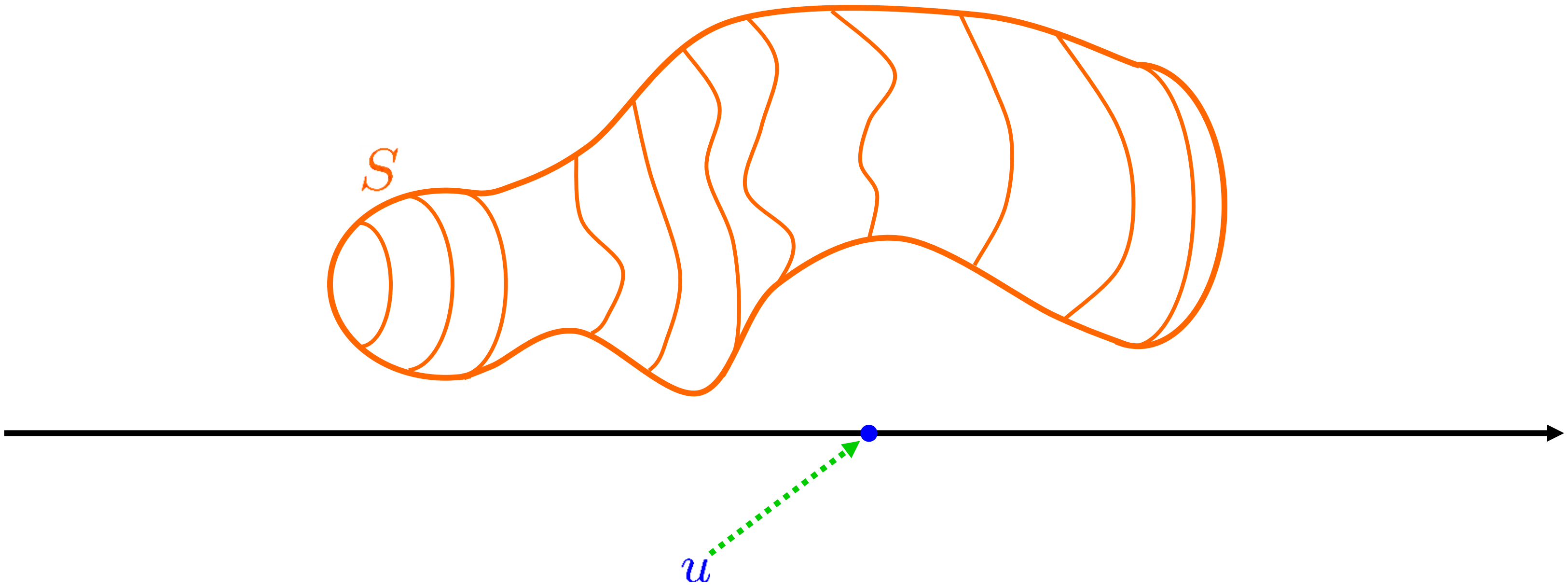
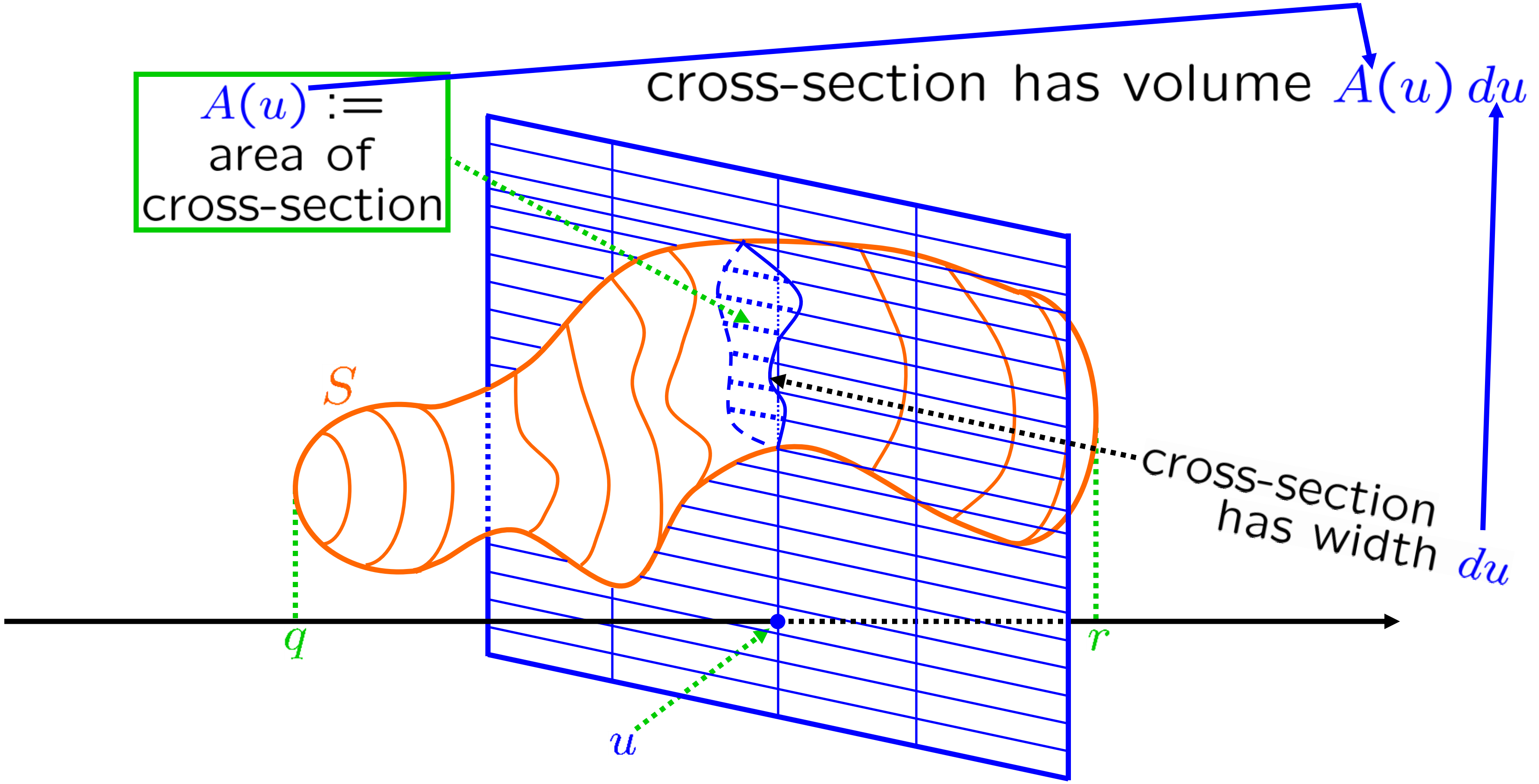
CALCULUS
Volume by slices



Goal: Compute the volume, V, inside the orange solid, S, above.



