4603 HW2

1. Let $S \subset \mathbb{R}$ be nonempty. Prove the following statements:

(i) S is countable if and only if there exists an injective function $f : S \to \mathbb{N}$.

(ii) S is countable if and only if there exists a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers such that $S = \{a_n : n \in \mathbb{N}\}$.

Solution. (i) Assume S is countable, that is, S is either finite or countably infinite. First suppose S is finite, and write $S = \{s_1, \ldots, s_n\}$. Then $f : S \to \mathbb{N}$, $f(s_j) = j$ is an injective function. Now suppose S is countably infinite. This means there exists a bijective function $f : S \to \mathbb{N}$, which of course is also an injection. Conversely, assume there exists an injective function $S \to \mathbb{N}$. Then $f : S \to f(S)$ is a bijective function, meaning $|S| = |f(S)|$. If $f(S)$ is finite then so is S and we are done. So assume $f(S)$ is infinite. Then by a theorem in class, $f(S)$ is countably infinite. (Recall we proved that infinite sets of natural numbers are countably infinite.) So S is countably infinite and we are done.

(ii) Assume S is countable. If S is finite, write $S = \{s_1, \ldots, s_n\}$, and define $a_1 = s_1, \ldots, a_n = s_n$ and $a_m = s_1$ for $m > n$. Then $S = \{a_n : n \in \mathbb{N}\}$ by construction. Now assume S is countably infinite. Then there is a bijection $f : S \to \mathbb{N}$. Define $a_n = f^{-1}(n)$; then $S = \{a_n : n \in \mathbb{N}\}$ as desired. Conversely, assume there exists a sequence $\{a_n\}_{n=1}^{\infty}$ such that $S = \{a_n : n \in \mathbb{N}\}$. Define $f : S \to \mathbb{N}$ by $f(x) = \min\{n \in \mathbb{N} : a_n = x\}$. If $x \neq y \in S$ then $\{n \in \mathbb{N} : a_n = x\}$ is disjoint from $\{n \in \mathbb{N} : a_n = y\}$, so that $f(x) \neq f(y)$. This shows f is injective.

2. Construct a bijective function $f : 2^{\mathbb{N}} \to S$, where S is the set of all sequences of 0's and 1's. Noting that each real number between 0 and 1 has a base two decimal expansion, what do you expect to be true about the cardinality of the set $(0, 1) \subset \mathbb{R}$?

Solution. Define $f : 2^{\mathbb{N}} \to S$ by $f(A) = \{a_n\}_{n=1}^{\infty}$ where $a_n = 1$ if $n \in A$, and $a_n = 0$ otherwise. If $A \neq B$ then either there exists $n \in A \setminus B$ or there exists $n \in B \setminus A$. In both cases the nth term of the sequences $f(A)$ and $f(B)$ are different, so that $f(A) \neq f(B)$. So f is injective. Now given a sequence $\{a_n\}_{n=1}^{\infty} \in S$, let $A = \{n \in \mathbb{N} : a_n = 1\}$. Then by construction $f(A) = \{a_n\}_{n=1}^{\infty}$. This shows f is surjective. We conlude f is bijective.

Noticing that any real number in $(0, 1)$ has a base two expansion (i.e. a representation as a sequence of 0's and 1's after a decimal point), one might expect that $|(0, 1)| = |2^{\mathbb{N}}|$. This is indeed true, though one must be careful because a given real number may have two different base two expansions.

3. Show that $\sqrt{3} \notin \mathbb{Q}$. Then show that $z = \sup\{x \in \mathbb{R} : x^2 < 3\}$ exists and satisfies $z^2 = 3$. Conclude that $\sqrt{3} \in \mathbb{R} \setminus \mathbb{Q}$.

Solution. Suppose that $\sqrt{3} = m/n$ with $m, n \in \mathbb{N}$ having no prime factors in common. Then $3 = m^2/n^2$ so $3n^2 = m^2$, which shows m^2 is divisible by 3. By Euclid's lemma m is also divisible by 3. So $m = 3k$ for some $k \in \mathbb{N}$. Now $3n^2 = (3k)^2$ so $n^2 = 3k^2$. Thus n is also divisible by 3, contradiction. So $\sqrt{3} \notin \mathbb{Q}$.

Now $\{x \in \mathbb{R} : x^2 < 3\}$ is nonempty since it contains 0, and is bounded above by 3 (for example). So $z = \sup\{x \in \mathbb{R} : x^2 < 3\}$ exists. Suppose $z \neq 3$; then either $z < 3$ or $z > 3$. Assume $z > 3$, and pick $\delta > 0$ such that $\delta < (z^2 - 3)/(2z)$. Then $(z - \delta)^2 > 3$, which implies $z - \delta$ is an upper bound of $\{x \in \mathbb{R} : x^2 < 3\}$, contrary to z being the least upper bound. Now assume $z < 3$, and pick $\delta \in (0, 1)$ such that $\delta < (3 - z^2)/(2z + 1)$. Then $(z + \delta)^2 < 3$, which means $z + \delta \in \{x \in \mathbb{R} : x^2 < 3\}$, contrary to z being an upper bound of $\{x \in \mathbb{R} : x^2 < 3\}$.

Using the least upper bound axiom of $\mathbb{R}$, we can conclude that $\sqrt{3} \in \mathbb{R}$. So $\sqrt{3} \in \mathbb{R} \setminus \mathbb{Q}$.

4. Let $S \subset \mathbb{R}$ be nonempty. A real number y is a *lower bound* of S if $x \geq y$ for all $x \in S$, and S is said to be *bounded below* if such a lower bound exists. Furthermore y is said to be the *greatest lower bound* of S if $y \geq z$ for every lower bound z of S. In this case we write $y = \inf S$ (inf is short for "infimum").

Use the least upper bound property of $\mathbb{R}$ to prove that every nonempty subset of $\mathbb{R}$ which is bounded below has a greatest lower bound.

Let $S \subset \mathbb{R}$ be nonempty and bounded below, say S contains y and is bounded below by w. Define $-S = \{-x : x \in S\}$. Then $-S$ is nonempty since it contains $-y$, and $-S$ is bounded above by $-w$: if $-x \in -S$ then $x \in S$ so that $x \geq w$ and then $-x \leq -w$. Thus, $-z \equiv \sup(-S)$ exists. We claim $z = \inf(S)$. Let $x \in S$. Then $-x \in -S$ so $-x \leq -z$, which implies $x \geq z$. So z is a lower bound of S. Now suppose v is another lower bound of S. The arguments above show that then $-v$ is an upper bound of $-S$, and so $-v \geq z$. Now $v \leq z$ which shows z is the greatest lower bound of S, i.e. $z = \inf(S)$.

5. Define the sequence $\{a_n\}_{n=1}^{\infty}$ by $a_n = 0$ if n is a power of 10, and $a_n = 1$ otherwise. Prove that $\{a_n\}_{n=1}^{\infty}$ does not converge to 1. How is this different from proving that $\{a_n\}_{n=1}^{\infty}$ does not converge?

Solution. Let $\epsilon = 1/2$ and let N be arbitrary. Let n be a power of 10 such that $n \geq N$. Now $n \geq N$ yet $|a_n - 1| = |0 - 1| \geq \epsilon$. This shows $\{a_n\}_{n=1}^{\infty}$ does not converge to 1.

To show that $\{a_n\}_{n=1}^{\infty}$ does not converge, one has to show it does not converge to L, for *any* L.