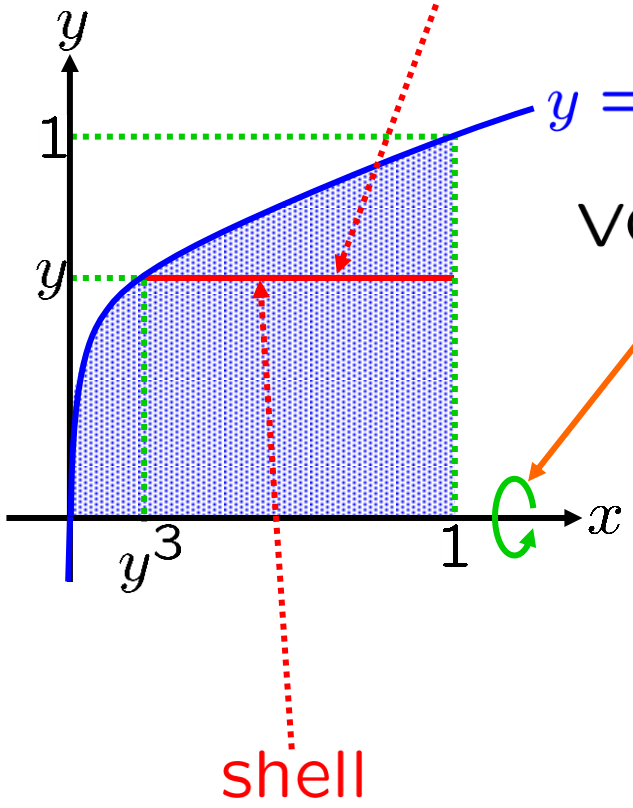
CALCULUS
Volume by cylindrical shells, problems

EXAMPLE: Use cylindrical shells to find the volume of the solid obtained by revolving, about the x-axis, the region under the curve $y = \sqrt[3]{x}$ from $x = 0$ to $x = 1$.

Redo the problem using the disk method.



AREA OF SHELL = [circumference][width]

$= [2\pi y][1 - y^3]$

$$\text{VOLUME} = \int_0^1 [2\pi y][1 - y^3]\, dy$$

$$= 2\pi \int_0^1 y - y^4\, dy$$

$$= 2\pi \left[\frac{1}{2} - \frac{1}{5}\right]$$

$$= 2\pi \left[\frac{3}{10}\right] = \frac{3\pi}{5}$$

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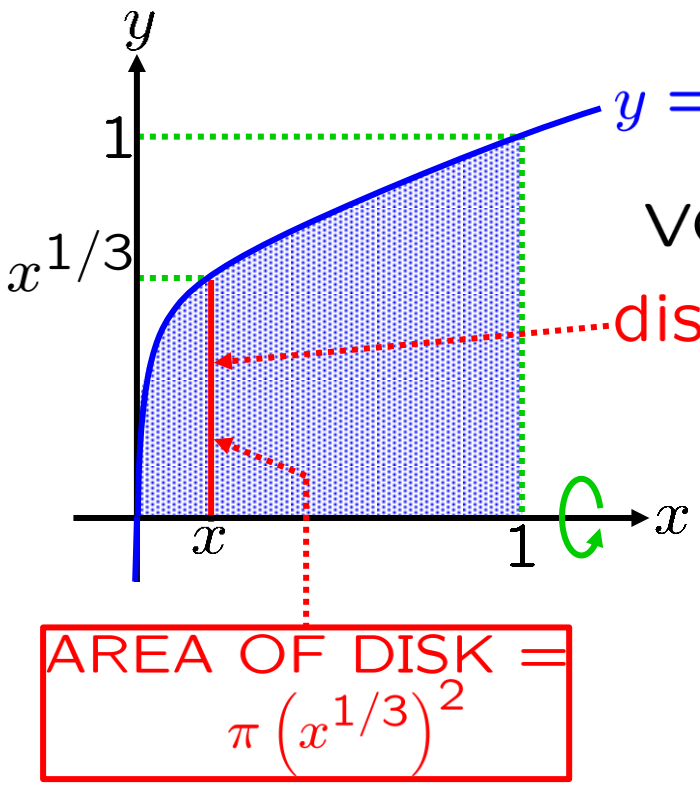


