Math 4603 Midterm 1 Name: ____________________

Instructions:

This exam contains 6 questions. Each question is worth 20 points. The exam is worth 120 points in total.

1. Prove that $\mathbb{Q}$ is countable.

Solution. Define $f : \mathbb{Q} \to \mathbb{Z} \times \mathbb{N}$ by $f(0) = (0, 1)$ and $f(n/m) = (n, m)$ where it is assumed $n \in \mathbb{Z} \setminus \{0\}$ and $m \in \mathbb{N}$ have no common divisors. It is clear that f is injective. From theorems in class[1], $\mathbb{Z} \times \mathbb{N}$ is countable. So there is an injective function $g : \mathbb{Z} \times \mathbb{N} \to \mathbb{N}$. Now the composition $g \circ f : \mathbb{Q} \to \mathbb{N}$ is injective, which proves the result.

[1] Theorem 1: $\mathbb{Z}$ is countable; Theorem 2: If S and T are countable, so is $S \times T$. See also HW2, Problem 1(i).

2. Use strong induction to show that every natural number $n \geq 2$ can be written as a product of prime powers.

Solution. 2 is considered a product of prime powers, where the product has only one term and the power is one. Assume that the numbers $2, 3, \ldots, n$ can each be written as a product of prime powers. If $n+1$ is prime then it is a product of prime powers[2]. If $n+1$ is not prime then

$$(*) \quad n + 1 = jk$$

where j, k are integers between 2 and n, inclusive. So j and k can each be written as products of prime powers, which along with $(*)$ means $n+1$ can be written as a product of prime powers.

[2]In the same way that 2 is.

3. Let $a_1 = 4$ and for $n \geq 1$,

$$a_{n+1} = \frac{a_n + \frac{2}{a_n}}{2}. \tag{1}$$

Prove that $\{a_n\}_{n=1}^{\infty}$ converges. Conclude that $\lim a_n = \sqrt{2}$.

Hint: You may use without justification the arithmetic-geometric mean inequality: for positive real numbers x and y,

$$\frac{x+y}{2} \geq \sqrt{xy}.$$

Solution. By the hint, $a_n \geq \sqrt{2}$ for all $n \geq 2$. As $a_1 = 4$ we see that $\{a_n\}_{n=1}^{\infty}$ is bounded below by $\sqrt{2}$. Also, for $n \geq 1$ we have

$$\frac{a_{n+1}}{a_n} = \frac{1 + \frac{2}{a_n^2}}{2} \leq \frac{1 + \frac{2}{\sqrt{2}^2}}{2} = 1,$$

which means $a_{n+1} \leq a_n$, that is, $\{a_n\}_{n=1}^{\infty}$ is decreasing. So $\{a_n\}_{n=1}^{\infty}$ has a limit, L. From equation (1) and the arithmetic properties of limits we see that

$$L = \frac{L + \frac{2}{L}}{2}.$$

Solving for L we obtain $L = \pm\sqrt{2}$. Since the sequence is bounded below by $\sqrt{2}$ we conclude $L = \sqrt{2}$.

4. Let $\{a_n\}_{n=1}^{\infty}$ be a Cauchy sequence. *Without using the fact that Cauchy sequences are convergent*, prove that $\{a_n : n \in \mathbb{N}\}$ has at most one accumulation point.

Solution. Assume $\{a_n : n \in \mathbb{N}\}$ has two accumulation points a and b, where $a < b$. Let $\epsilon = (b-a)/3$. Choose N such that $n, m \geq N$ imply $|a_n - a_m| < \epsilon$. Choose m such that $m \geq N$ and $a_m \in (a-\epsilon, a+\epsilon)$, and choose n such that $n \geq N$ and $a_n \in (b-\epsilon, b+\epsilon)$. Then

$$|a_n - a_m| = a_n - a_m > b - \epsilon - (a+\epsilon) = b - a - 2\epsilon = \epsilon,$$

contradiction.

5. Using the $\epsilon - N$ definition of convergence, prove that $\{\sqrt{4+1/n}\}_{n=1}^{\infty}$ converges to 2.

Hint: Note that

$$|\sqrt{4+1/n}-2| = \frac{|\sqrt{4+1/n}-2||\sqrt{4+1/n}+2|}{|\sqrt{4+1/n}+2|}.$$

Solution. Let $\epsilon > 0$ and choose $N > 1/(4\epsilon)$. Then $n \geq N$ implies

$$\begin{aligned}
|\sqrt{4+1/n}-2| &= \frac{|\sqrt{4+1/n}-2||\sqrt{4+1/n}+2|}{|\sqrt{4+1/n}+2|} \\
&= \frac{1/n}{\sqrt{4+1/n}+2} \\
&< \frac{1}{4n} \leq \frac{1}{4N} < \epsilon.
\end{aligned}$$

6. Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers with subsequences converging to L and M, where $L \neq M$. Show that then $\{a_n\}_{n=1}^{\infty}$ does not converge.

Solution. Suppose $\{a_n\}_{n=1}^{\infty}$ converges to L, and let $\{a_{n_k}\}_{n=1}^{\infty}$ be any subsequence of $\{a_n\}_{n=1}^{\infty}$. Let $\epsilon > 0$. Choose N such that $n \geq N$ implies $|a_n - L| < \epsilon$. Since $n_k \geq k$, we see that $k \geq N$ implies $|a_{n_k} - L| < \epsilon$. So every subsequence of $\{a_n\}_{n=1}^{\infty}$ converges to L, contradiction.