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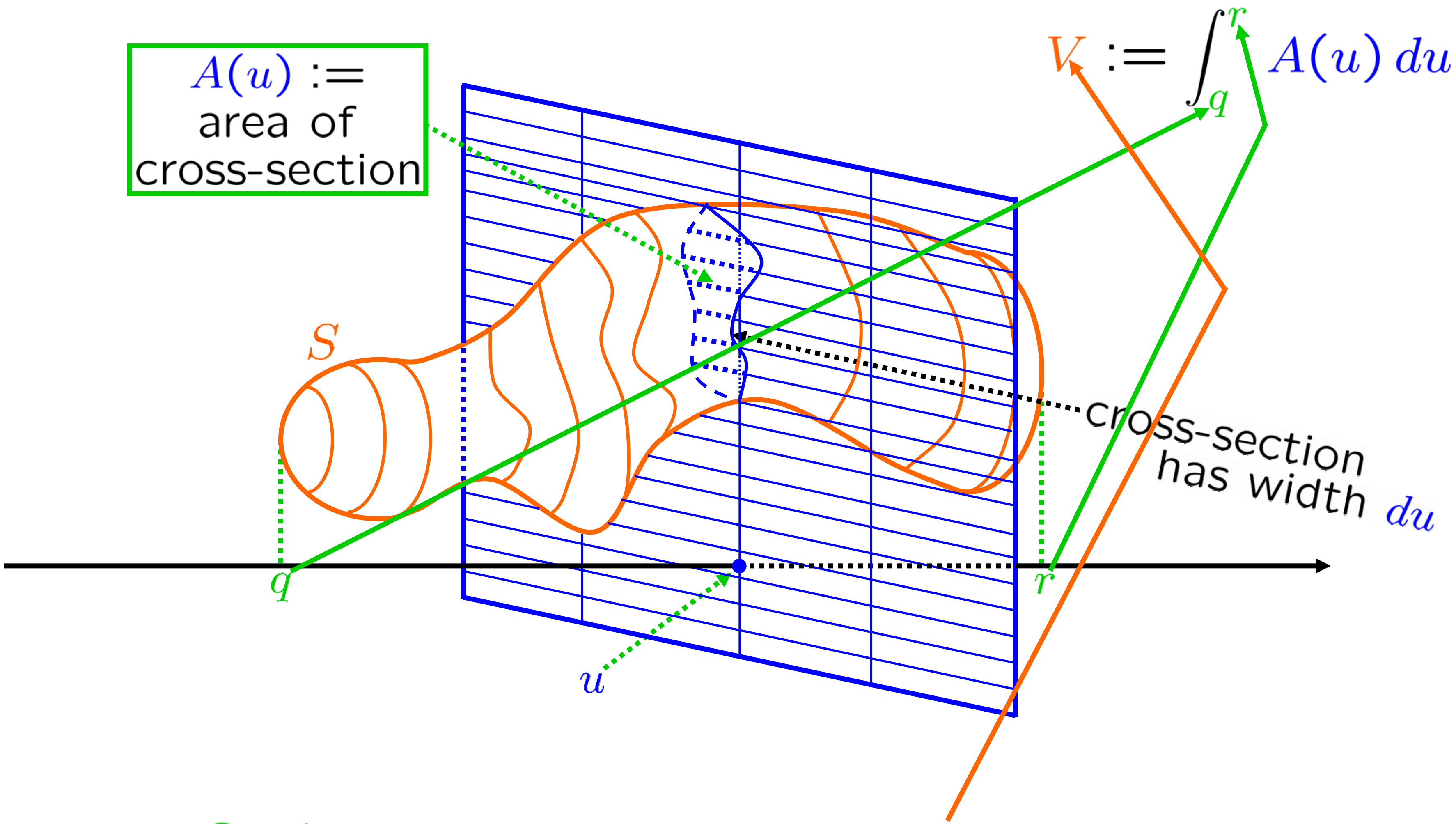

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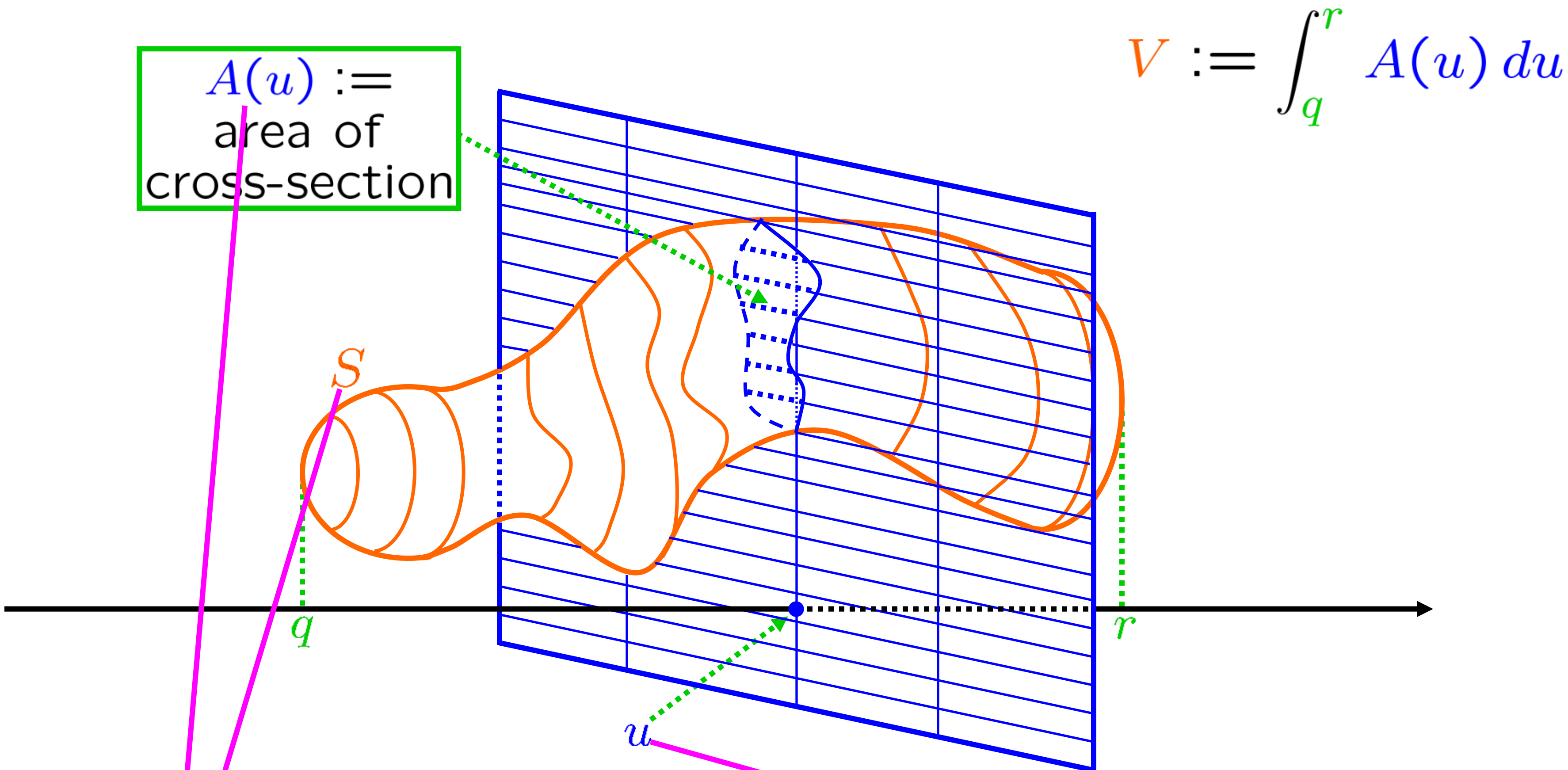
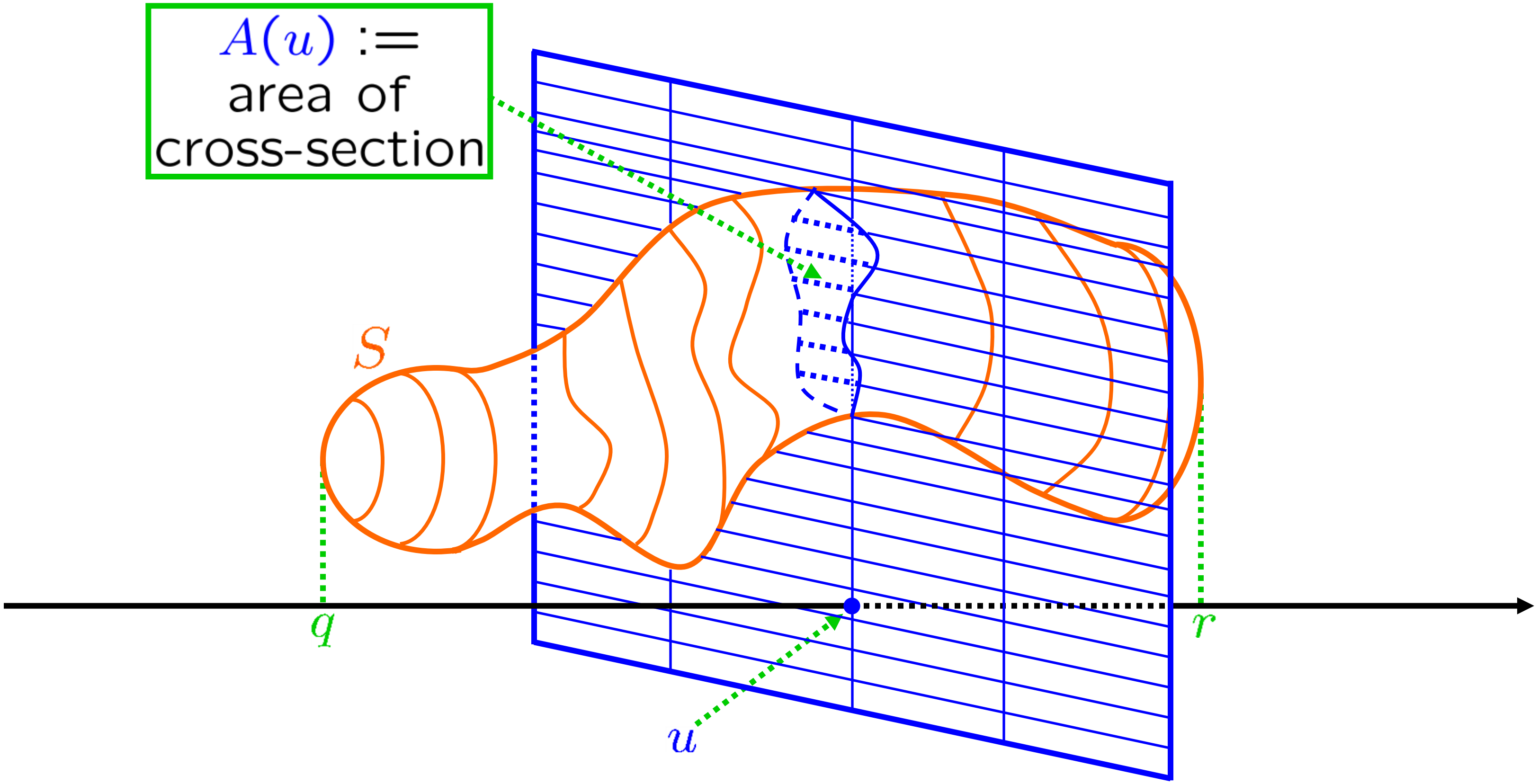