AREA OF DISK = $\pi \left(x^{1/3}\right)^2$

$$\text{VOLUME} = \int_0^1 [2\pi y][1 - y^3]\, dy$$

$$= 2\pi \int_0^1 y - y^4\, dy$$

$$= 2\pi \left[\frac{1}{2} - \frac{1}{5}\right]$$

$$= 2\pi \left[\frac{3}{10}\right] = \frac{3\pi}{5}$$

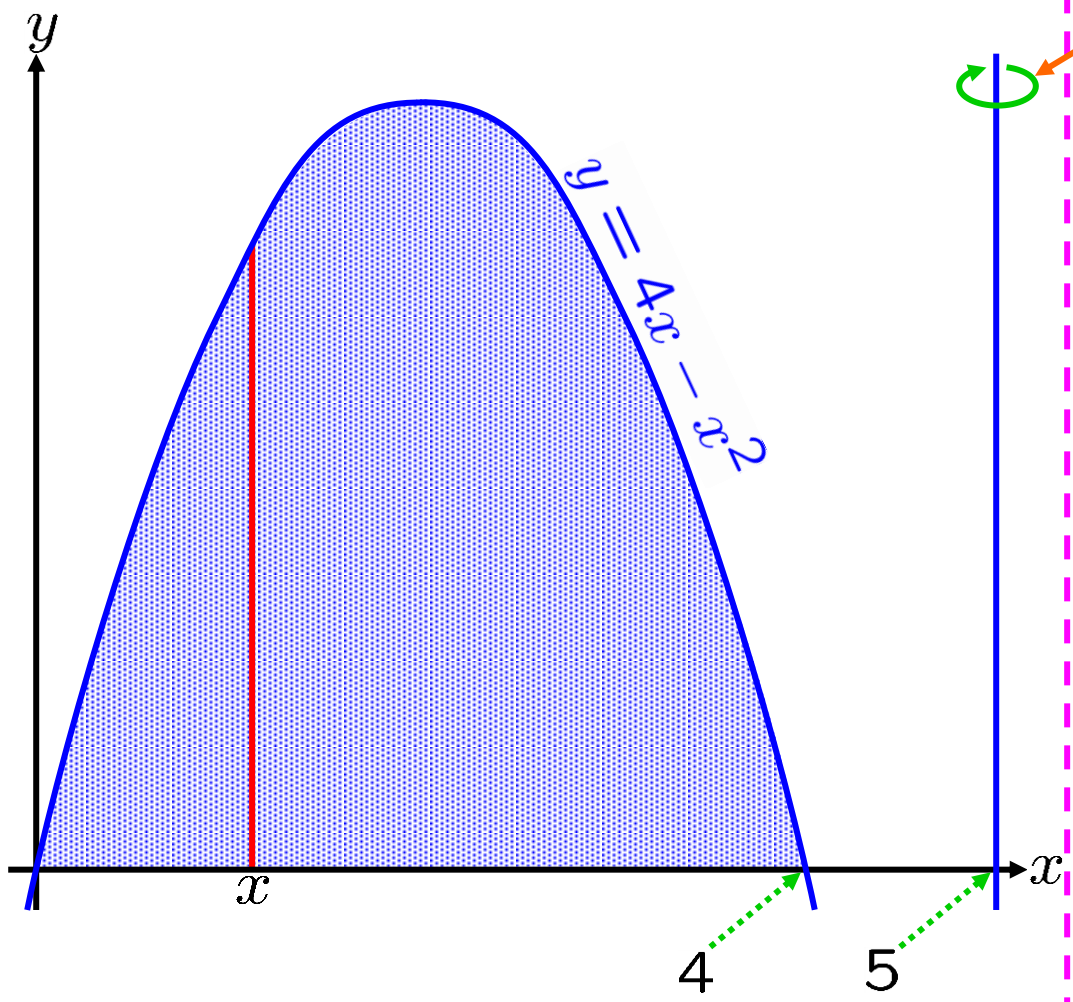
SKILL

disk and shell methods

$$\text{VOLUME} = \int_0^1 \pi \left(x^{1/3}\right)^2\, dx = \pi \int_0^1 x^{2/3}\, dx = \pi \left[\frac{1}{5/3}\right] = \frac{3\pi}{5}$$

EXAMPLE: Using the shell method, find the volume of the solid obtained by revolving, about the line $x = 5$, the region bounded by: $y = 4x - x^2$ and $y = 0$

Sketch the region.



