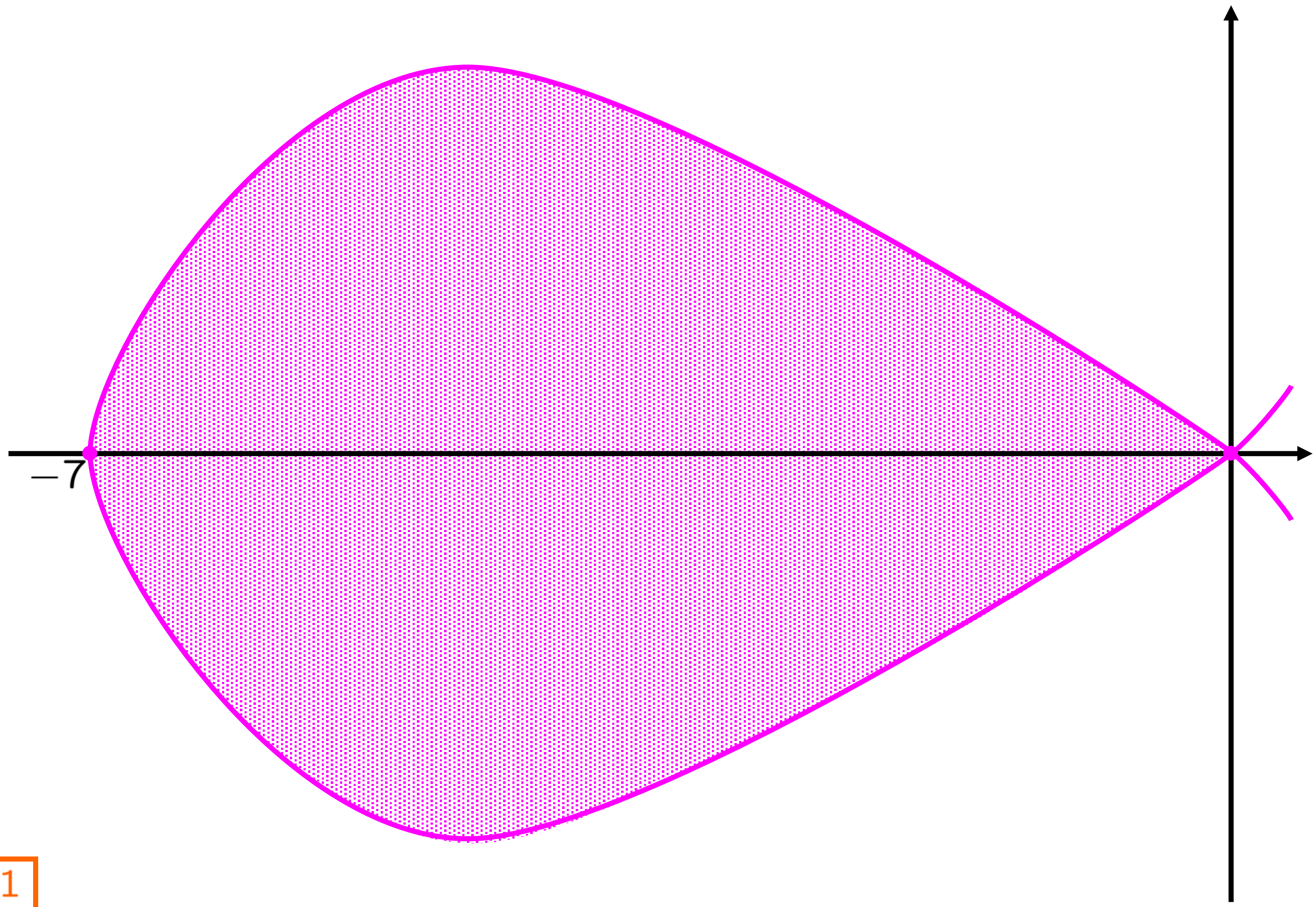


$$\left.\begin{array}{r} 157 \times 100 \\ + \\ 181 \times 100 \\ + \\ 57 \times 100 \\ + \\ 92 \times 100 \\ + \\ 178 \times 100 \\ + \\ 100 \times 100 \end{array}\right\} = 765 \times 100 = 76,500 \blacksquare$$

