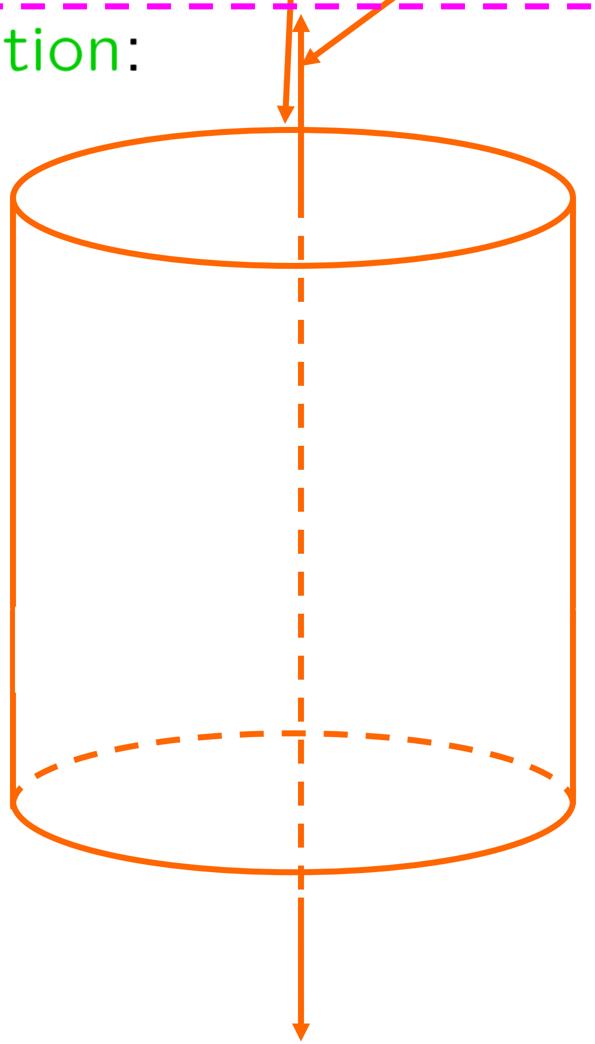
CALCULUS
Volume by slices and
the disk and washer methods,
problems

EXAMPLE: A wedge shape is cut out of a right circular cylinder of radius 7 by two cuts. One cut is perpendicular to the axis of the cylinder. The other intersects the first at an angle of 45° along a diameter of the cylinder. Find the volume of the wedge.

Solution:



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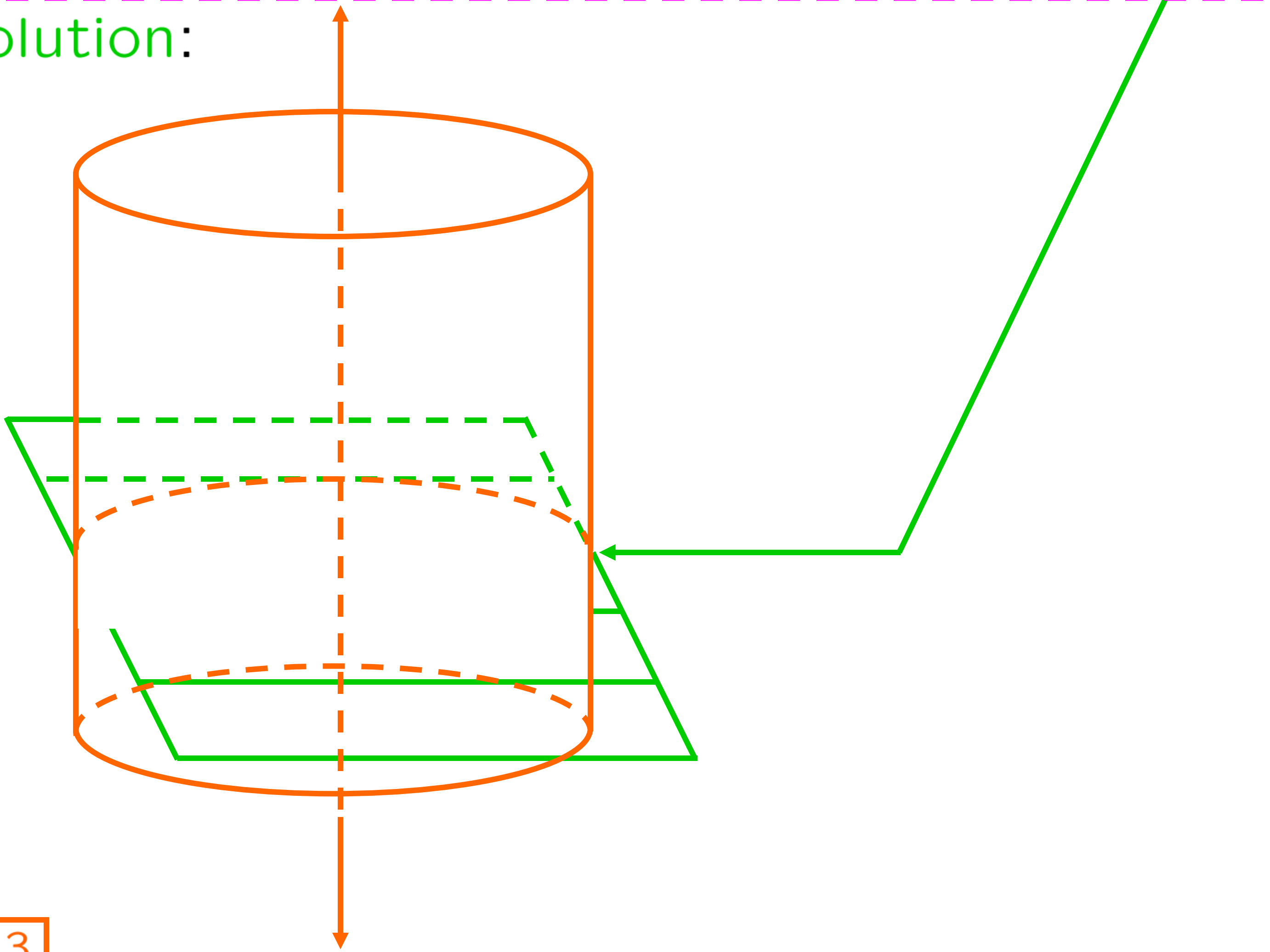