$$\int_0^4 [2\pi(5-x)][4x-x^2]\,dx$$

$$= 2\pi \int_0^4 [5-x][4x-x^2]\,dx$$

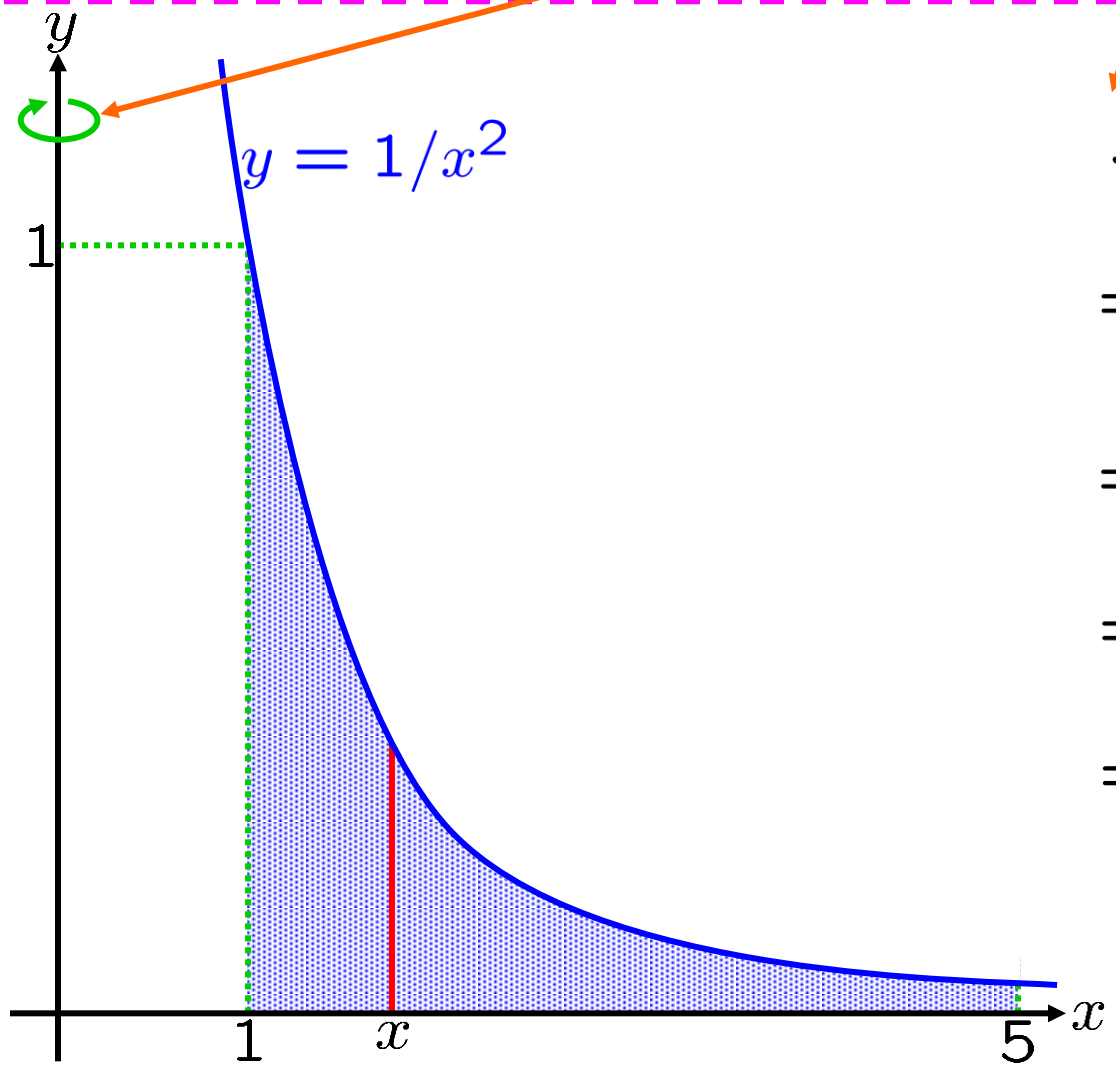
$$= 2\pi \int_0^4 x^3 - 9x^2 + 20x\,dx$$

$$= 2\pi \left[\frac{4^4}{4} - \frac{9\cdot 4^3}{3} + \frac{20\cdot 4^2}{2}\right]$$

■

SKILL
shell method

EXAMPLE: Using the shell method, find the volume of the solid obtained by revolving, about the y-axis, the region bounded by: $y = 1/x^2$, $y = 0$, $x = 1$ and $x = 5$.
Sketch the region.



