Math 4603 Midterm 2 Name: ________________________________

Instructions:

This exam contains 6 questions. Each question is worth 20 points. The exam is worth 120 points in total.

1. Define $f : (0, \infty) \to \mathbb{R}$ by $f(x) = \sqrt{x}$. Prove that f is continuous.

Hint: $|\sqrt{x} - \sqrt{a}| = |\sqrt{x} - \sqrt{a}|\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}$.

Solution. Let $a \in (0, \infty)$ and $\epsilon > 0$. Pick $\delta = \epsilon\sqrt{a}$. Then $x \in (0, \infty)$ and $|x - a| < \delta$ imply

$$|f(x) - f(a)| = |\sqrt{x} - \sqrt{a}| = \frac{|x-a|}{\sqrt{x}+\sqrt{a}} < \frac{\delta}{\sqrt{x}+\sqrt{a}} < \frac{\delta}{\sqrt{a}} = \epsilon.$$

2. Let $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$. We say that f *has the left-handed limit* L *at* a if for each $\epsilon > 0$, there is $\delta > 0$ such that $a - \delta < x < a$ implies $|f(x) - L| < \epsilon$.

Let f be increasing. Prove that f has a left-handed limit at a.

Solution. We will show f has the limit $L = \sup\{f(z) : z < a\}$ at a. Let $\epsilon > 0$. As $L - \epsilon$ is not an upper bound of $\{f(z) : z < a\}$, we may pick $y < a$ such that $f(y) > L - \epsilon$. Choose $\delta = a - y$. Then $a - \delta < x < a$ implies $y < x < a$ and so

$$L - \epsilon < f(y) \leq f(x) \leq L < L + \epsilon.$$

3. Let $E \subset \mathbb{R}$. Assume that if $x_n \in E$ for all n and $\lim x_n = x$, then $x \in E$. Prove that E is closed.

Solution. Let x be an accumulation point of E. We must show $x \in E$. For each n, pick x_n in E such that $x_n \in (x - 1/n, x + 1/n)$. So $\{x_n\}_{n=1}^{\infty}$ is a sequence in E which clearly converges to x. By the assumption above, $x \in E$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Let $a \in \mathbb{R}$ and define

$$U = \{x \in \mathbb{R} \,:\, f(x) \neq a\}.$$

Show that U is open.

Solution. Let $x \in U$. So $f(x) \neq a$. Let $\epsilon = |f(x) - a|$ and pick $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. Then $y \in (x - \delta, x + \delta)$ implies

$$|f(y) - a| \geq |f(x) - a| - |f(x) - f(y)| = \epsilon - |f(y) - f(x)| > 0,$$

that is, $f(y) \neq a$. This shows that $(x - \delta, x + \delta) \subset U$, so U is open.

5. Define $f : [-1, 1] \to \mathbb{R}$ by $f(x) = x^3$. Prove that f is uniformly continuous.
Hint: $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$.

Solution. Let $\epsilon > 0$ and pick $\delta = \epsilon/3$. Then $x, y \in [-1, 1]$ and $|x - y| < \delta$ imply

$$|f(x) - f(y)| = |x^3 - y^3| \leq |x - y||x^2 + xy + y^2| < \delta(x^2 + |xy| + y^2) \leq 3\delta = \epsilon.$$

6. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is unbounded on every open interval. Prove that your function has this property.

Hint: Consider a modification of the Riemann function.

Solution. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} m, & x = n/m \in \mathbb{Q},\ n \in \mathbb{Z} \setminus \{0\},\ m \in \mathbb{N},\ m, n \text{ have no common divisors} \\ 0, & x \notin \mathbb{Q} \text{ or } x = 0 \end{cases}$$

Let (a, b) be an open interval. Suppose that M is an upper bound for f on (a, b), that is, $f(x) \leq M$ for all $x \in (a, b)$. Then

$$(a, b) \cap \mathbb{Q} \subset \{n/m \in \mathbb{Q} \,:\, 1 \leq m \leq M,\ ma < n < mb\}$$

The set on the RHS above is finite, a contradiction to the fact that (a, b) contains infinitely many rational numbers.