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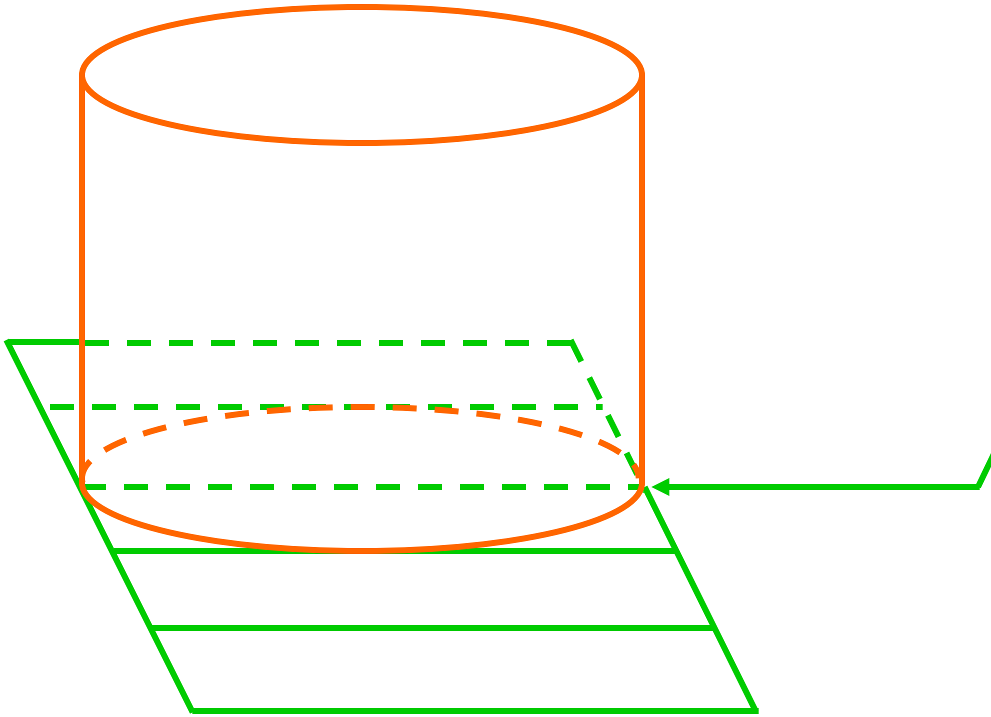

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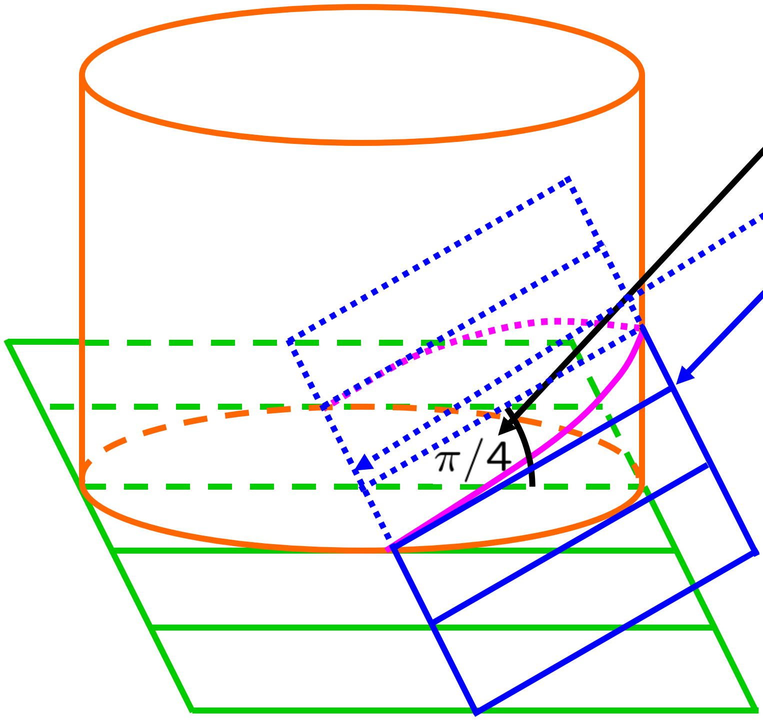

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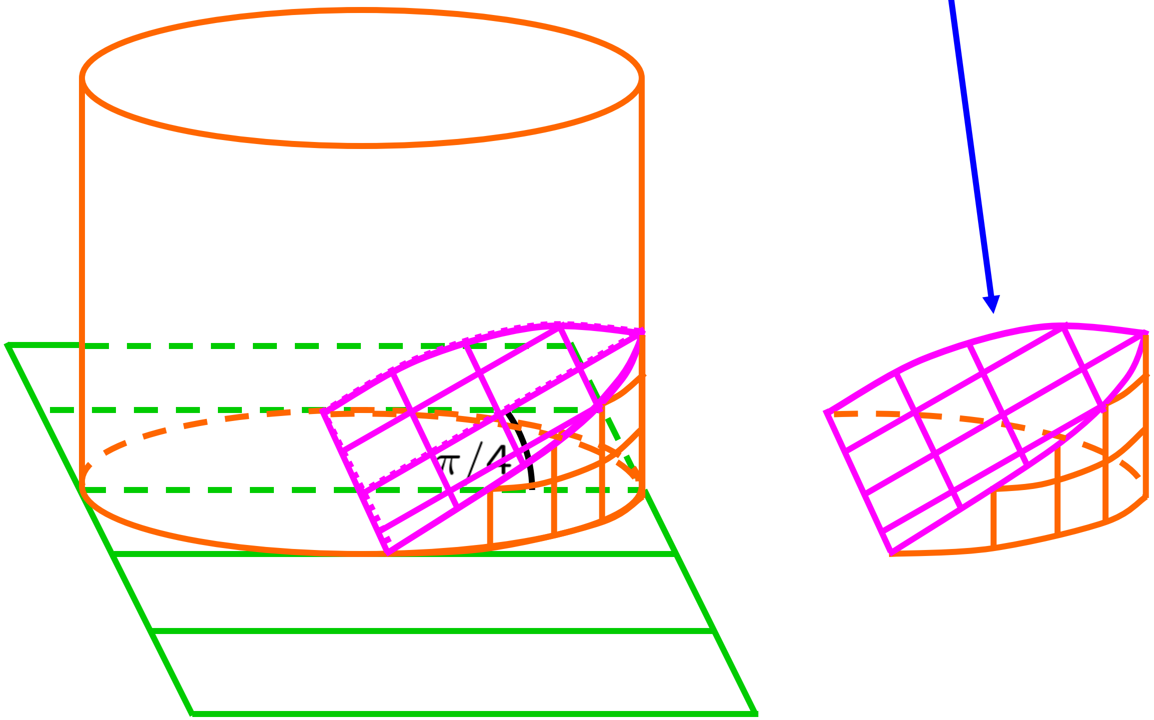