$$\int_1^5 [2\pi x][1/x^2]\,dx$$

$$= 2\pi \int_1^5 (1/x)\,dx$$

$$= 2\pi \left[\ln(|x|)\right]_{x:\to 1}^{x:\to 5}$$

$$= 2\pi \left[(\ln 5) - (\ln 1)\right]$$

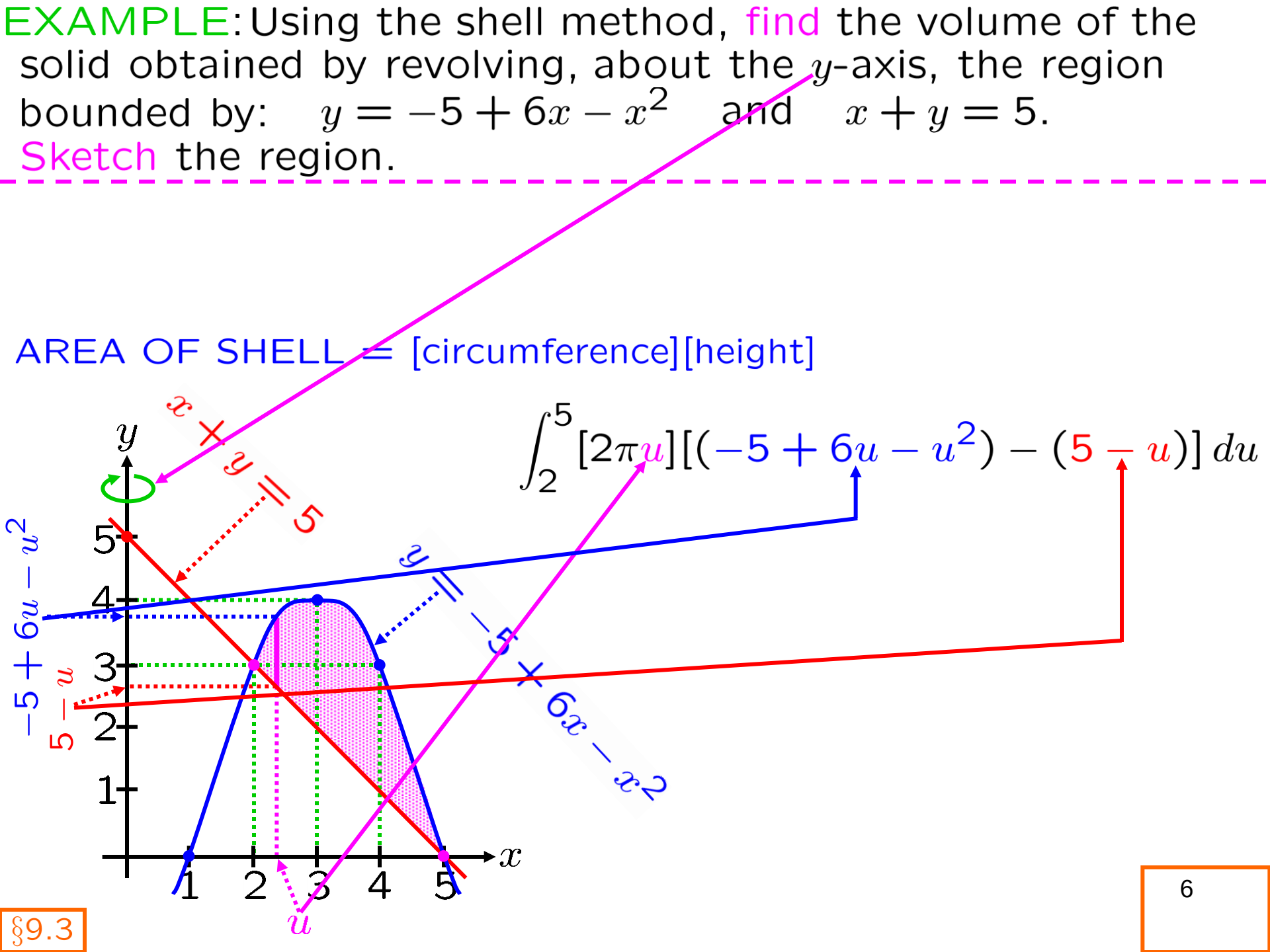
$$= 2\pi(\ln 5)$$ ■

SKILL
shell method

EXAMPLE: Using the shell method, find the volume of the solid obtained by revolving, about the y-axis, the region bounded by: $y = -5 + 6x - x^2$ and $x + y = 5$. Sketch the region.

AREA OF SHELL = [circumference][height]

$$\int_2^5 [2\pi u][(-5 + 6u - u^2) - (5 - u)]\, du$$



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$$\int_2^5 [2\pi u][(-5 + 6u - u^2) - (5 - u)]\, du$$

$$= 2\pi \int_2^5 [u][(-5 + 6u - u^2) - (5 - u)]\, du$$

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$$= 2\pi \int_2^5 [u][(-5 + 6u - u^2) - (5 - u)]\, du$$

