Math 4603 Midterm 3 Name: ____________________________

Instructions:

This exam contains 6 questions. Each question is worth 20 points. The exam is worth 120 points in total.

1. Define $f : (0, \infty) \to \mathbb{R}$ by $f(x) = 1/x$. Use the definition of derivative to show that f is differentiable and $f'(x) = -1/x^2$.

Solution. Let $x \in (0, \infty)$ and compute:

$$f'(x) = \lim_{y \to x} \frac{1/y - 1/x}{y - x} = \lim_{y \to x} \frac{x - y}{xy(y - x)} = \lim_{y \to x} -\frac{1}{xy} = -\frac{1}{x^2},$$

where the last equality comes from continuity of the function $g(y) \equiv -1/(xy)$ on $(0, \infty)$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and suppose that $f'(x) > 0$ for all x. Show that f is strictly increasing, that is, $f(x) < f(y)$ whenever $x < y$.

Solution. Let $x < y$. By MVT, there is $c \in (x, y)$ such that

$$\frac{f(y) - f(x)}{y - x} = f'(c) > 0.$$

Thus $f(y) - f(x) = f'(c)(y - x) > 0$; that is, $f(x) < f(y)$.

3. Define $f : [-1,1] \to \mathbb{R}$ by $f(x) = \sqrt{1-x^2}$. Show that f is uniformly continuous[1] but not Lipschitz continuous.

Solution. We have seen that $g(x) = \sqrt{x}$ is continuous on $[0,\infty)$, and $h(x) = 1-x^2$ is everywhere continuous since it is a polynomial. Since $h([-1,1]) = [0,1] \subset [0,\infty)$, it follows that $f = g \circ h$ is continuous. As $[-1,1]$ is compact, f is uniformly continuous.

Let $M > 0$ and pick $0 < x < 1 - 1/M^2$. Then

$$\left|\frac{f(x)-f(1)}{x-1}\right| = \frac{\sqrt{1-x^2}}{1-x} = \sqrt{\frac{1+x}{1-x}} > \sqrt{\frac{1}{1-x}} > \sqrt{M^2} = M,$$

so $|f(x) - f(1)| > M|x-1|$. Since M was arbitrary, f is not Lipschitz continuous.

[1] Do not use the definition to show this. What do we know about compositions of continuous functions?

4. Let $\phi(x) = 2x - x\ln x$. Prove[2] that ϕ is a contraction mapping on $[2, e]$. If $x_0 = 2$ and $x_{n+1} = \phi(x_n)$ for $n \geq 0$, to what real number does $\{x_n\}_{n=0}^{\infty}$ converge?

Solution. Note that $\phi'(x) = 1 - \ln x \in [0, 1 - \ln 2]$ for $x \in [2, e]$. In particular, ϕ is Lipschitz continuous with constant $K = 1 - \ln 2 < 1$, and ϕ is increasing so

$$\phi([2, e]) \subset [\phi(2), \phi(e)] = [4 - 2\ln 2, e] \subset [2, e] \tag{1}$$

where we have used the fact that $\ln 2 < \ln e = 1$, so that $4 - 2\ln 2 > 2$. We conclude that ϕ is a contraction mapping on $[2, e]$. So $\{x_n\}$ converges to x_*, the unique solution of $\phi(x_*) = x_*$ in $[2, e]$. From the equality in (1) we see that $x_* = e$.

[2] Use without justification the following facts about $f(x) = \ln x$ for $x > 0$: $f'(x) = 1/x$, f is strictly increasing, $f(1) = 0$, and $f(e) = 1$. Also you may take for granted that $e > 2$.

5. Let $f : [a, a+h] \to \mathbb{R}$ be such that f'' exists on $[a, a+h]$, with $f'(a) = 0$ and $f''(a) < 0$. Use Taylor's theorem to show that if f'' is continuous at a, then f has a local maximum at a.

Solution. By Taylor's theorem, for each $t \in [0, h]$ there exists $c \in (a, a+t)$ such that

$$f(a+t) = f(a) + \frac{1}{2} f''(c) t^2.$$

Thus,

$$\lim_{t \to 0} \frac{f(a+t) - f(a)}{t^2} = \lim_{t \to 0} \frac{f''(c)}{2} = \frac{f''(a)}{2} < 0. \tag{2}$$

The second equality in (2) comes by continuity of f'' at a. Since $t^2 > 0$ for all t, equation (2) implies that $f(a+t) - f(a) < 0$ for t in a δ-interval around 0, i.e. f has a local maximum at a.

6. Define $f : [0,3] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \\ 1, & 2 < x \le 3 \end{cases}$$

and let $P = \{0, 1, 2, 3\}$. Compute $U(P, f)$ and $L(P, f)$.

Solution. $U(P, f) = 5$ and $L(P, f) = 2$.