4603 HW10

1. Use IVT and MVT to show that $x^3 + x + 1 = 0$ has exactly one solution.

2. Define $f : (0, \infty) \to \mathbb{R}$ by $f(x) = \log_b x$. Taking for granted the fact that f' exists and is continuous, prove that $f'(x) = c/x$ where c is a constant.

3. *Newton's method* attempts to find solutions to $f(x) = 0$ via the successive approximations

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad x_0 = \text{initial guess.}$$

Let $f(x) = x^2 - 2$ and $x_0 = 2$. Use the contraction mapping theorem to prove that Newton's method will succeed in finding $\sqrt{2}$, that is, $\lim_{n\to\infty} x_n = \sqrt{2}$. Then use the theorem to estimate $\sqrt{2}$ to within 10^{-3}.

4. Let $f : [a, b] \to [a, b]$ be surjective and differentiable, such that $0 < m \leq f'(x) < M$ for all $x \in [a, b]$. Let $y_* \in [a, b]$ and define $\phi : [a, b] \to \mathbb{R}$ by

$$\phi(x) = x - \frac{f(x) - y_*}{M}.$$

Prove that ϕ is a contraction mapping on $[a, b]$. If $x_0 \in [a, b]$ and $x_{n+1} = \phi(x_n)$ for $n \geq 0$, what can be said about $x_* \equiv \lim_{n\to\infty} x_n$?

5. A function $f : (a, b) \to \mathbb{R}$ is called uniformly differentiable if it is differentiable and for each $\epsilon > 0$, there is $\delta > 0$ such that $x, y \in (a, b)$ and $0 < |x - y| < \delta$ imply

$$\left| \frac{f(y) - f(x)}{y - x} - f'(x) \right| < \epsilon.$$

Prove that if f is uniformly differentiable, then f' is uniformly continuous. Is the converse true? Prove it, or provide a counterexample.