DEFINITION OF VOLUME:

Let S be a solid that lies between the planes through q and r.
If the cross-sectional area of S in the plane through u is $A(u)$, where A is a continuous function,
then the **volume of** S is $\int_q^r A(u)\,du$.



NOTE: The line need not be horizontal, and,
if horizontal, need not point to the right.

We could use other variables besides u.

In the next example (Cavalieri's Principle),
we'll use a vertical line pointing downward,
and the variable "z".

EXAMPLE: Prove **Cavalieri's principle**, which asserts that

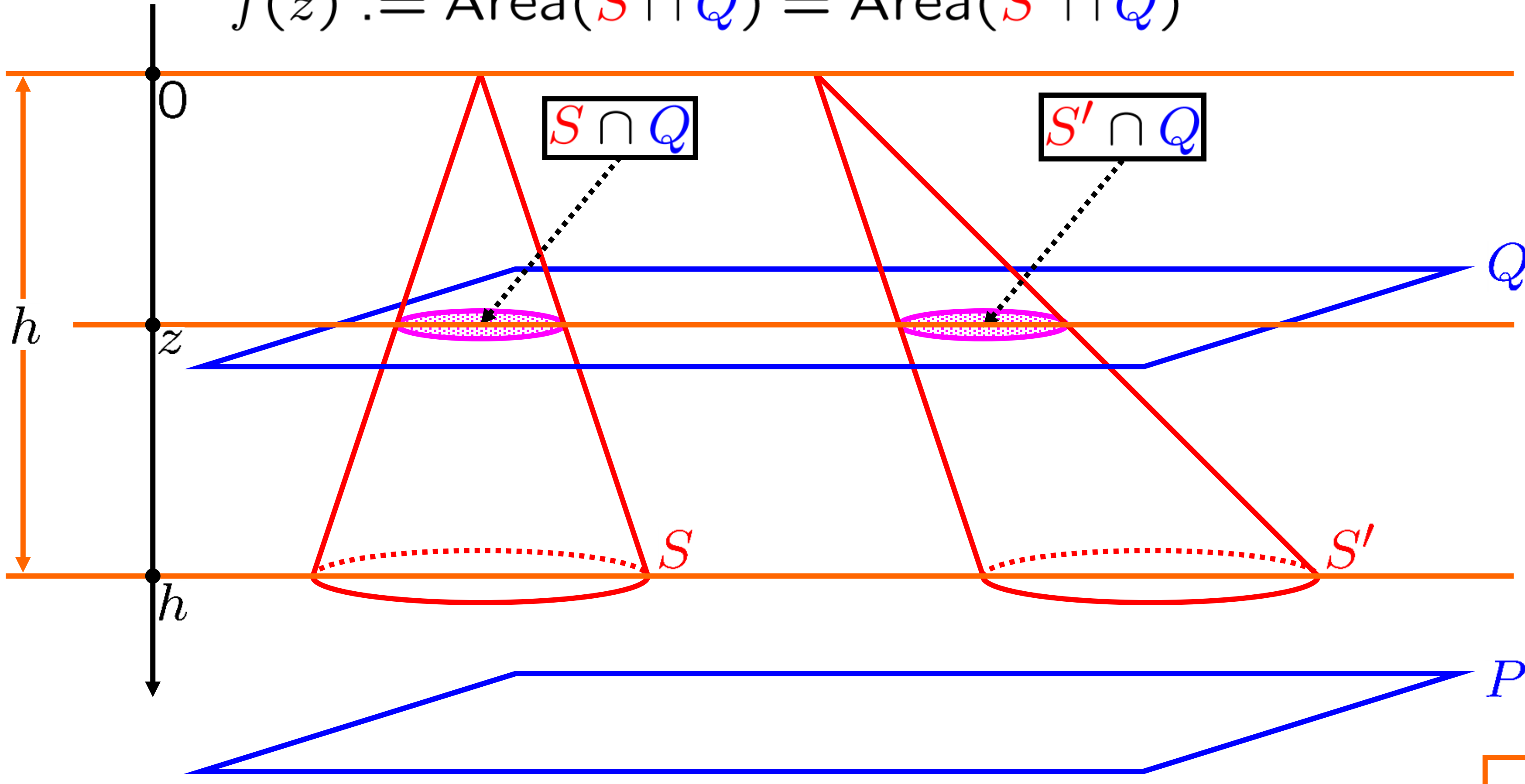
if P is a plane, if S and S' are solids

and if, for any plane Q parallel to P,

$S \cap Q$ has the same area as $S' \cap Q$,

then S has the same volume as S'.