$$= 2\pi \int_2^5 [u][-10 + 7u - u^2]\, du$$

$$= 2\pi \int_2^5 [-10u + 7u^2 - u^3]\, du$$

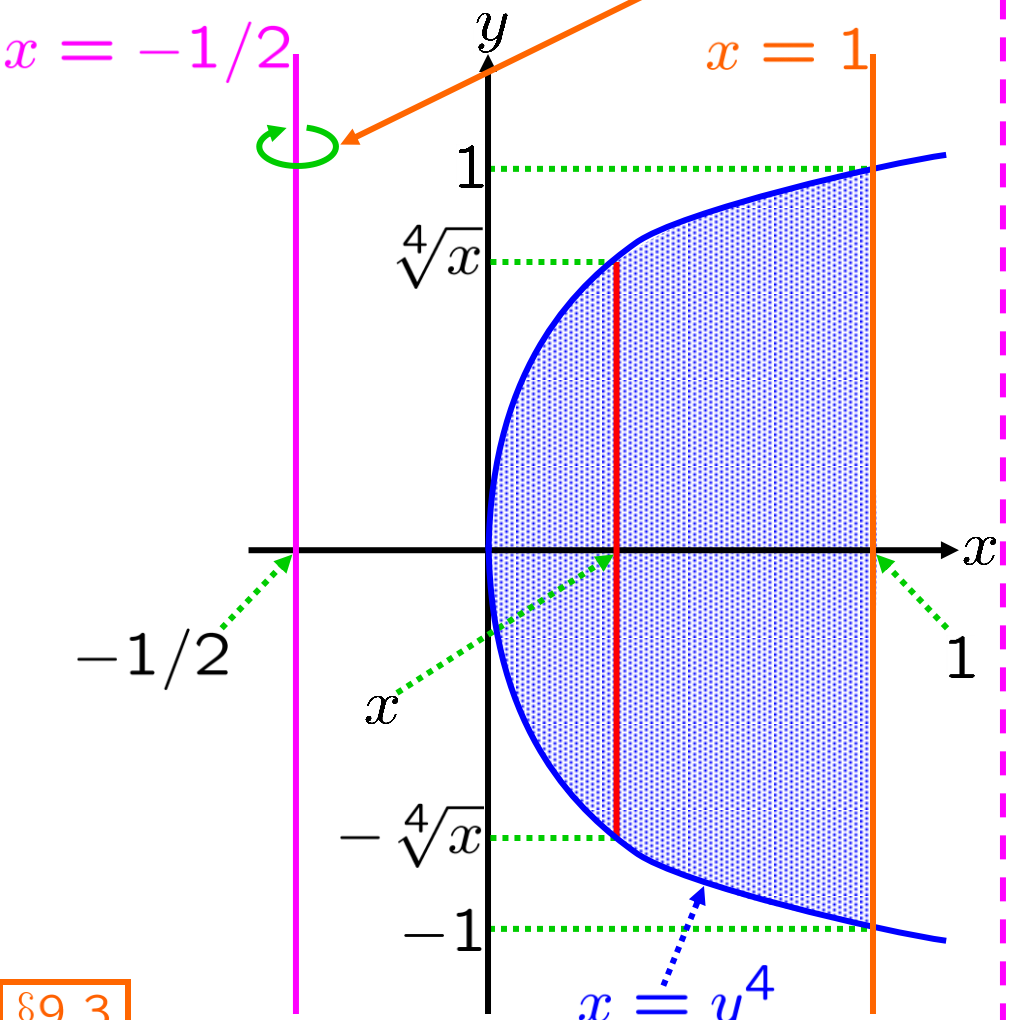
$$= 2\pi \left[-10\frac{u^2}{2} + 7\frac{u^3}{3} - \frac{u^4}{4}\right]_{u:\to 2}^{u:\to 5}$$

$$= 2\pi \left[\left(-10\frac{5^2}{2} + 7\frac{5^3}{3} - \frac{5^4}{4}\right) - \left(-10\frac{2^2}{2} + 7\frac{2^3}{3} - \frac{2^4}{4}\right)\right] = \cdots$$

SKILL
shell method

EXAMPLE: Using the shell method, find the volume of the solid obtained by revolving, about $x = -1/2$, the region bounded by: $x = y^4$ and $x = 1$. Sketch the region.

$$x = y^4 \quad \Leftrightarrow \quad \pm\sqrt[4]{x} = y$$



$$\int_0^1 [2\pi(x + (1/2))][2\sqrt[4]{x}]\,dx$$

$$= 4\pi \int_0^1 [x + (1/2)][x^{1/4}]\,dx$$

$$= 4\pi \int_0^1 \left[x^{5/4} + \frac{x^{1/4}}{2}\right] dx$$

$$= 4\pi \left[\frac{x^{9/4}}{9/4} + \frac{x^{5/4}}{2(5/4)}\right]_{x:\to 0}^{x:\to 1}$$

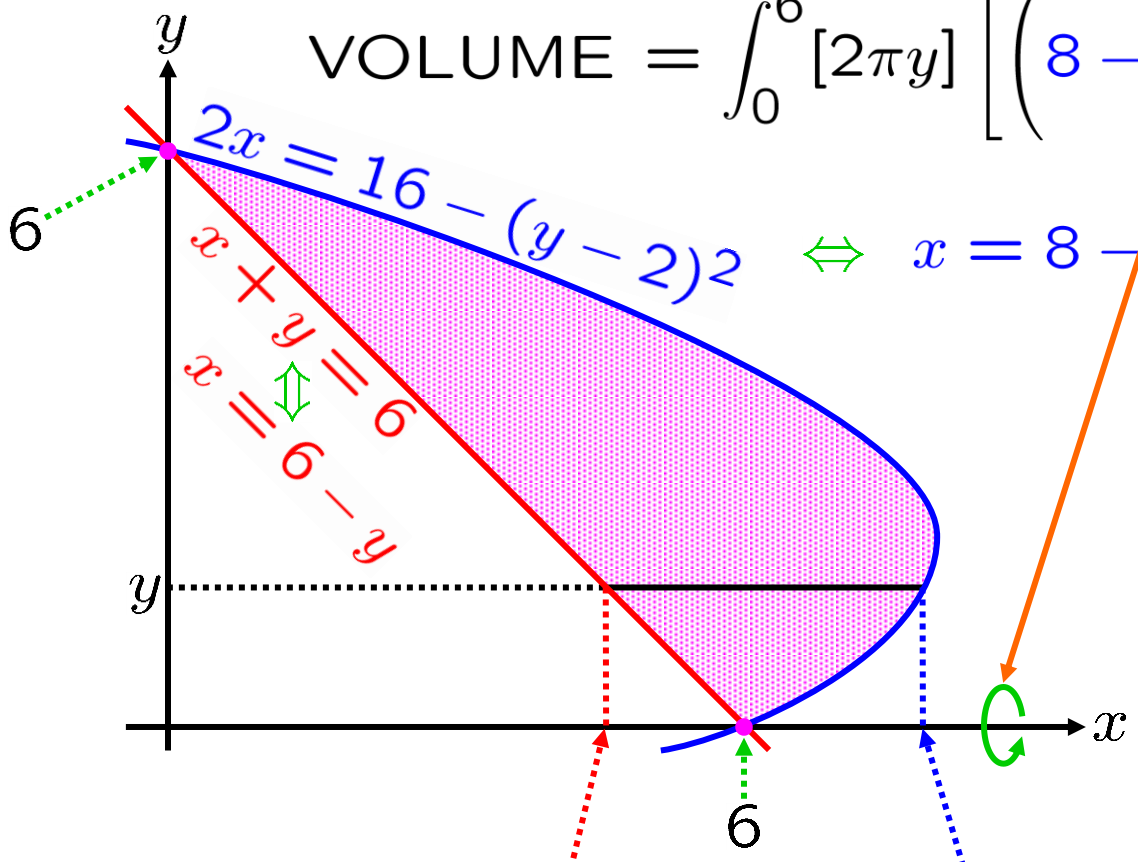
$$= 4\pi \left[\frac{1}{9/4} + \frac{1}{2(5/4)}\right] \blacksquare$$

