4603 HW5

1. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2, & x \neq 2 \\ 11, & x = 2 \end{cases}.$$

Prove that $\lim_{x \to 2} f(x) = 4$.

2. Let $f, g : D \to \mathbb{R}$, let a be an accumulation point of D, and assume $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} g(x) = M$. Prove that then $\lim_{x \to a} (fg)(x) = LM$.

3. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 1$ if $x \in \mathbb{Q}$ and $f(x) = 0$ if $x \notin \mathbb{Q}$. Let a be any real number. Show that f does not have a limit at a.

4. Give an example of an increasing function $f : [a, b] \to \mathbb{R}$ such that the set

$$S = \{x \in [a, b] \, : \, f \text{ does not have a limit at } x\}$$

is countably infinite.

5. Let $f : [a, b] \to \mathbb{R}$ and assume f has a limit at $c \in (a, b)$. Define

$$U(x) = \inf\{f(y) \, : \, y > x\}$$
$$L(x) = \sup\{f(y) \, : \, y < x\}.$$

Must it be true that $U(c) = L(c)$? Prove it, or provide a counterexample.