$f(z) := \text{Area}(S \cap Q) = \text{Area}(S' \cap Q)$



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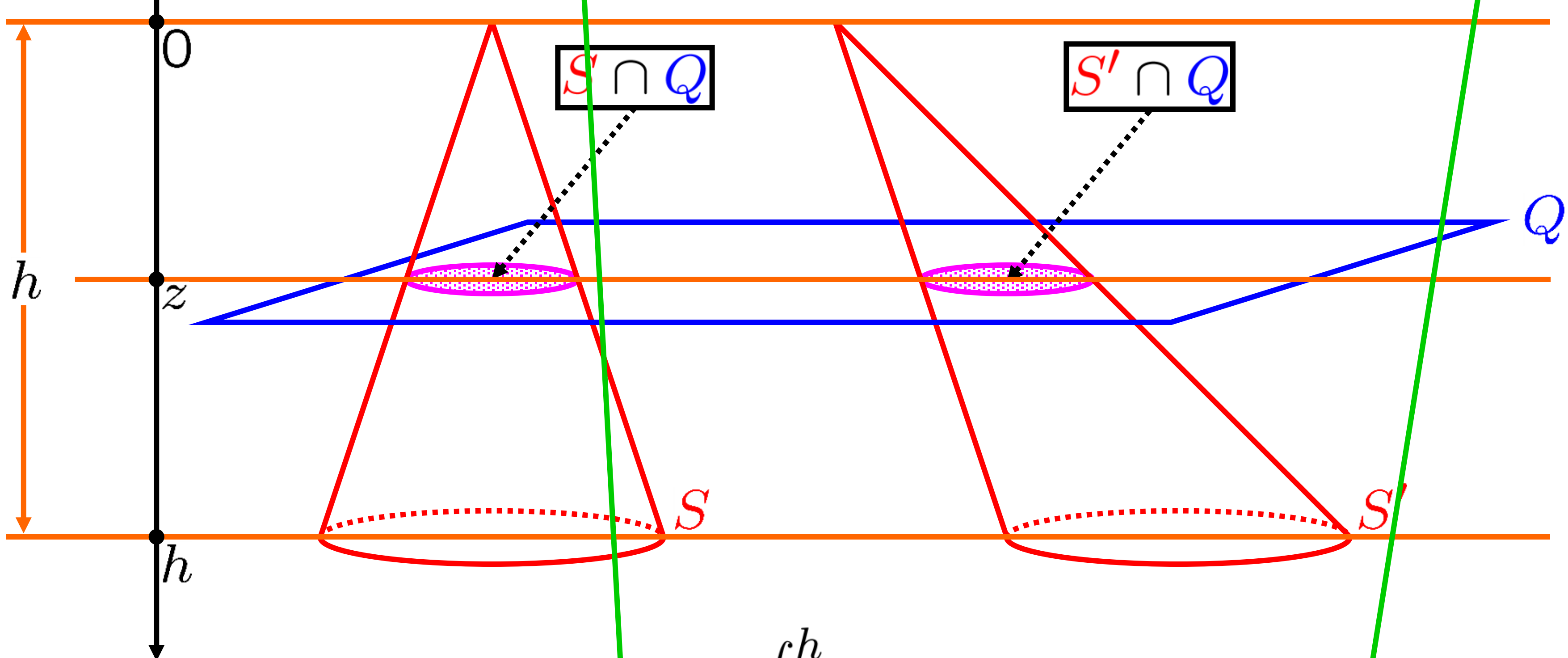
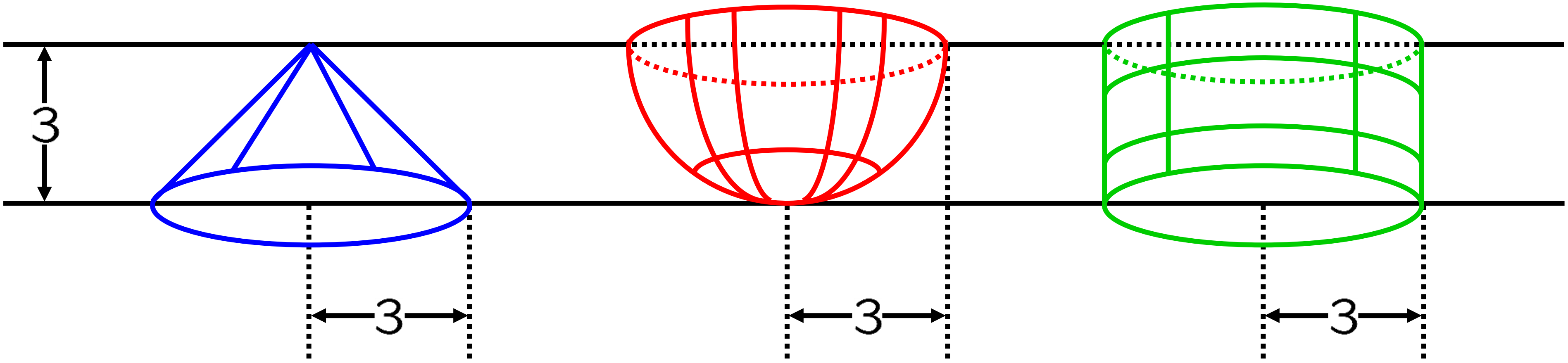


Volume$(S) = \int_0^h f(z)\, dz =$ Volume(S') QED

EXAMPLE: Using Cavalieri's principle, show, in the figures below, that the sum of the volumes of the cone and hemisphere is equal to the volume of the cylinder.



