4603 HW11

1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(0) = f'(0) = 0$ and $f''(x) - f(x) = 0$ for all $x \in \mathbb{R}$. Use Taylor's theorem to show that $f(x) = 0$ for all $x \in \mathbb{R}$.

Solution. Note that $f''' = (f')'' = f'$, $f'''' = (f'')'' = f''$,... and in general

$$f^{(n)} = \begin{cases} f, & n \text{ even} \\ f', & n \text{ odd} \end{cases}$$

Since $f(0) = f'(0) = 0$, we conclude that every Taylor polynomial for f centered at $a = 0$ is the zero function. Let $x > 0$. For every n,

$$f(x) = \frac{f^{(n+1)}(c)}{(n+1)!}\, x^{n+1}$$

for some $c \in (0, x)$. Since $f^{(n+1)}$ is differentiable, it is continuous, so on the compact set $[0, x]$ it is bounded, say $|f^{(n+1)}| \leq M$. Thus,

$$|f(x)| \leq \frac{Mx^{n+1}}{(n+1)!} \tag{1}$$

The limit as $n \to \infty$ of the RHS of (1) is 0. Thus, $f(x) = 0$. The arguments for $x < 0$ are identical.

2. Define $f : [0, 1] \to \mathbb{R}$ by $f(x) = 0$ if $x \in \mathbb{Q}$ and $f(x) = 1$ if $x \notin \mathbb{Q}$. Prove that f is not Riemann integrable.

Solution. Let $P = \{x_0, \ldots, x_n\}$ be any partition of $[0, 1]$. Note that each interval $[x_{i-1}, x_i]$ contains both rational and irrational points, so $m_i = 0$ and $M_i = 1$ for $i = 1, \ldots, n$. Thus, $U(P, f) - L(P, f) = 1$. As P was arbitrary, f is not Riemann integrable.

3. Let $f : [a, b] \to \mathbb{R}$ be a continuous, nonnegative function. Suppose $f(c) > 0$ for some $c \in [a, b]$. Prove that then $\int_a^b f\, dx > 0$.

Solution. Suppose for simplicity that $c \in (a, b)$. Using continuity of f, pick $0 < \delta < \min\{|c - a|, |c - b|\}$ such that $x \in (c - \delta, c + \delta)$ implies $f(x) > f(c)/2$. Let $P = \{a, c - \delta, c + \delta, b\}$. Since f is nonnegative, $L(P, f) \geq (2\delta)f(c)/2 = \delta f(c) > 0$. Since $L(P, f) \leq \int_a^b f\, dx$ we are done. The cases where $c = a$ or $c = b$ are similar.