4603 HW7

1. If $f+g$ is continuous a, must it be true that both f and g are continuous at a? Prove it, or provide a counterexample.

Solution. No. Let $f(x)=0$ if $x<0$ and $f(x)=1$ if $x\geq 0$, and let $g=-f$. Then $f+g\equiv 0$ is continuous at 0 but neither f nor g is continuous at 0.

2. Define $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=x^2$. Prove that f is continuous.

Solution. Let $a\in\mathbb{R}$ and $\epsilon>0$. Let $M=\max\{|2a-1|,|2a+1|\}$ and choose $\delta=\min\{1,\epsilon/M\}$. Then $|x-a|<\delta$ implies

$$|f(x)-f(a)|=|x^2-a^2|=|x-a||x+a|<\delta M<\epsilon.$$

3. Define $f:(0,1)\to\mathbb{R}$ by $f(x)=1/x$. Prove that f is continuous but not uniformly continuous.

Solution. Let $a\in(0,1)$ and $\epsilon>0$. Choose $\delta=\min\{a/2,\epsilon a^2/2\}$. Then $x\in(0,1)$ and $|x-a|<\delta$ implies

$$|f(x)-f(a)|=\left|\frac{1}{x}-\frac{1}{a}\right|=\frac{|x-a|}{xa}<\frac{\delta}{a^2/2}<\epsilon.$$

This proves continuity of f. Now let $\epsilon=1$ and let $\delta\in(0,1)$. Pick $x=\delta/2$ and $y=\delta$. Then $x,y\in(0,1)$ and $|x-y|<\delta$ but

$$|f(x)-f(y)|=\left|\frac{1}{x}-\frac{1}{y}\right|=\frac{|x-y|}{xy}=\frac{\delta/2}{\delta^2/2}=\frac{1}{\delta}\geq\epsilon.$$

This shows f is not uniformly continuous.

4. Prove that if $E\subset R$ is nonempty, closed and bounded above, then $\sup E$ is an element of E.

Solution. Let $z=\sup E$ and assume $z\notin E$. Let $\epsilon>0$. Since $z-\epsilon$ is not an upper bound of E, there is $x\in E$ such that $z-\epsilon<x<z$. Thus $(z-\epsilon,z+\epsilon)$ has a point of E besides z, showing z is an accumulation point of E. Since E is closed it contains z, contradiction.

5. Prove the following statements about open sets in $\mathbb{R}$:

(i) Any union of open sets is open.
(ii) A finite intersection of open sets is open.
(iii) A set is open if and only if it is a union of open intervals.

Solution. (i) Let $\{U_\alpha\}_{\alpha \in A}$ be a collection of open sets and let $U = \cup_{\alpha \in A} U_\alpha$. Let $x \in U$. Then $x \in U_\alpha$ for some $\alpha \in A$. Pick $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset U_\alpha$. Since $U_\alpha \subset U$, $(x - \epsilon, x + \epsilon) \subset U$, showing U is open.

(ii) Let $U_1, \ldots, U_n$ be open sets and $U = U_1 \cap \ldots \cap U_n$. Let $x \in U$. Then $x \in U_j$ for $j = 1, \ldots, n$. For each $j = 1, \ldots, n$, pick ϵ_j such that $(x - \epsilon_j, x + \epsilon_j) \subset U_j$. Let $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\}$. Then for each $j = 1, \ldots, n$ we have $(x - \epsilon, x + \epsilon) \subset (x - \epsilon_j, x + \epsilon_j) \subset U_j$. Thus $(x - \epsilon, x + \epsilon) \subset U_1 \cap \ldots \cap U_n$, proving $U_1 \cap \ldots \cap U_n$ is open.

(iii) Assume U is open. For each $x \in U$ pick ϵ_x such that $(x - \epsilon_x, x + \epsilon_x) \subset U$. Then $U = \cup_{x \in U} (x - \epsilon_x, x + \epsilon_x)$. The converse follows from part (i).