4603 HW6

1. If $f + g$ has a limit at a, must it be true that both f and g have limits at a? Prove it, or give a counterexample.

2. Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & x \notin \mathbb{Q} \\ 1/\sqrt{m}, & x = n/m \in \mathbb{Q} \text{ where } n \neq 0 \text{ and } n, m \text{ have no common divisors} \\ 1, & x = 0 \end{cases}$$

Let $a \in [0, 1]$. Prove that $\lim_{x \to a} f(x) = f(a)$ if and only if $a \notin \mathbb{Q}$.

3. Let $f : [a, b] \to \mathbb{R}$ be a bounded function, define $g : (a, b) \to \mathbb{R}$ by $g(x) = \sup\{f(y) : y < x\}$, and let $c \in (a, b)$. Prove that if $\lim_{x \to c} f(x) = f(c)$, then $\lim_{x \to c} g(x) = g(c)$.

4. Let f and g be defined as in Problem 3, and let $c \in (a, b)$. If f has a limit at c, must g have a limit at c? Either prove it, or provide a counterexample.

5. Prove that if $f : [a, b] \to \mathbb{R}$ is increasing, then f has a limit at b.