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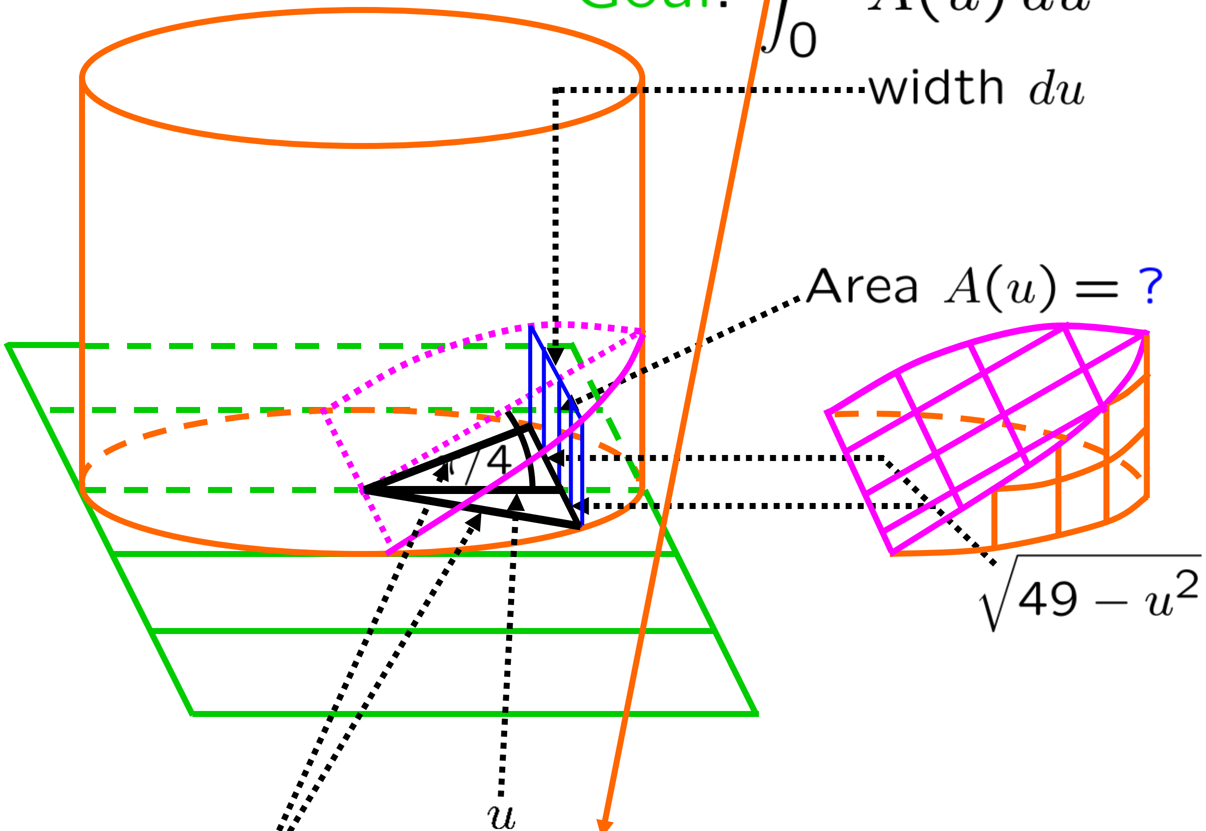
Solution:

Goal: $\int_0^7 A(u)\,du$



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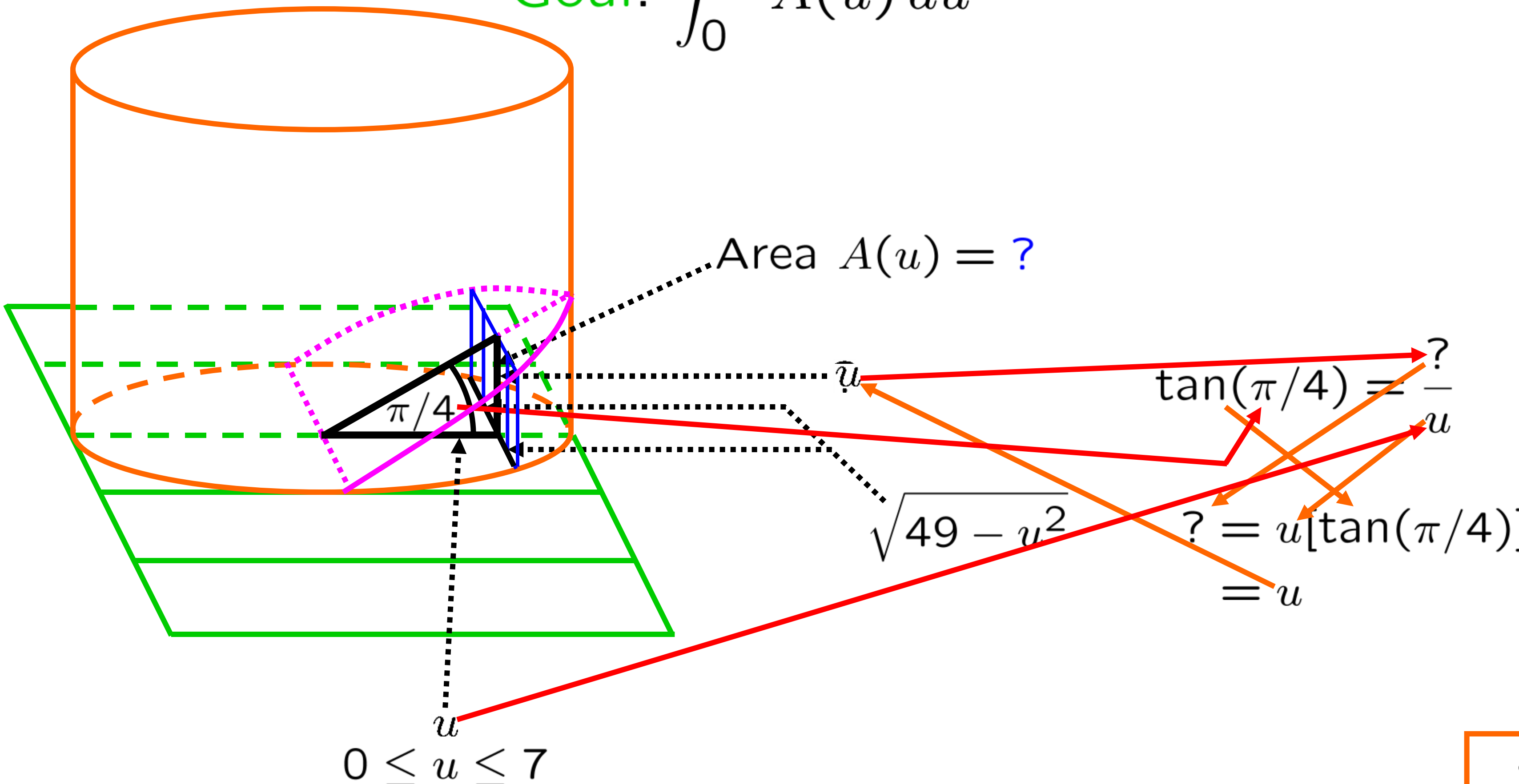

