4603 HW6

1. If $f+g$ has a limit at a, must it be true that both f and g have limits at a? Prove it, or give a counterexample.

Solution. No, for example define $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=0$ for $x<0$ and $f(x)=1$ for $x\geq 0$, and let $g=-f$. Then $f+g\equiv 0$, so $f+g$ has a limit at 0, but neither f nor g has a limit at 0.

2. Define $f:[0,1]\to\mathbb{R}$ by

$$f(x)=\begin{cases}0, & x\notin\mathbb{Q}\\ 1/\sqrt{m}, & x=n/m\in\mathbb{Q}\text{ where } n\neq 0 \text{ and } n,m \text{ have no common divisors}\\ 1, & x=0\end{cases}$$

Let $a\in[0,1]$. Prove that $\lim_{x\to a} f(x)=f(a)$ if and only if $a\notin\mathbb{Q}$.

Solution. Let $\epsilon>0$ and consider $S=\{x\in[0,1]\,:\,f(x)\geq\epsilon\}$. Observe that S is finite: if $x\in S$, then x is rational and can be written $x=n/m$ in lowest terms with $1\leq m\leq 1/\epsilon^2$ and $0\leq n\leq m$. Write $S=\{x_1,\ldots,x_k\}$ and let $\delta=\min_{x_j\neq a}|x_j-a|$. Then $x\in[0,1]$ and $0<|x-a|<\delta$ implies $x\neq x_j$ for $j=1,\ldots,k$, so that $|f(x)-0|=f(x)<\epsilon$. This shows $\lim_{x\to a} f(x)=0$. The result follows from definition of f.

3. Let $f:[a,b]\to\mathbb{R}$ be a bounded function, define $g:(a,b)\to\mathbb{R}$ by $g(x)=\sup\{f(y)\,:\,y<x\}$, and let $c\in(a,b)$. Prove that if $\lim_{x\to c} f(x)=f(c)$, then $\lim_{x\to c} g(x)=g(c)$.

Solution. We recall the following simple fact:

$$(*)\text{ If } x,y\in(a,b) \text{ with } x<y, \text{ then } f(x)\leq g(y).$$

Assume $\lim_{x\to c} f(x)=f(c)$. We will consider two cases: $g(c)=f(c)$ and $g(c)\neq f(c)$. Assume $g(c)=f(c)$. Let $\epsilon>0$ and choose $\delta>0$ such that $|x-c|<\delta$ implies[1] $|f(x)-f(c)|<\epsilon$. Pick u,v such that $c-\delta<u<c<v<c+\delta$. Note that $f(x)\leq g(c)$ for $x<c$ and $f(x)<f(c)+\epsilon$ for $c\leq x<v$. Thus $x\in(u,v)$ and $(*)$ imply

$$g(c)-\epsilon=f(c)-\epsilon<f(u)\leq g(x)\leq\max\{g(c),f(c)+\epsilon\}=g(c)+\epsilon.$$

It easily follows that $\lim_{x\to c} g(x)=g(c)$. Now assume $g(c)\neq f(c)$ and let $\epsilon=|f(c)-g(c)|$. Choose δ such that $|x-c|<\delta$ implies $|f(x)-f(c)|<\epsilon$. Then either $f(x)>g(c)$ for all $x\in(c-\delta,c+\delta)$, or $f(x)<g(c)$ for all $x\in(c-\delta,c+\delta)$. The former is impossible by $(*)$, so the latter must hold, which implies g is constant on $(c-\delta,c+\delta)$. It easily follows that $\lim_{x\to c} g(x)=g(c)$.

[1] Since $c\in(a,b)$, $\delta>0$ can be chosen small enough so that $|x-c|<\delta$ also implies $x\in[a,b]$.

4. Let f and g be defined as in Problem 3, and let $c \in (a, b)$. If f has a limit at c, must g have a limit at c? Either prove it, or provide a counterexample.

Solution. No. Define $f : [-1, 1] \to \mathbb{R}$ by $f(x) = 0$ for $x \neq 0$ and $f(0) = 1$. Then $g : (-1, 1) \to \mathbb{R}$ has the formula $g(x) = 0$ for $x \leq 0$ and $g(x) = 1$ for $x > 0$. So f has a limit at 0, but g does not have a limit at 0.

5. Prove that if $f : [a, b] \to \mathbb{R}$ is increasing, then f has a limit at b.

Solution. Observe that f is bounded above by $f(b)$. Let $\epsilon > 0$ and $L = \sup\{f(x) : x \in [a, b)\}$. Pick $y \in [a, b)$ such that $f(y) > L - \epsilon$, and let $\delta = |b - y|$. Then $x \in [a, b]$ and $0 < |x - b| < \delta$ implies $y < x < b$, so that

$$L - \epsilon < f(y) \leq f(x) \leq L < L + \epsilon.$$