4603 HW3

1. Prove that Cauchy sequences are bounded.

Solution. Let $\{a_n\}_{n=1}^{\infty}$ be Cauchy. Choose N such that $n, m \geq N$ implies $|a_n - a_m| < 1$. Then in particular $n \geq N$ implies $a_N - 1 < a_n < a_N + 1$. So $\{a_n\}_{n=1}^{\infty}$ is bounded above by $\max\{a_1, \ldots, a_{N-1}, a_N + 1\}$ and below by $\min\{a_1, \ldots, a_{N-1}, a_N - 1\}$.

2. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $\{a_n : n \in \mathbb{N}\}$ has exactly one accumulation point, L. Must it be true that $\lim a_n = L$? Prove it, or give a counterexample.

Solution. This is false. For example let $a_n = 1/n$ if n is odd, and $a_n = 1$ if n is even. Then $\{a_n : n \in \mathbb{N}\}$ has exactly one accumulation point, 0. But $\{a_n\}_{n=1}^{\infty}$ does not converge.

3. Let $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$ for $n \geq 1$. Prove that $\lim a_n = 2$.

Solution. Note that $a_1 = \sqrt{2} < a_2 = \sqrt{2 + \sqrt{2}}$. Assume $a_{n-1} < a_n$. Then $a_n = \sqrt{2 + a_{n-1}} < \sqrt{2 + a_n} = a_{n+1}$. By induction, $\{a_n\}_{n=1}^{\infty}$ is increasing. Now note that $a_1 \leq 2$ and suppose $a_n \leq 2$. Then $a_{n+1} = \sqrt{2 + a_n} \leq \sqrt{2 + 2} = 2$. By induction $\{a_n\}_{n=1}^{\infty}$ is bounded above by 2. We can conclude that $\{a_n\}_{n=1}^{\infty}$ converges, say to L. By arithmetic properties of limits, $\{\sqrt{2 + a_n}\}_{n=1}^{\infty}$ converges to $\sqrt{2 + L}$. But $\sqrt{2 + a_n} = a_{n+1}$ which shows $\{\sqrt{2 + a_n}\}_{n=1}^{\infty}$ converges to L, so $L = \sqrt{2 + L}$. We conclude $L = 2$.

4. If S is a set of real numbers, let A_S be the set of all accumulation points of S. Give an example of a bounded countably infinite set $S \subset \mathbb{R}$ such that A_S is countably infinite.

Solution. Consider

$$S = \{1/n + 1/m : n, m \in \mathbb{N}\}.$$

Then S is countably infinite and so is

$$A_S = \{1/n : n \in \mathbb{N}\} \cup \{0\}.$$

5. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $a_n \neq 0$ for all n and $\lim a_n = L$, where $L \neq 0$. Prove that $\lim a_n^{-1} = L^{-1}$.

Solution. Let $A = \inf\{|a_n| : n \in \mathbb{N}\}$. We first show that $A > 0$. Choose M such that $n \geq M$ implies $|a_n - L| < |L|/2$. Then $|a_n| > |L|/2$ for $n \geq M$ and so $A \geq \min\{|a_1|, \ldots, |a_M|, |L|/2\} > 0$. Let $\epsilon > 0$, and choose N such that $n \geq N$ implies $|a_n - L| < \epsilon|AL|$. Then $n \geq N$ implies

$$\left|a_n^{-1} - L^{-1}\right| = \frac{|a_n - L|}{|a_n L|} \leq \frac{|a_n - L|}{|AL|} < \epsilon.$$