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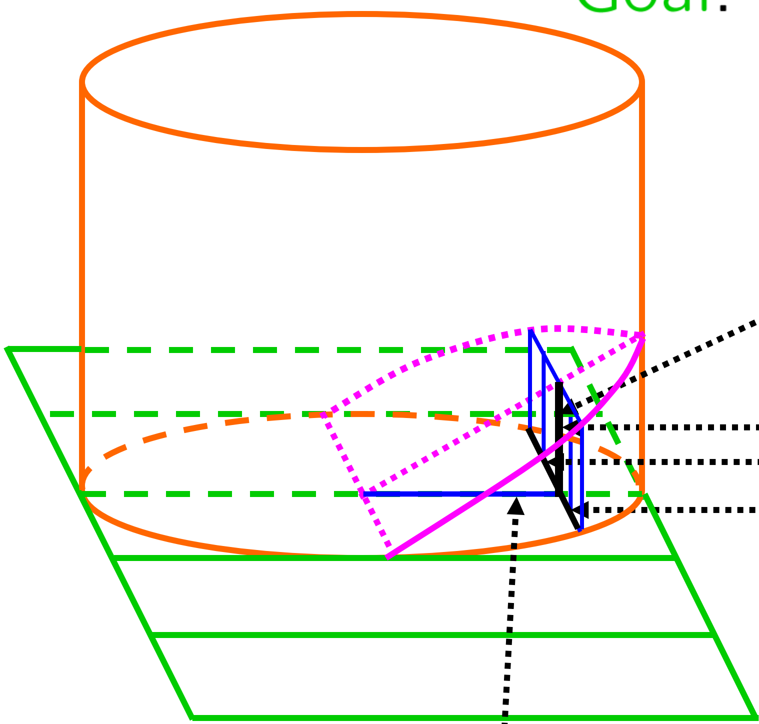
Solution:

