# 4603 HW10

1. Use IVT and MVT to show that $x^3 + x + 1 = 0$ has exactly one solution.

*Solution.* Let $f(x) = x^3 + x + 1$. Note that $f(-1) = -1 < 0$ and $f(1) = 3 > 0$ so by IVT there is a solution to $x^3 + x + 1 = 0$ between $-1$ and 1. Suppose there are two such solutions, say $u$ and $v$, with $u \neq v$. Then there is $c$ between $u$ and $v$ such that $f'(c) = [f(v) - f(u)]/(v - u) = 0$. But $f'(c) = 3c^2 + 1 > 0$, contradiction.

2. Define $f : (0, \infty) \to \mathbb{R}$ by $f(x) = \log_b x$. Taking for granted the fact that $f'$ exists and is continuous, prove that $f'(x) = c/x$ where $c$ is a constant.

*Solution.* Let $x \in (0, \infty)$ and observe that

$$\begin{aligned} \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{f\left(1 + \frac{h}{x}\right)}{h} \\ &= \lim_{h\to 0} \frac{f'\left(1 + \frac{h}{x}\right)\frac{1}{x}}{1} \\ &= \frac{f'(1)}{x} \end{aligned} \tag{1}$$

where the first equality in (1) comes from algebraic properties of logarithms[1], the second equality comes from L'Hospital's rule and the chain rule, and the last equality comes from the assumption that $f'$ is continuous.

L'Hospital's rule is justified here because: the numerator and denominator of

$$\frac{f\left(1 + \frac{h}{x}\right)}{h}$$

are both equal to 0 when $h = 0$, both are differentiable functions of $h$, and the derivative of the denominator is always nonzero (it equals 1). The second equality in (1) is thus true *a posteriori*, that is, after we find that the limit in the second line of (1) exists.

3. *Newton's method* attempts to find solutions to $f(x) = 0$ via the successive approximations

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad x_0 = \text{initial guess.} \tag{2}$$

Let $f(x) = x^2 - 2$ and $x_0 = 2$. Use the contraction mapping theorem to prove that Newton's method will succeed in finding $\sqrt{2}$, that is, $\lim_{n\to\infty} x_n = \sqrt{2}$. Then use the theorem to estimate $\sqrt{2}$ to within $10^{-3}$.

*Solution.* Define

$$\phi(x) = \frac{1}{2}\left(x + \frac{2}{x}\right). \tag{3}$$

[1] Namely $\log_b u - \log_b u = \log_b \frac{u}{v}$ for any $u, v > 0$.

Thus $x_{n+1} = \phi(x_n)$ for $n \geq 0$, where $\{x_n\}_{n=0}^{\infty}$ are defined as in (2) and $x_0 = 2$. If we can show $\phi$ is a contraction mapping on $[\sqrt{2}, 2]$ with constant $K$, then

$$|x_n - x_*| \leq \frac{K^n}{1-K}|x_1 - x_0| \tag{4}$$

where $\phi(x_*) = x_*$, and in particular, $\lim x_n = x_*$. It is easy to check from (3) that $\phi(x_*) = x_*$ implies $x_* = \pm\sqrt{2}$; since $x_n \geq 0$ for all $n$ we can conclude $x_* = \sqrt{2}$.

Thus, it only remains to show $\phi$ is a contraction mapping on $[\sqrt{2}, 2]$ and to use (4) to estimate $\sqrt{2}$ to within $10^{-3}$. Note that $\phi'(x) = 1/2 - 1/x^2$, so $0 \leq \phi'(x) \leq 1/4$ for $x \in [\sqrt{2}, 2]$. In particular $\phi$ is increasing, so

$$\phi([\sqrt{2}, 2]) = [\phi(\sqrt{2}), \phi(2)] = [\sqrt{2}, 3/2] \subset [\sqrt{2}, 2].$$

Thus, $\phi$ is a contraction mapping with constant $K = 1/4$. So,

$$|x_n - \sqrt{2}| \leq \frac{2}{3}\left(\frac{1}{4}\right)^n. \tag{5}$$

The right hand side of (5) is less than $10^{-3}$ when $n = 5$. We compute

$$x_5 = 1.414213562373095...$$

In fact, $x_5$ is within $10^{-15}$ of $\sqrt{2}$! (So the estimate in (5) is not necessarily optimal...)

4. Let $f : [a, b] \to [a, b]$ be surjective and differentiable, such that $0 < m \leq f'(x) < M$ for all $x \in [a, b]$. Let $y_* \in [a, b]$ and define $\phi : [a, b] \to \mathbb{R}$ by

$$\phi(x) = x - \frac{f(x) - y_*}{M}. \tag{6}$$

Prove that $\phi$ is a contraction mapping on $[a, b]$. If $x_0 \in [a, b]$ and $x_{n+1} = \phi(x_n)$ for $n \geq 0$, what can be said about $x_* \equiv \lim_{n\to\infty} x_n$?

*Solution.* Note that $\phi'(x) = 1 - f'(x)/M$, so $0 \leq \phi'(x) \leq 1 - m/M$ for all $x \in [a, b]$. In particular $\phi$ is increasing. Observe that $f$ is also increasing, and so since $f$ is surjective, we must have $f(a) = a$ and $f(b) = b$. Now $\phi([a, b]) = [\phi(a), \phi(b)]$ where

$$\phi(a) = a - \frac{a - y_*}{M} \geq a$$
$$\phi(b) = b - \frac{b - y_*}{M} \leq b,$$

which shows $\phi([a, b]) \subset [a, b]$. Thus $\phi$ is a contraction mapping (with constant $1 - m/M$). So $\phi(x_*) = x_*$, which from (6) implies $f(x_*) = y_*$.

5. A function $f : (a, b) \to \mathbb{R}$ is called uniformly differentiable if it is differentiable and for each $\epsilon > 0$, there is $\delta > 0$ such that $x, y \in (a, b)$ and $0 < |x - y| < \delta$ imply

$$\left|\frac{f(y) - f(x)}{y - x} - f'(x)\right| < \epsilon.$$

Prove that if $f$ is uniformly differentiable, then $f'$ is uniformly continuous. Is the converse true? Prove it, or provide a counterexample.

*Solution.* Suppose $f$ is uniformly differentiable. Let $\epsilon > 0$. Pick $\delta > 0$ such that $x, y \in (a, b)$ and $0 < |x - y| < \delta$ imply

$$\left| \frac{f(y) - f(x)}{y - x} - f'(x) \right| < \epsilon/2.$$

Then $x, y \in (a, b)$ and $0 < |x - y| < \delta$ imply

$$|f'(x) - f'(y)| \leq \left| f'(x) - \frac{f(y) - f(x)}{y - x} \right| + \left| \frac{f(x) - f(y)}{x - y} - f'(y) \right| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Conversely, suppose $f'$ is uniformly continuous. Let $\epsilon > 0$ and pick $\delta > 0$ such that $x, y \in (a, b)$ and $|x - y| < \delta$ imply $|f'(x) - f'(y)| < \epsilon$. Let $x, y \in (a, b)$ be such that $0 < |x - y| < \delta$. By MVT there exists $c$ between $x$ and $y$ such that

$$f'(c) = \frac{f(y) - f(x)}{y - x}.$$

Since $c \in (a, b)$ and $0 < |x - c| < |x - y| < \delta$, we have

$$\left| \frac{f(y) - f(x)}{y - x} - f'(x) \right| = \left| f'(c) - f'(x) \right| < \epsilon$$

as required.