4603 HW12

1. Define $f : [0, 1] \to \mathbb{R}$ by $f(x) = x^2$. Use FTC to show that $\int_0^1 f\,dx = 1/3$.

Compare this to Problem 4 of HW11, where you computed this integral by hand.

Solution. Define $F(x) = x^3/3$. Then $F'(x) = x^2$. As F' is continuous, it is (Riemann) integrable. So by FTC1,

$$\int_0^1 x^2\,dx = \int_0^1 F'(x)\,dx = F(1) - F(0) = 1/3.$$

2. Define

$$f(x) = \int_1^x \frac{1}{t}\,dt.$$

Without assuming any facts about logarithms, prove that f is differentiable and strictly increasing. Then prove that $f(xy) = f(x) + f(y)$ for all $x, y > 0$.

Solution. Fix y and let $g(x) = f(xy) = \int_1^{xy} \frac{1}{t}\,dt$. Note that the integrand in the formulas defining f and g is continuous. So by FTC2 and the chain rule,

$$g'(x) = \frac{1}{xy}y = \frac{1}{x} = f'(x).$$

Thus g and f differ by a unique constant of integration. Since this constant depends on the original fixed value of y, it is really a unique function of y:

$$g(x) = f(xy) = f(x) + C(y).$$

By symmetry, identical arguments give

$$g(y) = f(xy) = f(y) + D(x).$$

where D is a unique function. Thus

$$f(x) + C(y) = f(y) + D(x) \tag{1}$$

Observe that $C = D = f$ satisfies (1). The result follows.

3. Show that

$$\frac{1}{3\sqrt{2}} \leq \int_0^1 \frac{x^2}{\sqrt{1+x^2}}\,dx \leq \frac{1}{3}.$$

Hint: If $f(x) \leq g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

Solution. Note that for $x \in [0,1]$,

$$\frac{1}{\sqrt{2}} \leq \frac{1}{\sqrt{1+x^2}} \leq 1.$$

Thus,

$$\frac{1}{3\sqrt{2}} = \int_0^1 \frac{x^2}{\sqrt{2}}\,dx \leq \int_0^1 \frac{x^2}{\sqrt{1+x^2}}\,dx \leq \int_0^1 x^2\,dx = \frac{1}{3}.$$

4. Suppose that $f : [a,b] \to \mathbb{R}$ is integrable and nonnegative. Prove that $\sqrt{f}$ is integrable.

Solution. Say $f([a,b]) \subset [c,d]$ with $c \geq 0$. Define $\phi : [c,d] \to \mathbb{R}$ by $\phi(x) = \sqrt{x}$. Note that ϕ is continuous, so $\sqrt{f} = \phi \circ f$ is integrable.

5. Suppose $f, g : [a,b] \to \mathbb{R}$ are differentiable and f', g' are integrable. Prove that

$$\int_a^b f'g\,dx = f(b)g(b) - f(a)g(a) - \int_a^b g'f\,dx.$$

Solution. By the chain rule,

$$(fg)' = f'g - g'f.$$

As f, g are differentiable, they are continuous, hence integrable. So since sums and products of integrable functions are integrable, $f'g - g'f$ is integrable. Thus $(fg)'$ is also integrable and

$$\int_a^b (fg)'\,dx = \int_a^b f'g\,dx - \int_a^b g'f\,dx$$

The result follows by applying FTC1:

$$\int_a^b (fg)'\,dx = f(b)g(b) - f(a)g(a).$$