Goal: $\displaystyle\int_0^7 A(u)\,du = 2\int_0^7 u\sqrt{49-u^2}\,du$

$v := 49 - u^2$
$dv = -2u\,du$

Area $A(u) = \left[?2\sqrt{49-u^2}\right][u]$

$\tan(\pi/4) = \dfrac{?}{u}$



u

$\sqrt{49-u^2}$

$? = u[\tan(\pi/4)]$
$= u$

u
$0 \le u \le 7$

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$v := 49 - u^2$

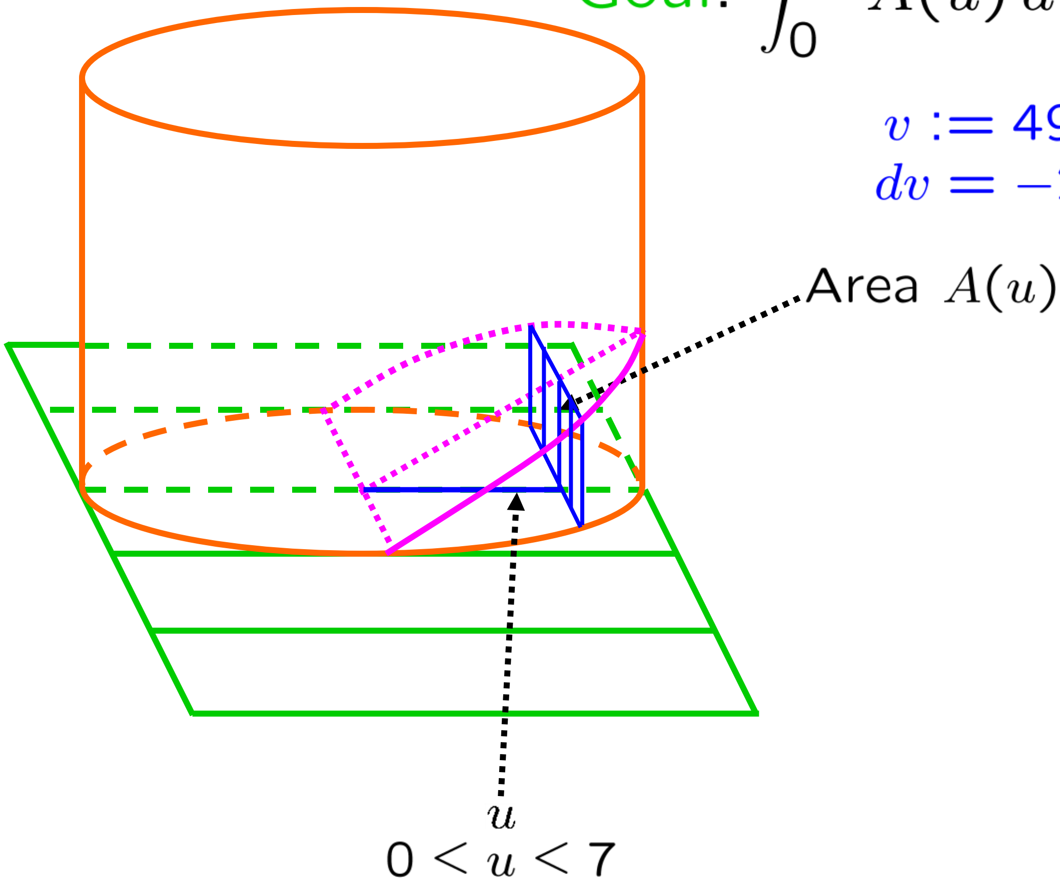
$dv = -2u\,du$

$= 2\int_{49}^{0} \sqrt{v}\,\frac{dv}{+2}$

$= \int_0^{49} v^{1/2}\,dv$

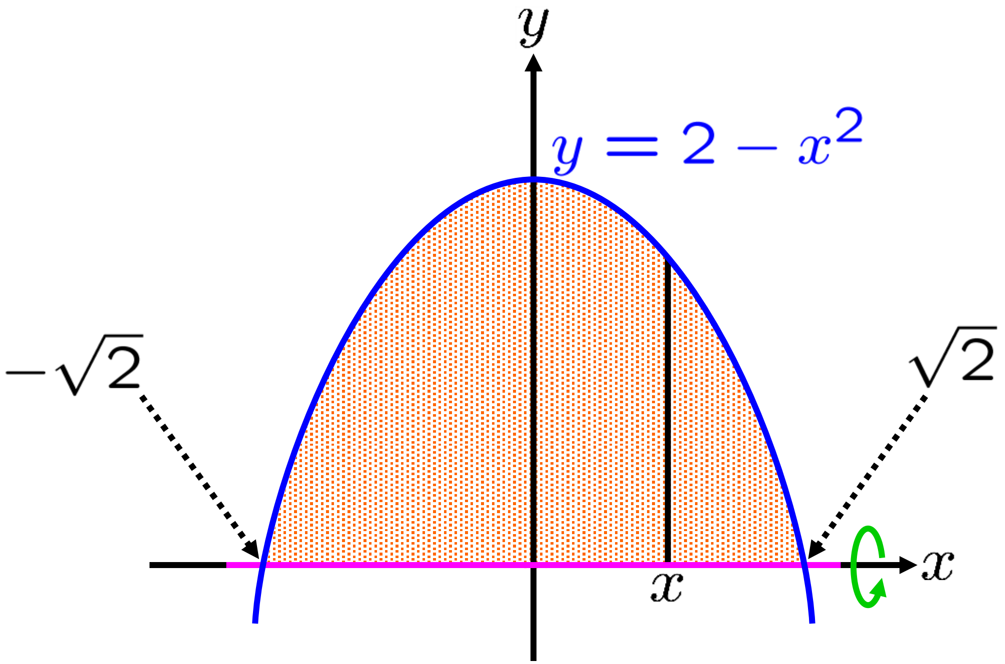
$= \frac{(49)^{3/2}}{3/2}$ $\quad 7^3 = 343$

$= \frac{686}{3}$ ■



SKILL
vol solid

EXAMPLE: Find the volume of the solid obtained by revolving the region bdd by $y = 2 - x^2$ and $y = 0$ about the x-axis.



SYMMETRY

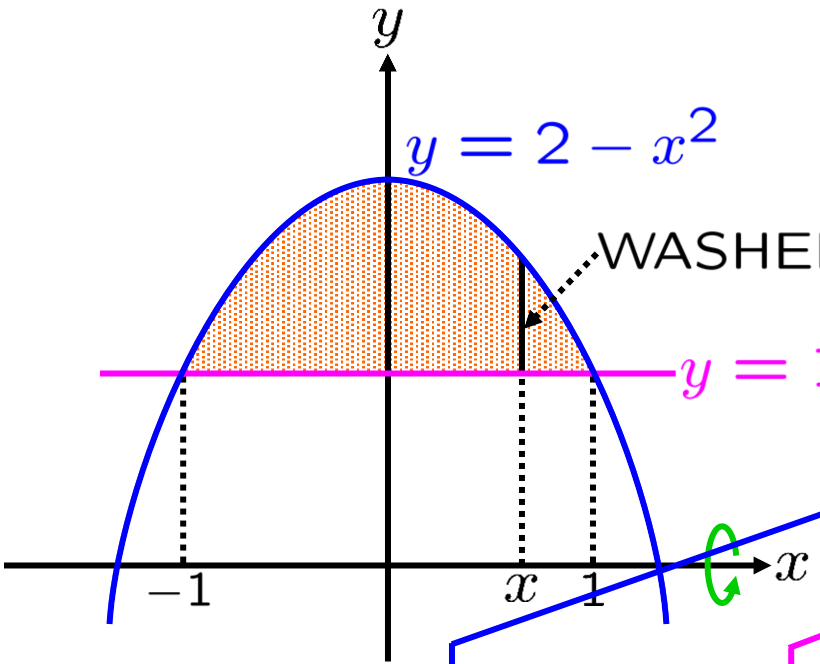
$$\int_{-\sqrt{2}}^{\sqrt{2}} \pi(2 - x^2)^2\, dx = 2\int_{0}^{\sqrt{2}} \pi(2 - x^2)^2\, dx$$

$$= 2\pi\int_{0}^{\sqrt{2}} 4 - 4x^2 + x^4\, dx$$

$$= 2\pi\left[4\sqrt{2} - \frac{4(\sqrt{2})^3}{3} + \frac{(\sqrt{2})^5}{5}\right]$$ ■

SKILL
disk method

EXAMPLE: Find the volume of the solid obtained by revolving the region bdd by $y = 2 - x^2$ and $y = 1$ about the x-axis.



