4603 HW5

1. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2, & x \neq 2 \\ 11, & x = 2 \end{cases}$$

Prove that $\lim_{x\to 2} f(x) = 4$.

Solution. Let $\epsilon > 0$ and choose $\delta = \min\{1, \epsilon/5\}$. Then $0 < |x-2| < \delta$ implies

$$|f(x) - 4| = |x^2 - 4| = |x-2||x+2| < \delta|x+2| < 5\delta \leq \epsilon.$$

2. Let $f, g : D \to \mathbb{R}$, let a be an accumulation point of D, and assume $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$. Prove that then $\lim_{x\to a}(fg)(x) = LM$.

Solution. Let $\epsilon > 0$. Since f has a limit at a, it is bounded in a δ-interval around a. Choose $\delta_0 > 0$ and $B > 0$ such that $x \in D$ and $0 < |x-a| < \delta_0$ implies $|f(x)| \leq B$. Choose $\delta_f > 0$ and $\delta_g > 0$ such that

$$x \in D \text{ and } 0 < |x-a| < \delta_f \implies |f(x) - L| < \frac{\epsilon}{2|M|}$$
$$x \in D \text{ and } 0 < |x-a| < \delta_g \implies |g(x) - M| < \frac{\epsilon}{2B}$$

Let $\delta = \min\{\delta_0, \delta_f, \delta_g\}$. Then $x \in D$ and $0 < |x-a| < \delta$ implies

$$\begin{aligned} |f(x)g(x) - LM| &= |f(x)(g(x) - M) + M(f(x) - L)| \\ &\leq |f(x)||g(x) - M| + |M||f(x) - L| \\ &< B\frac{\epsilon}{2B} + |M|\frac{\epsilon}{2|M|} = \epsilon. \end{aligned}$$

3. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 1$ if $x \in \mathbb{Q}$ and $f(x) = 0$ if $x \notin \mathbb{Q}$. Let a be any real number. Show that f does not have a limit at a.

Solution. Note that if $\epsilon = 1/2$ then $(L-\epsilon, L+\epsilon)$ will not contain both 0 and 1. For any $\delta > 0$, there is $x \in \mathbb{Q}$ and $y \notin \mathbb{Q}$ such that $x, y \in (-\delta, \delta) \setminus \{a\}$. (From a theorem in class, between any two distinct real numbers there is a rational and irrational number.) Then $0 < |x-a| < \delta$ and $0 < |y-a| < \delta$, but either $f(x) = 1 \notin (L-\epsilon, L+\epsilon)$ or $f(y) = 0 \notin (L-\epsilon, L+\epsilon)$.

4. Give an example of an increasing function $f : [a,b] \to \mathbb{R}$ such that the set

$$S = \{x \in [a,b] \,:\, f \text{ does not have a limit at } x\}$$

is countably infinite.

Solution. One such function would be $f : [0,1] \to \mathbb{R}$,

$$f(x) = \begin{cases} 1/n, & x \in \left(\frac{1}{n+1}, \frac{1}{n}\right], \quad n = 1, 2, \dots \\ 0, & x = 0 \end{cases}$$

For this function we have $S = \{1/n : n = 2, 3, 4, \dots\}$.

5. Let $f : [a,b] \to \mathbb{R}$ and assume f has a limit at $c \in (a,b)$. Define

$$U(x) = \inf\{f(y) : y > x\}$$
$$L(x) = \sup\{f(y) : y < x\}.$$

Must it be true that $U(c) = L(c)$? Prove it, or provide a counterexample.

Solution. No. For example, define $f : [0,2] \to \mathbb{R}$ by $f(x) = 1$ if $x = 0$ and $f(x) = 0$ otherwise. Then f has a limit at 1, but $U(1) = 0 \neq 1 = L(1)$.