4603 HW1

1. Give examples of the following:
   (i) A function which is bijective;
   (ii) A function which is neither injective nor surjective;
   (iii) A function which is injective but not surjective;
   (iv) A function which is surjective but not injective.

Solution. (i) $f : \{0,1\} \to \{0,1\}$, $f(0) = 0$, $f(1) = 1$. (ii) $f : \{0,1\} \to \{0,1\}$, $f(0) = f(1) = 0$. (iii) $f : \{0,1\} \to \{0,1,2\}$, $f(0) = 0$, $f(1) = 1$. (iv) $f : \{0,1,2\} \to \{0,1\}$, $f(0) = f(1) = 0$, $f(2) = 1$.

2. Let $f : S \to T$ be a function and $A \subset S$. Prove that $A \subset f^{-1}(f(A))$ and that $A = f^{-1}(f(A))$ if f is injective.

Solution. Let $x \in A$. Then $f(x) \in f(A)$, so $x \in f^{-1}(f(A))$. This proves $A \subset f^{-1}(f(A))$. Now assume f is injective and let $x \in f^{-1}(f(A))$. Then $f(x) \in f(A)$. Assume that $x \notin A$. As f is injective, $\{f(x)\}$ is disjoint from $f(A)$, contradiction. So $x \in A$ and this shows $f^{-1}(f(A)) \subset A$. We conclude $A = f^{-1}(f(A))$.

3. Let $f : S \to T$ be a function and $B \subset T$. Prove that $B \supset f(f^{-1}(B))$ and that $B = f(f^{-1}(B))$ if f is surjective.

Solution. Let $y \in f(f^{-1}(B))$. Suppose that $y \notin B$. Then $f^{-1}(\{y\})$ is disjoint from $f^{-1}(B)$. Thus $y = f(f^{-1}(\{y\}))$ is disjoint from $f(f^{-1}(B))$, contradiction. So $y \in B$ which shows $B \supset f(f^{-1}(B))$. Now assume f is surjective and let $y \in B$. Then there exists $x \in S$ such that $f(x) = y$. Note that $x \in f^{-1}(\{y\}) \subset f^{-1}(B)$, so $f(x) = y \in f(f^{-1}(B))$. This shows $B \subset f(f^{-1}(B))$ so that $B = f(f^{-1}(B))$.

4. Use the well-ordering principle to prove the principle of strong induction.

Solution. Let $P(n)$ be a statement for every n such that $P(1)$ is true and such that for each $k \in \mathbb{N}$, if $P(1), P(2), \ldots, P(k)$ are all true then so is $P(k+1)$. Assume however that $P(n)$ is false for some n and let

$$S = \{n \in \mathbb{N} : P(n) \text{ is false}\}.$$

By assumption S is nonempty, so by the well-ordering principle it has a least element n_0. Since $P(1)$ is true we have $n_0 \geq 2$. Now observe that $P(1), P(2), \ldots, P(n_0 - 1)$ are all true yet $P(n_0)$ is false, which contradicts our above assumption.

5. Use strong induction to prove that every natural number can be written as a sum of distinct

powers of two.

Solution. Let $P(n)$ be the statement "n can be written as a sum of distinct powers of two." Note that $P(1)$ is true since $1 = 2^0$. Assume that $P(1), P(2), \ldots, P(k)$ are all true. Let 2^m be the largest power of 2 such that $k+1-2^m \geq 1$. So $k+1-2^m \in \{1, 2, \ldots, k\}$ and $k+1-2^{m+1} \leq 0$. In particular $k+1 \leq 2^{m+1}$. If $k+1 = 2^{m+1}$ we are done, so assume $k+1 < 2^{m+1}$. Then

$$k + 1 - 2^m < 2^{m+1} - 2^m = 2^m. \tag{1}$$

Now $k + 1 - 2^m$ can be written as a sum of distinct powers of two, and all of these powers are less than 2^m. It follows that $k + 1$ can be written as a sum of distinct powers of two.