Outer radius $= 2 - x^2$

Inner radius $= 1$

$$\text{Volume} = \int_{-1}^{1} \pi(2 - x^2)^2 - \pi(1)^2 \, dx$$

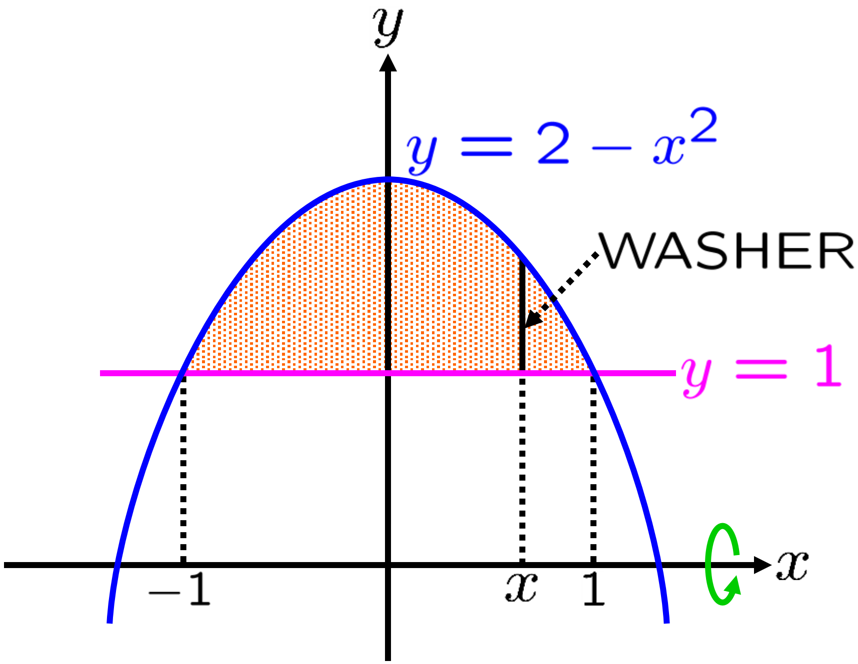
$$= \pi \int_{-1}^{1} (2 - x^2)^2 - 1 \, dx$$

SYMMETRY

$$= 2\pi \int_{0}^{1} (2 - x^2)^2 - 1 \, dx$$

$$= 2\pi \int_{0}^{1} (4 - 4x^2 + x^4) - 1 \, dx$$

EXAMPLE: Find the volume of the solid obtained by revolving the region bdd by $y = 2 - x^2$ and $y = 1$ about the x-axis.



Outer radius $= 2 - x^2$

Inner radius $= 1$

$$\text{Volume} = 2\pi \int_0^1 (4 - 4x^2 + x^4) - 1\, dx$$

$$= 2\pi \int_0^1 3 - 4x^2 + x^4\, dx$$

$$= 2\pi \int_0^1 (4 - 4x^2 + x^4) - 1\, dx$$

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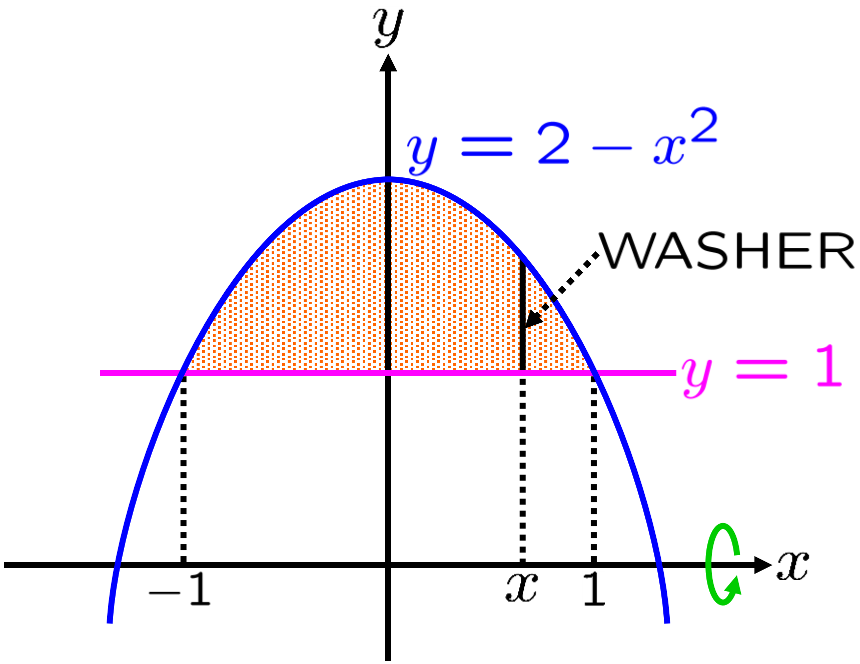


Outer radius $= 2 - x^2$

Inner radius $= 1$

$$\text{Volume} = 2\pi \int_0^1 (4 - 4x^2 + x^4) - 1\, dx$$

$$= 2\pi \int_0^1 3 - 4x^2 + x^4\, dx$$

$$= 2\pi \left[3x - \frac{4x^3}{3} + \frac{x^5}{5}\right]_{x:\to 0}^{x:\to 1} = 2\pi \left[3 - \frac{4}{3} + \frac{1}{5}\right] ■$$

SKILL
washer method

EXAMPLE: Find the volume of the solid obtained by revolving the region bounded by

$y = 2x^2$ and $y = 20 - 3x^2$

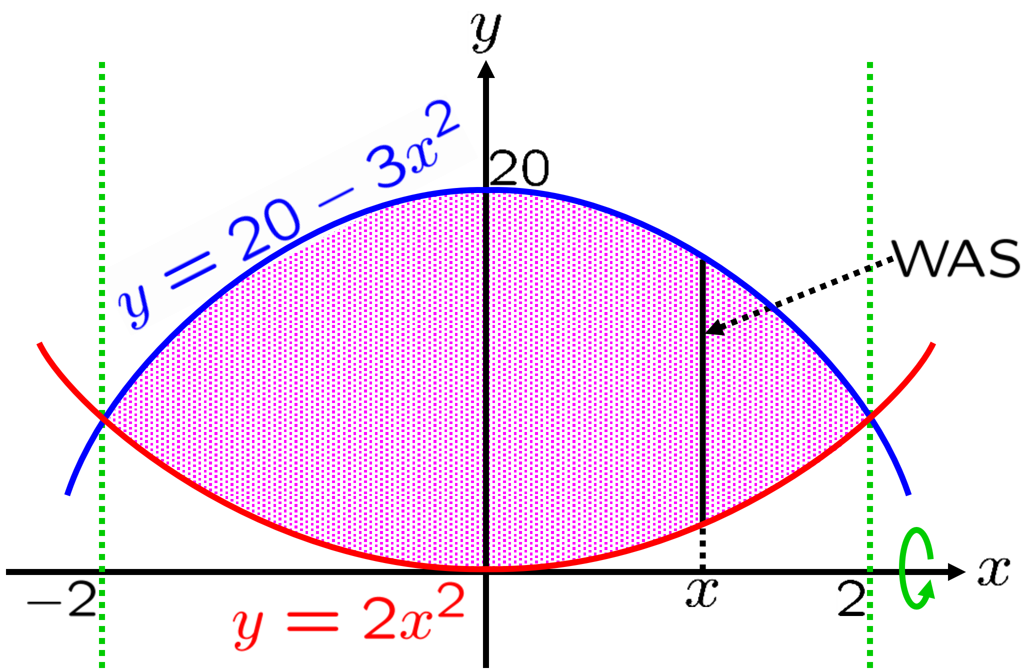
about the x-axis.