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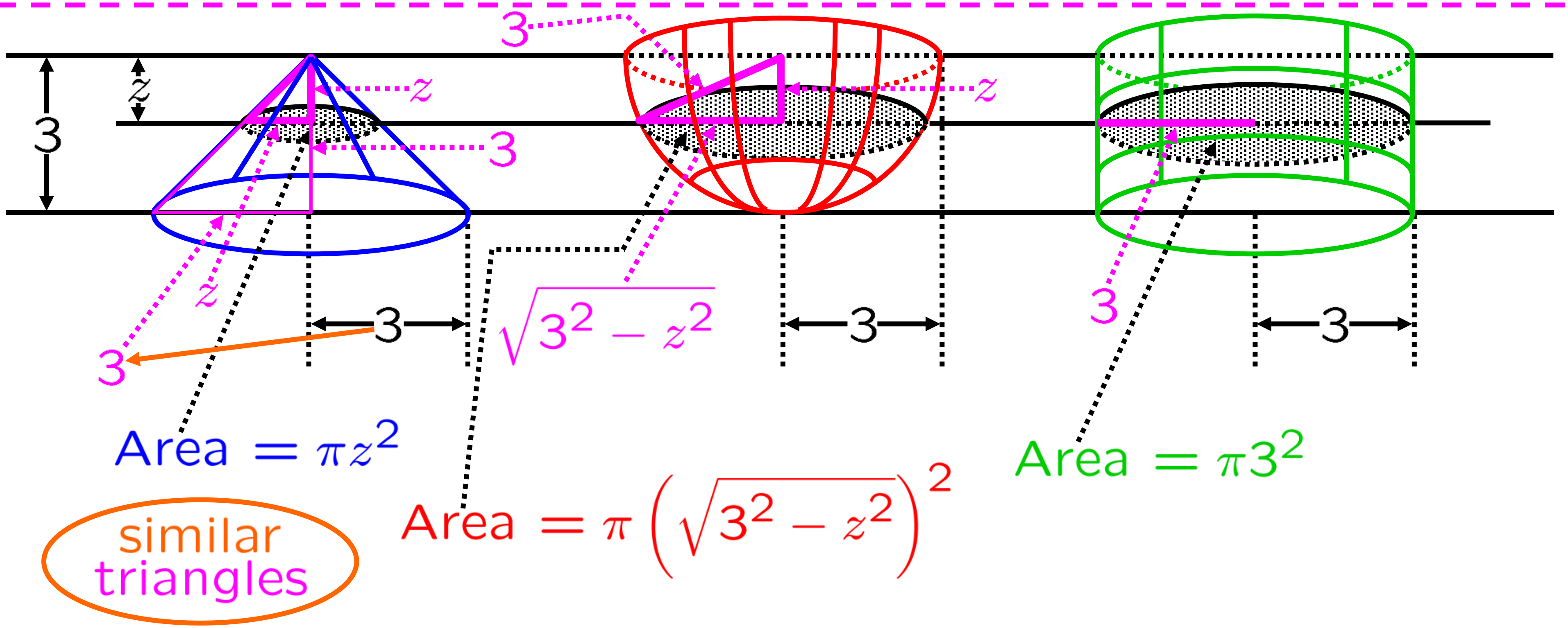

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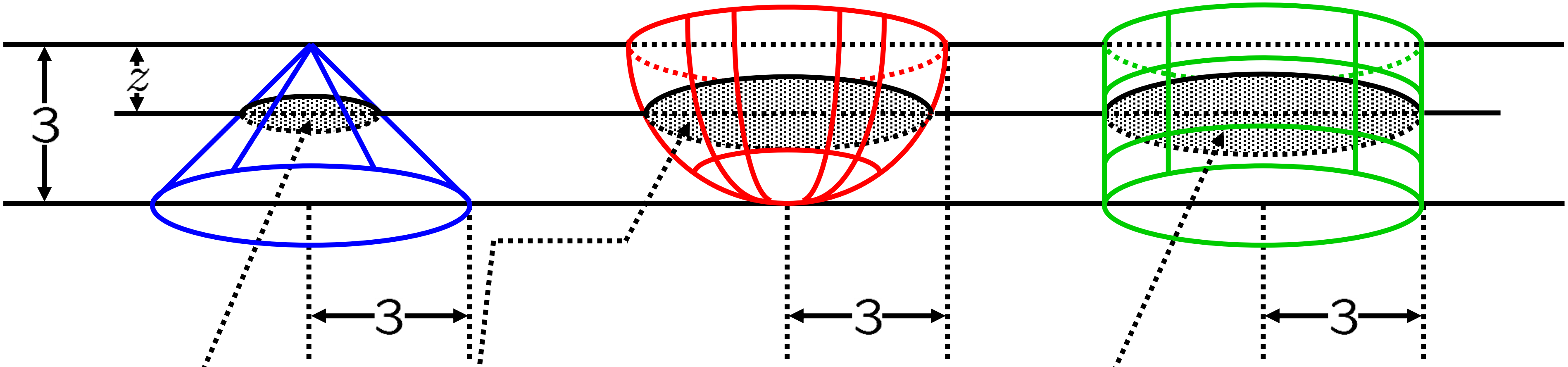

Area $= \pi z^2$

Area $= \pi \left(\sqrt{3^2 - z^2} \right)^2$

Area $= \pi 3^2$

Vol(cone)	Vol(hemisphere)	Vol(cylinder)
$\Vert$	$\Vert$	$\Vert$
$\int_0^3 \pi z^2 \, dz$	$\int_0^3 \pi \left(\sqrt{3^2 - z^2} \right)^2 dz$	$\int_0^3 \pi 3^2 \, dz$

$$\pi z^2 \quad + \quad \pi \left(\sqrt{3^2 - z^2} \right)^2 \quad = \quad \pi 3^2$$

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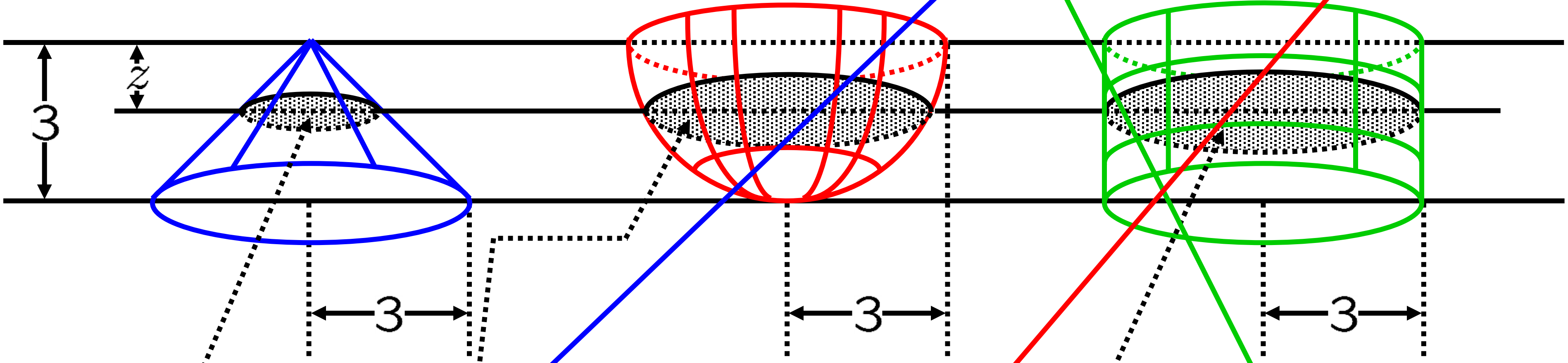

Area $= \pi z^2$

Area $= \pi \left(\sqrt{3^2 - z^2}\right)^2$

Area $= \pi 3^2$

Vol(cone) + Vol(hemisphere) = Vol(cylinder)

QED

$$\int_0^3 \pi z^2 \, dz + \int_0^3 \pi \left(\sqrt{3^2 - z^2}\right)^2 dz = \int_0^3 \pi 3^2 \, dz$$

$$\pi z^2 + \pi \left(\sqrt{3^2 - z^2}\right)^2 = \pi 3^2$$

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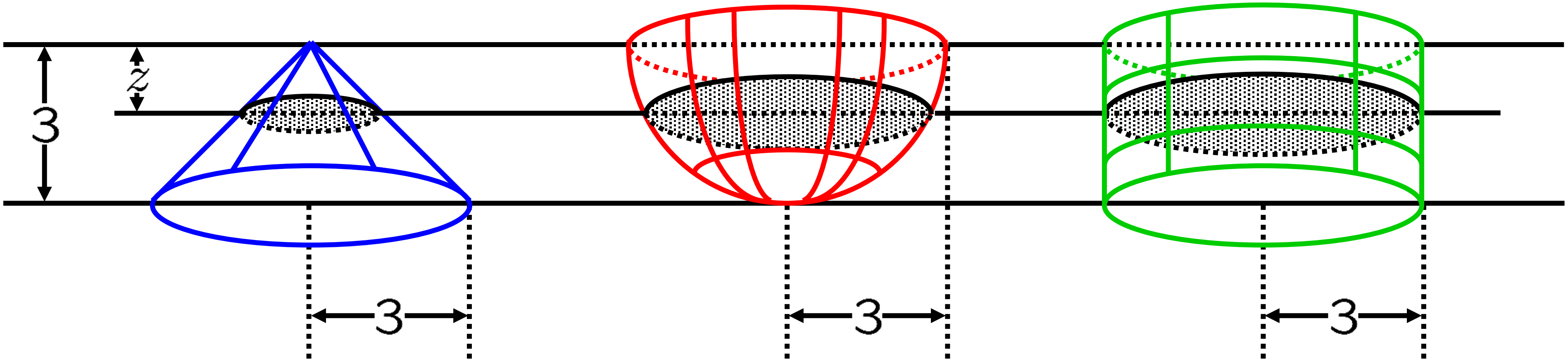