SKILL
shell method

EXAMPLE: Using the shell method, find the volume of the solid obtained by revolving, about the x-axis, the region bounded by: $x + y = 6$ and $2x = 16 - (y-2)^2$

Sketch the region.

$$\text{VOLUME} = \int_0^6 [2\pi y]\left[\left(8 - \frac{(y-2)^2}{2}\right) - (6-y)\right] dy$$



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$$= \pi \int_0^6 2y \left[\left(8 - \frac{y^2 - 4y + 4}{2} \right) - (6-y) \right] dy$$

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$$= \pi \int_0^6 2y \left[\left(\frac{16}{2} - \frac{y^2 - 4y + 4}{2} \right) - (6 - y) \right] dy$$

$$= \pi \int_0^6 2y \left[\left(\frac{16 - y^2 + 4y - 4}{2} \right) - \left(\frac{12 - 2y}{2} \right) \right] dy$$

$$= \pi \int_0^6 2y \left[\frac{\left(16 - y^2 + 4y - 4 \right) - (12 - 2y)}{2} \right] dy$$

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Sketch the region.

$$\text{VOLUME} = \pi \int_0^6 2y \left[\frac{\left(16 - y^2 + 4y - 4\right) - (12 - 2y)}{2} \right] dy$$

$$= \pi \int_0^6 y \left[16 - y^2 + 4y - 4 - 12 + 2y \right] dy$$

$$= \pi \int_0^6 y \left[-y^2 + 6y \right] dy$$

$$= \pi \int_0^6 -y^3 + 6y^2 \, dy$$

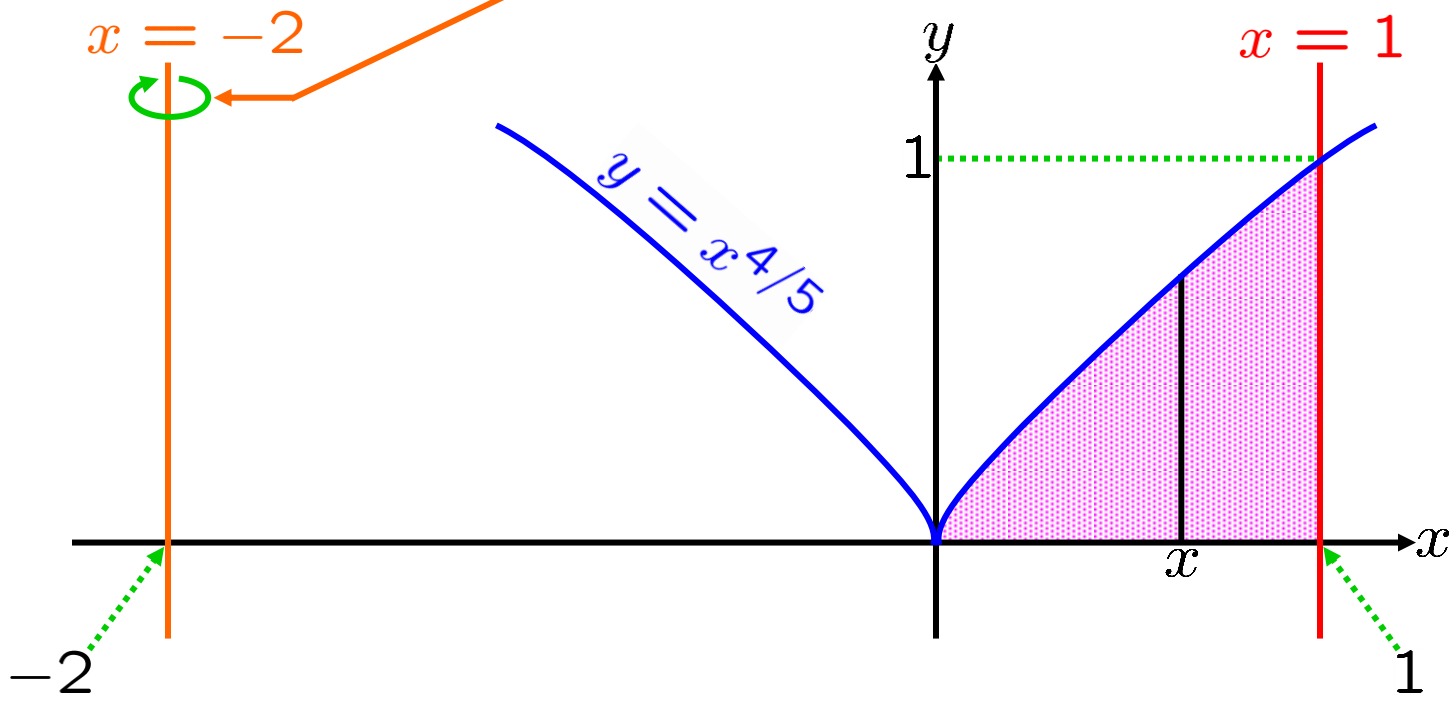
$$= \pi \left[-\frac{6^4}{4} + \frac{6 \cdot 6^3}{3} \right]$$