$$\int_{-2}^{2} [\pi(20 - 3x^2)^2 - \pi(2x^2)^2]\, dx$$

||

⋮

■



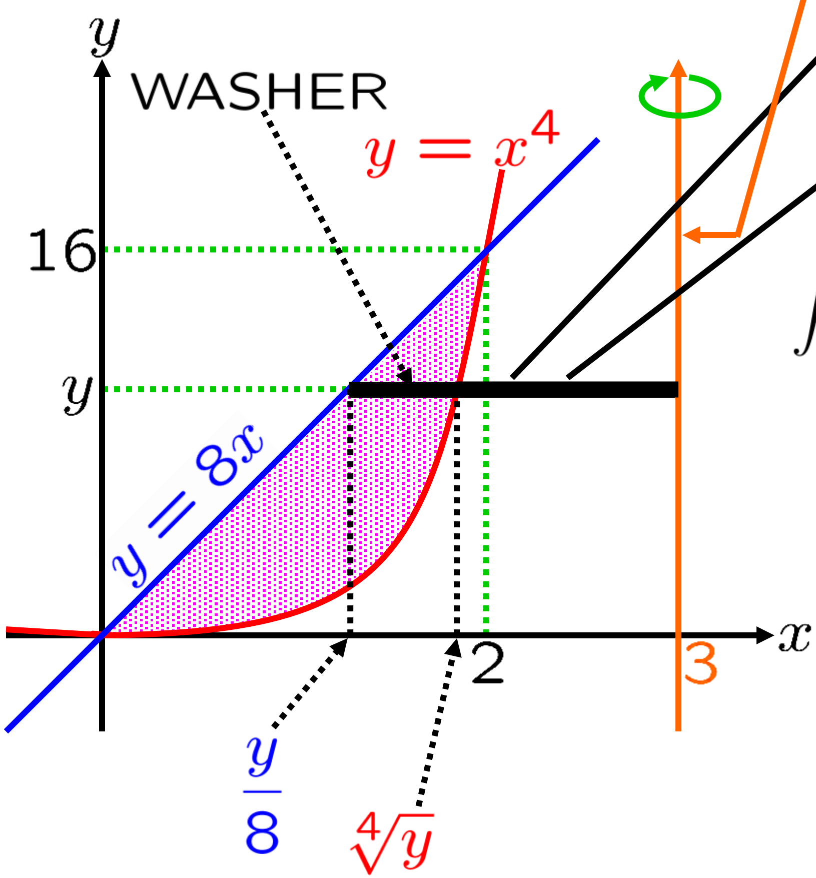
SKILL
washer method

$[2x^2 = 20 - 3x^2]$ iff $[x = 2$ or $x = -2]$

EXAMPLE: Set up an integral that computes the volume the volume of the solid obtained by revolving the region bounded by

$$y = x^4 \quad \text{and} \quad y = 8x$$

about the line $x = 3$. Don't evaluate the integral.



Outer radius $= 3 - \dfrac{y}{8}$

Inner radius $= 3 - \sqrt[4]{y}$

$$\int_0^{16} \pi \left(3 - \frac{y}{8}\right)^2 - \pi \left(3 - \sqrt[4]{y}\right)^2 \, dy$$

SKILL
washer method

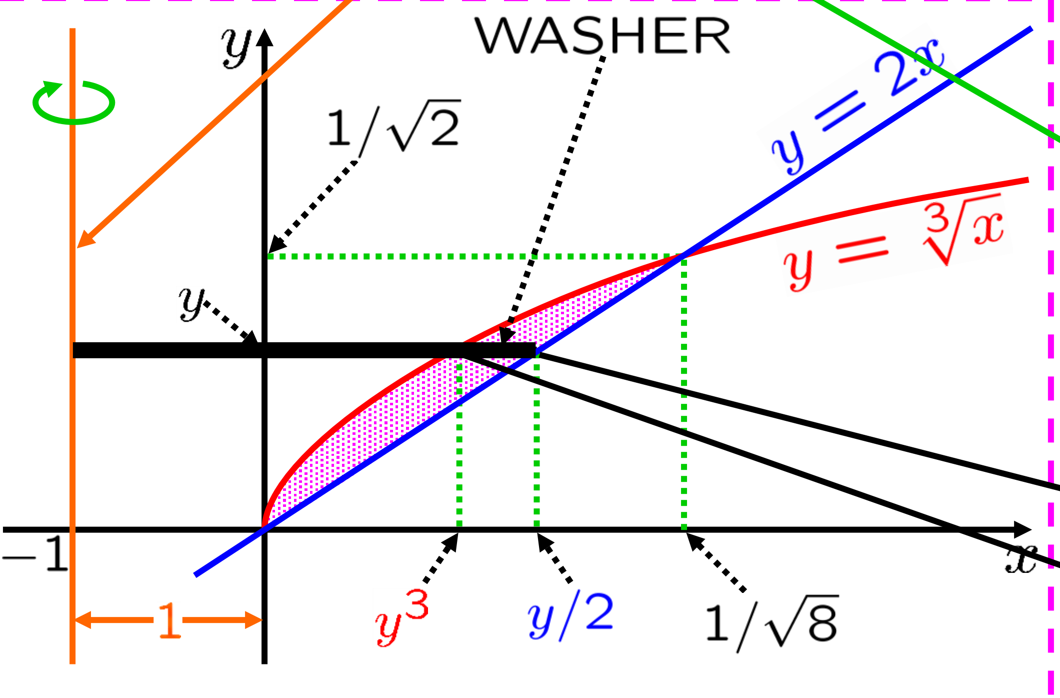
EXAMPLE: Find the volume of the solid obtained by revolving the region in the first quadrant bounded by

$$y = \sqrt[3]{x} \quad \text{and} \quad y = 2x$$

about the line $x = -1$.