NOTE: If you can compute (e.g., cone and cylinder) the volume of two of these figures, (e.g., sphere) then you can compute the volume of the third.

Vol(cone) + Vol(hemisphere) = Vol(cylinder)

QED

$$\int_0^3 \pi z^2\,dz + \int_0^3 \pi\left(\sqrt{3^2 - z^2}\right)^2 dz = \int_0^3 \pi 3^2\,dz$$

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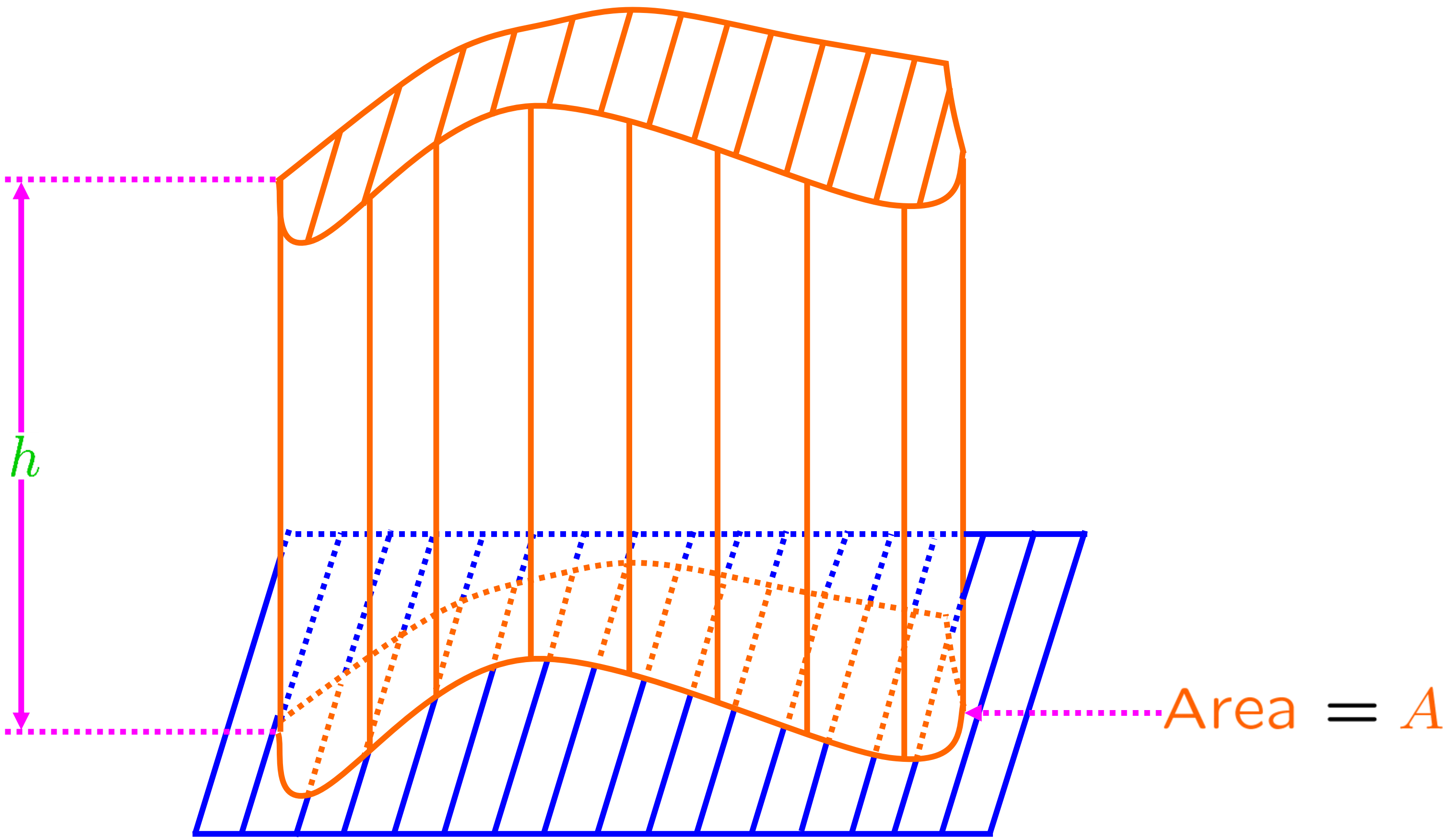
EXAMPLE: Find the volume of the generalized cylinder of height h and base area A.