$$= \pi 6^4 \left[-\frac{1}{4} + \frac{1}{3} \right] = 6^4 \left[\frac{1}{12} \right] \pi = 108\pi$$ ■

SKILL
shell method

EXAMPLE: Using the shell method, find the volume of the solid obtained by revolving, about the line $x = -2$, the region bounded by: $y = x^{4/5}$, $y = 0$ and $x = 1$. Sketch the region.

$$\text{VOLUME} = \int_0^1 [2\pi(x+2)][x^{4/5}]\,dx$$



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$$\text{VOLUME} = \int_0^1 [2\pi(x+2)][x^{4/5}]\,dx$$

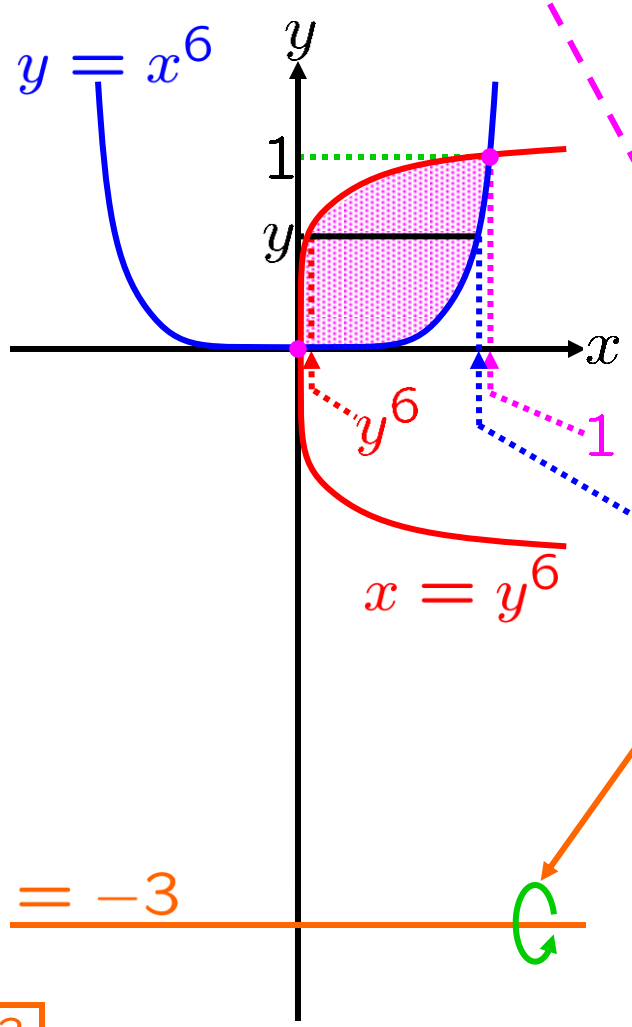
$$= 2\pi \int_0^1 [x+2][x^{4/5}]\,dx$$

$$= 2\pi \int_0^1 x^{9/5} + 2x^{4/5}\,dx$$

$$= 2\pi \left[\frac{1}{14/5} + \frac{2}{9/5}\right] = \cdots \blacksquare$$

SKILL
shell method

EXAMPLE: Using the shell method, find the volume of the solid obtained by revolving, about the line $y = -3$, the region bounded by: $y = x^6$ and $x = y^6$.
Sketch the region.