$$\int_0^{1/\sqrt{2}} \left[\pi(1 + (y/2))^2 - \pi(1 + y^3)^2\right] dy = \cdots \blacksquare$$

SKILL
washer method



Outer radius $= 1 + (y/2)$

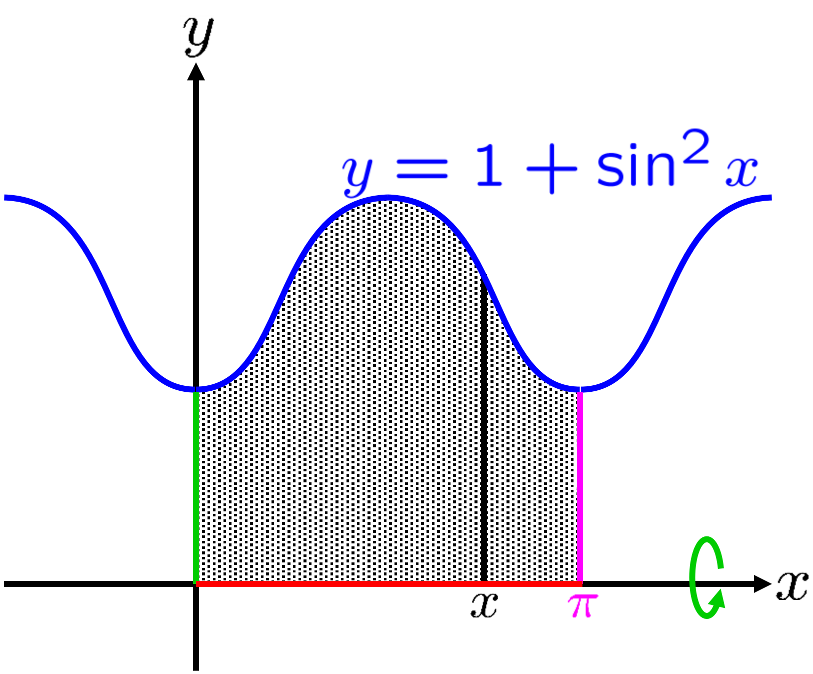
Inner radius $= 1 + y^3$

EXAMPLE: Describe a solid whose volume is

$$\pi \int_0^\pi \left[1 + \sin^2 x\right]^2\, dx.$$

$$\pi \int_0^\pi \left[1 + \sin^2 x\right]^2\, dx = \int_0^\pi \pi\left[1 + \sin^2 x\right]^2\, dx$$

is the volume of the solid obtained by revolving,
about the x-axis,
the region bounded by
$y = 1 + \sin^2 x$, $y = 0$, $x = 0$ and $x = \pi$. ■



SKILL
region from integral

EXAMPLE: Describe a solid whose volume is $\pi \int_4^8 e^{2x}\, dx$.

$$\pi \int_4^8 e^{2x}\, dx = \int_4^8 \pi \left(e^x\right)^2 \, dx$$

is the volume of the solid obtained by revolving,

about the x-axis,

the region bounded by

$y = e^x$, $y = 0$, $x = 4$ and $x = 8$. ■

Difficult to draw this with a uniform scale on the y-axis.

SKILL
region from integral

EXAMPLE: Describe a solid whose volume is

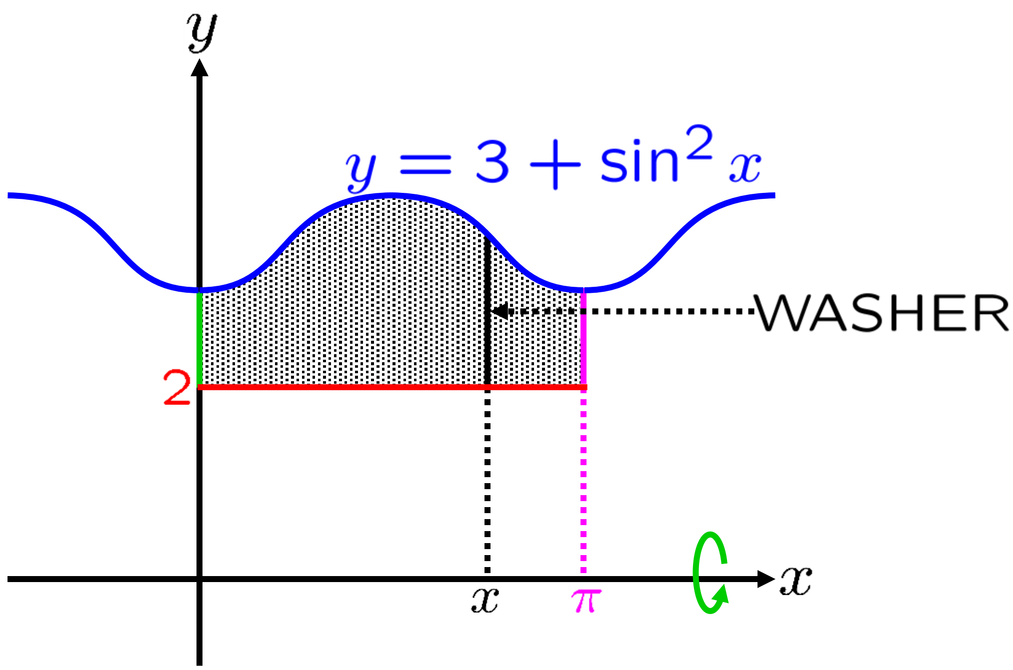
$$\pi \int_0^\pi \left[3 + \sin^2 x\right]^2 - 4 \, dx.$$

$$\pi \int_0^\pi \left[3 + \sin^2 x\right]^2 - 4 \, dx = \int_0^\pi \pi \left[3 + \sin^2 x\right]^2 - \pi 2^2 \, dx$$

is the volume of the solid obtained by revolving, about the x-axis, the region bounded by

$y = 3 + \sin^2 x$, $y = 2$, $x = 0$ and $x = \pi$. ■

SKILL
region from integral



EXAMPLE: Describe a solid whose volume is $\pi \int_3^7 y^4 - 9\,dy$.

$$\pi \int_3^7 y^4 - 9\,dy = \int_3^7 \pi \left(y^2\right)^2 - \pi 3^2\,dy$$

is the volume of the solid obtained by revolving,
about the y-axis,
the region bounded by
$x = y^2$, $x = 3$, $y = 3$ and $y = 7$. ■

SKILL
region from integral