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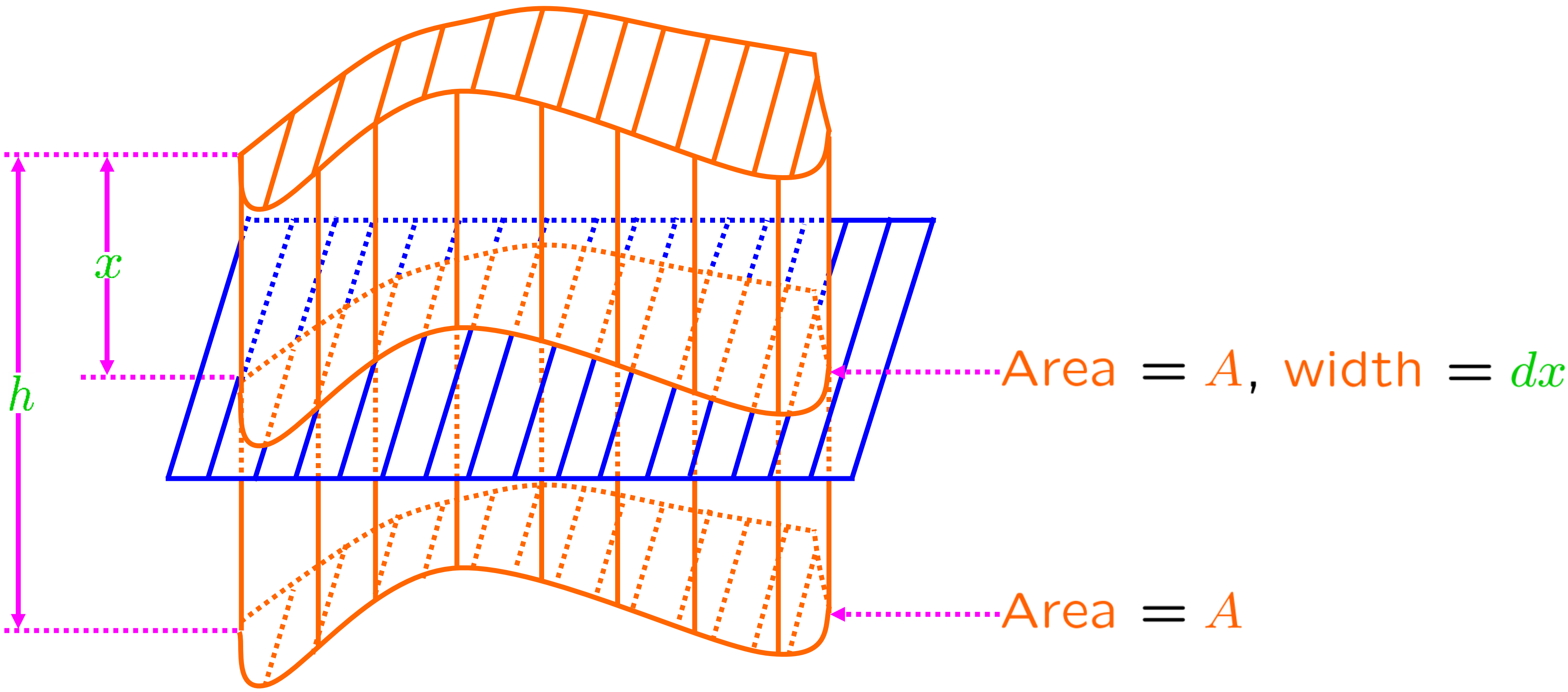
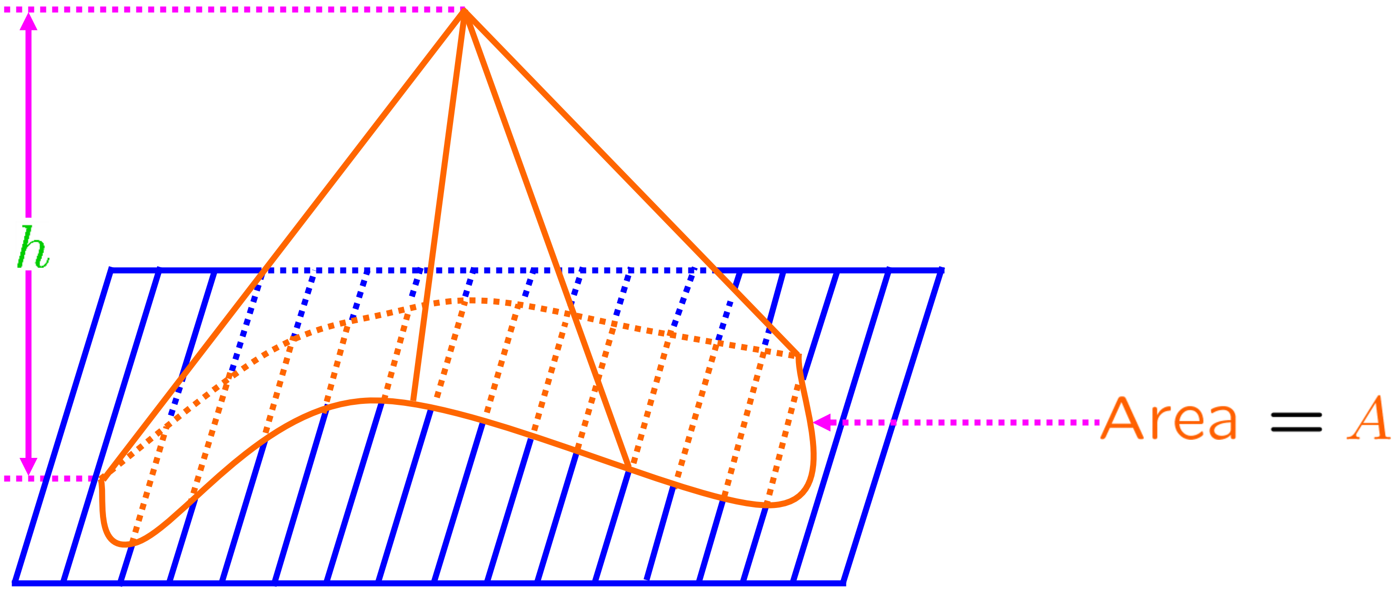


Volume of generalized cylinder $= \int_0^h A\,dx = Ah$ ■

EXAMPLE: Find the volume of the generalized cone of height h and base area A.



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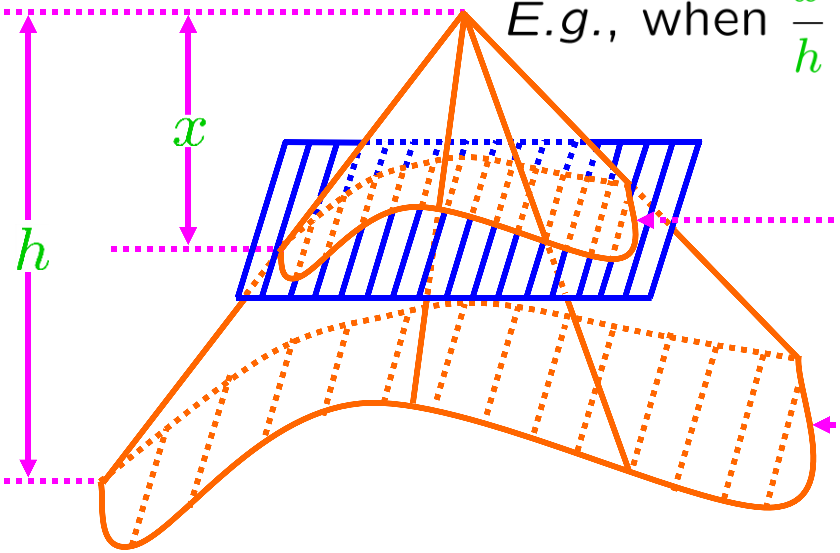


E.g., when $\frac{x}{h} = \frac{1}{2}$, the cross-section has $\frac{1}{4}$ the area A of the base.

