4603 HW4

1. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Prove that if x is an accumulation point of $\{a_n : n \in \mathbb{N}\}$, then $\{a_n\}_{n=1}^{\infty}$ has a subsequence converging to x.

Solution. Let x be an accumulation point of $\{a_n : n \in \mathbb{N}\}$. Then for each $\epsilon > 0$, $(x - \epsilon, x + \epsilon)$ contains infinitely many of the a_n's. Choose n_1 such that $a_{n_1} \in (x-1, x+1)$. Then choose $n_2 > n_1$ such that $a_{n_2} \in (x-1/2, x+1/2)$. Then choose $n_3 > n_2$ such that $a_{n_3} \in (x-1/3, x+1/3)$... in this way we obtain a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ such that $a_{n_k} \in (x - 1/k, x + 1/k)$ for all k. To see that $\{a_{n_k}\}_{k=1}^{\infty}$ converges to x, let $\epsilon > 0$ and choose $N > 1/\epsilon$. Then $k \geq N$ implies $|a_{n_k} - x| < 1/k \leq 1/N < \epsilon$ as desired.

2. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 0$ if $x < 0$ and $f(x) = 1$ if $x \geq 0$. Show that f has no limit at 0 in the following ways:

(i) By using the $\epsilon - \delta$ definition of limits of functions;

(ii) By using the sequential definition of limits of functions.

Solution. (i) Suppose f has the limit L at zero. Then there is $\delta > 0$ such that $x \in (-\delta, \delta) \setminus \{0\}$ implies $f(x) \in (L - 1/2, L + 1/2)$. Note that $(L - 1/2, L + 1/2)$ cannot contain both 0 and 1. But $\pm\delta/2 \in (-\delta, \delta) \setminus \{0\}$ and $f(-\delta/2) = 0$, $f(\delta/2) = 1$, contradiction.

(ii) Note that $\{1/n\}_{n=1}^{\infty}$ and $\{-1/n\}_{n=1}^{\infty}$ are both sequences with only nonzero values which converge to zero. However, $f(1/n) = 1$ for all n and $f(-1/n) = 0$ for all n, so $\{f(1/n)\}_{n=1}^{\infty}$ converges to 1, while $\{f(-1/n)\}_{n=1}^{\infty}$ converges to 0. Thus f has no limit at 0.

3. Let a be a positive real number. Prove that $\lim_{x \to a} \sqrt{x} = \sqrt{a}$.

Solution. Let $\epsilon > 0$ and choose $\delta = \epsilon\sqrt{a}$. Then $0 < |x - a| < \delta$ implies

$$
\begin{aligned}
|\sqrt{x} - \sqrt{a}| &= |\sqrt{x} - \sqrt{a}| \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}} \\
&= \frac{|x - a|}{\sqrt{x} + \sqrt{a}} \leq \frac{|x - a|}{\sqrt{a}} < \frac{\delta}{\sqrt{a}} < \epsilon.
\end{aligned}
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