# 4603 HW2

1. Let $S \subset \mathbb{R}$ be nonempty. Prove the following statements:

(i) $S$ is countable if and only if there exists an injective function $f : S \to \mathbb{N}$.

(ii) $S$ is countable if and only if there exists a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers such that $S = \{a_n : n \in \mathbb{N}\}$.

2. Construct a bijective function $f : 2^{\mathbb{N}} \to S$, where $S$ is the set of all sequences of 0's and 1's. Noting that each real number between 0 and 1 has a base two decimal expansion, what do you expect to be true about the cardinality of the set $(0, 1) \subset \mathbb{R}$?

3. Show that $\sqrt{3} \notin \mathbb{Q}$. Then show that $z = \sup\{x \in \mathbb{R} : x^2 < 3\}$ exists and satisfies $z^2 = 3$. Conclude that $\sqrt{3} \in \mathbb{R} \setminus \mathbb{Q}$.

4. Let $S \subset \mathbb{R}$ be nonempty. A real number $y$ is a *lower bound* of $S$ if $x \geq y$ for all $x \in S$, and $S$ is said to be *bounded below* if such a lower bound exists. Furthermore $y$ is said to be the *greatest lower bound* of $S$ if $y \geq z$ for every lower bound $z$ of $S$. In this case we write $y = \inf S$ (inf is short for "infimum").

Use the least upper bound property of $\mathbb{R}$ to prove that every nonempty subset of $\mathbb{R}$ which is bounded below has a greatest lower bound.

5. Define the sequence $\{a_n\}_{n=1}^{\infty}$ by $a_n = 0$ if $n$ is a power of 10, and $a_n = 1$ otherwise. Prove that $\{a_n\}_{n=1}^{\infty}$ does not converge to 1. How is this different from proving that $\{a_n\}_{n=1}^{\infty}$ does not converge?