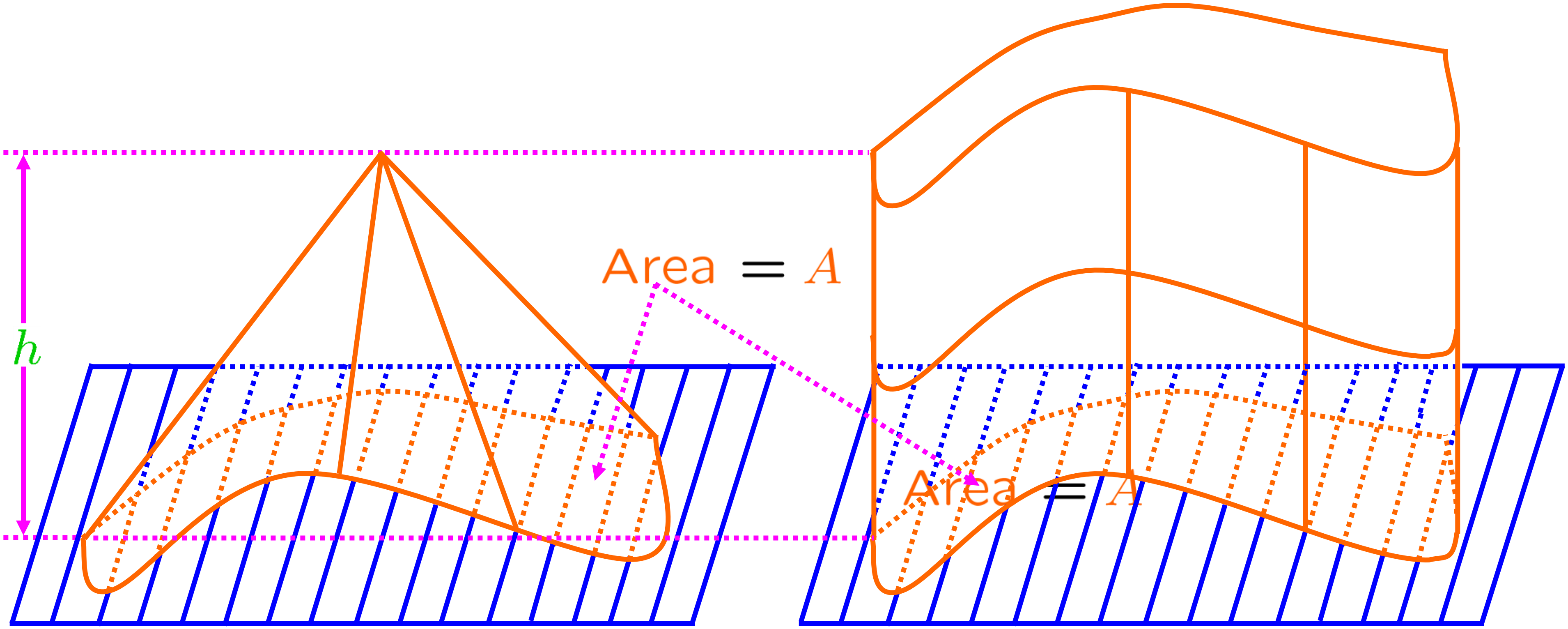
Area $= A\left[\frac{x}{h}\right]^2$, width $= dx$

Area $= A$

$$\text{Volume of generalized cone} = \int_0^h A\left[\frac{x}{h}\right]^2 dx$$

$$= \frac{A}{h^2}\int_0^h x^2\,dx$$

$$= \frac{A}{h^2}\left[\frac{x^3}{3}\right]_{x:\to 0}^{x:\to h} = \frac{A}{h^2}\left[\frac{h^3}{3}\right] = \frac{Ah}{3}$$ ■



Volume of generalized cone

$$= \frac{Ah}{3}$$

Volume of generalized cylinder

$$= Ah$$

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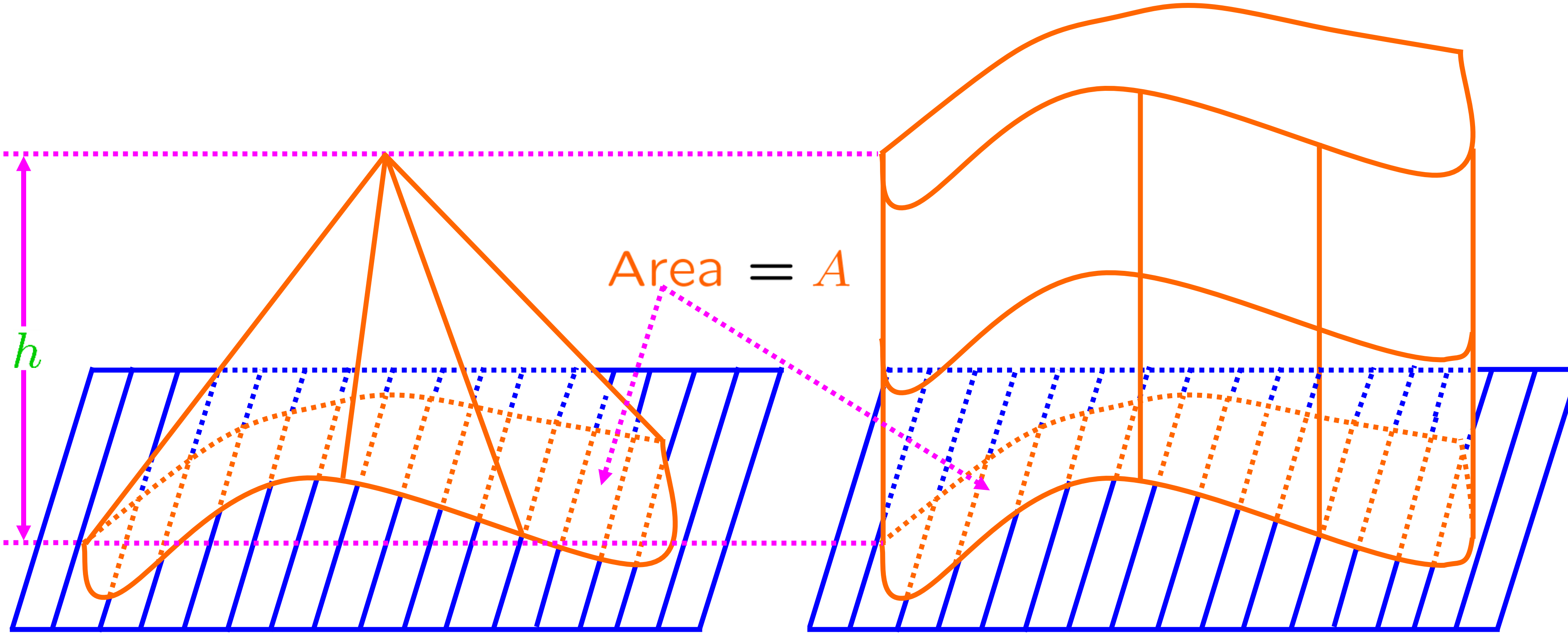


Volume of generalized cone

$$= \frac{Ah}{3}$$

Volume of generalized cylinder

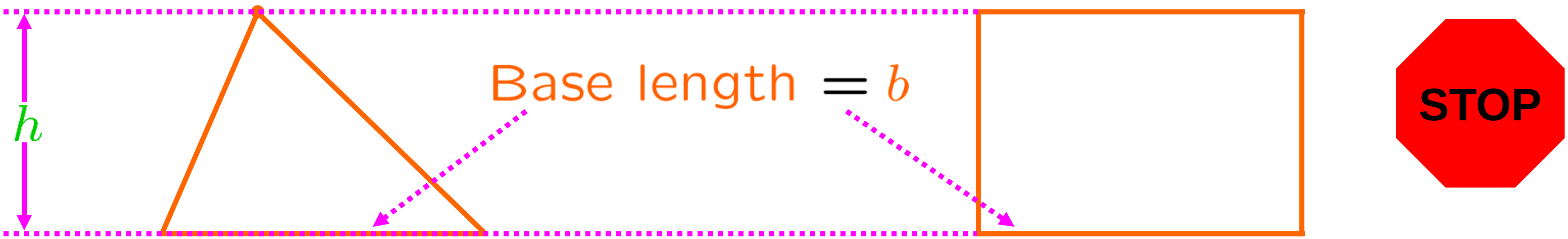
$$= Ah$$

In 3-D, $\quad \text{Vol(generalized cone)} = \dfrac{\text{Vol(generalized cylinder)}}{3}$

In n-D, $\text{Vol(generalized cone)} = \dfrac{\text{Vol(generalized cylinder)}}{n}$

A 2-D cone is a triangle. A 2-D cylinder is a rectangle.

$\text{Area(triangle)} = \dfrac{bh}{2}$ $\text{Area(rectangle)} = bh$



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In 3-D, $\text{Vol(generalized cone)} = \dfrac{\text{Vol(generalized cylinder)}}{3}$