SKILL
estimate area,
given cross-sections

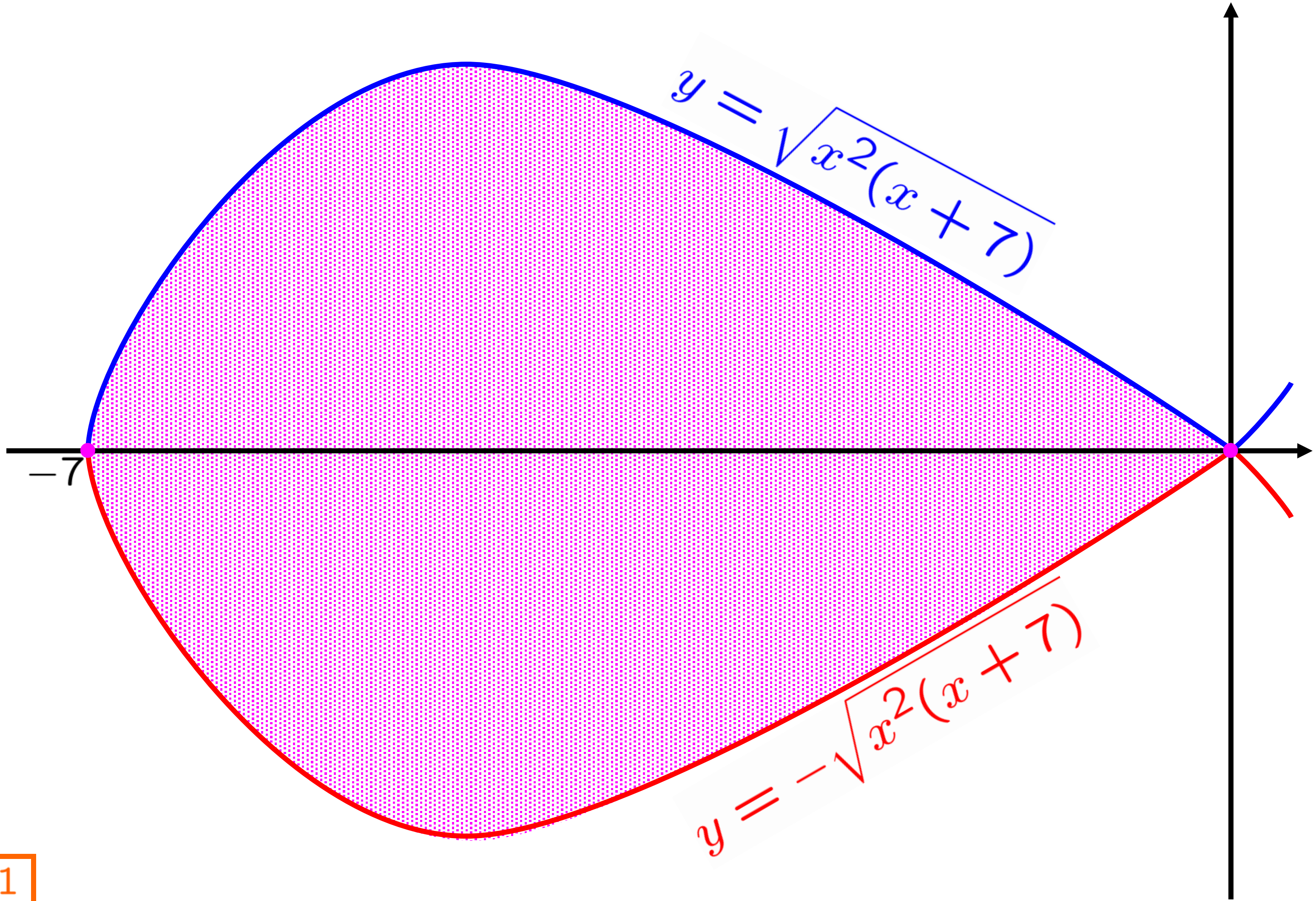
§9.1

EXAMPLE: Part of the graph of $y^2 = x^2(x+7)$ forms a loop. Find the area enclosed by that loop.



EXAMPLE: Part of the graph of $y^2 = x^2(x+7)$ forms a loop. Find the area enclosed by that loop.

$$\int_{-7}^{0} \left| \left(\sqrt{x^2(x+7)}\right) + \left(+\sqrt{x^2(x+7)}\right) \right| dx$$



EXAMPLE: Part of the graph of $y^2 = x^2(x+7)$ forms a loop. Find the area enclosed by that loop.

$$\int_{-7}^{0} \left(\sqrt{x^2(x+7)}\right) + \left(+\sqrt{x^2(x+7)}\right)\, dx$$

$$= 2\int_{-7}^{0} \sqrt{x^2(x+7)}\, dx$$

$\sqrt{x^2} = -x, \quad \forall x < 0$

$$= 2\int_{-7}^{0} \sqrt{x^2}\sqrt{x+7}\, dx$$

$$= 2\int_{-7}^{0} (-x)\sqrt{x+7}\, dx$$

$$= -2\int_{-7}^{0} x\sqrt{x+7}\, dx$$

$x = u - 7$
$u := x + 7$
$du = dx$

$$= -2\int_{-7+7}^{0+7} (u-7)\sqrt{u}\, du$$

$$= -2\int_{0}^{7} (u-7)u^{1/2}\, du$$

EXAMPLE: Part of the graph of $y^2 = x^2(x+7)$ forms a loop. Find the area enclosed by that loop.

$$\int_{-7}^{0} \left(\sqrt{x^2(x+7)}\right) + \left(+\sqrt{x^2(x+7)}\right)\, dx$$

$$u := x+7$$
$$du = dx$$

$$= -2\int_0^7 (u-7)u^{1/2}\, du$$

$$= -2\int_0^7 u^{3/2} - 7u^{1/2}\, du$$

$$u := x+7$$
$$du = dx$$

$$= -2\int_0^7 (u-7)u^{1/2}\, du$$

EXAMPLE: Part of the graph of $y^2 = x^2(x+7)$ forms a loop. Find the area enclosed by that loop.

$$\int_{-7}^{0} \left(\sqrt{x^2(x+7)}\right) + \left(+\sqrt{x^2(x+7)}\right)\,dx$$

$$u := x + 7$$
$$du = dx$$

$$= -2\int_0^7 (u-7)u^{1/2}\,du$$

$$= -2\int_0^7 u^{3/2} - 7u^{1/2}\,du$$

$$= -2\left[\frac{u^{5/2}}{5/2} - 7\left(\frac{u^{3/2}}{3/2}\right)\right]_{u:\to 0}^{u:\to 7}$$

$$= -2\left[\frac{7^{5/2}}{5/2} - 7\left(\frac{7^{3/2}}{3/2}\right)\right]$$

$$= -2\left[\frac{2}{5} - \frac{2}{3}\right] \cdot 7^{5/2}$$

SKILL
area between curves

$$= -2\left[\frac{6}{15} - \frac{10}{15}\right] \cdot 7^{5/2} = \frac{8}{15} \cdot 7^{5/2} \blacksquare$$

SKILL

area between curves

Whitman problems

§9.1, p. 182, #1-12