$$\text{VOLUME} = \int_0^1 [2\pi(y+3)][y^{1/6} - y^6]\,dy$$

$$= 2\pi \int_0^1 y^{7/6} - y^7 + 3y^{1/6} - 3y^6\,dy$$

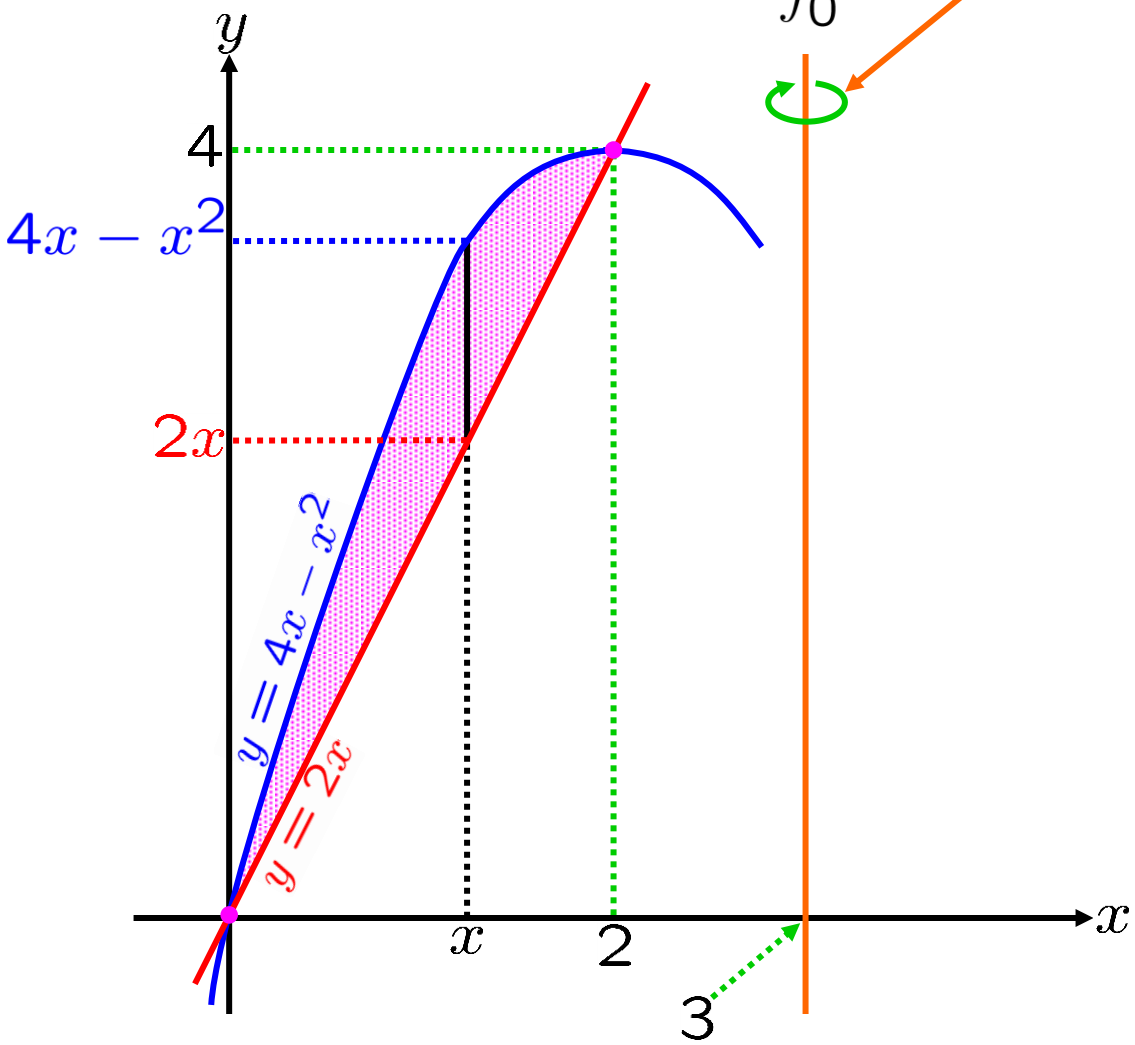
$$= 2\pi \left[\frac{1}{13/6} - \frac{1}{8} + \frac{3}{7/6} - \frac{3}{7}\right]$$

$= \cdots$ ■

SKILL
shell method

EXAMPLE: Using the shell method, find the volume of the solid obtained by revolving, about the line $x = 3$, the region bounded by: $y = 2x$ and $y = 4x - x^2$.
Sketch the region.

$$\text{VOLUME} = \int_0^2 [2\pi(3 - x)][(4x - x^2) - 2x]\,dx$$



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Sketch the region.

$$\text{VOLUME} = \int_0^2 [2\pi(3-x)][(4x-x^2)-2x]\,dx$$

$$= 2\pi \int_0^2 [3-x][2x-x^2]\,dx$$

$$= 2\pi \int_0^2 6x - 5x^2 + x^3\,dx$$

$$= 2\pi \left[\frac{6\cdot 2^2}{2} - \frac{5\cdot 2^3}{3} + \frac{2^4}{4}\right]$$

$= \cdots$ ■

SKILL
shell method

STOP