4603 HW8

1. Let A and B be disjoint subsets of $\mathbb{R}$, and $f : A \cup B \to \mathbb{R}$ a continuous function. Assume f is uniformly continuous on A and on B. Must it be true that f is uniformly continuous on $A \cup B$? Prove it or provide a counterexample.

Solution. No. Let $A = (0, 1)$ and $B = (1, 2)$, and define $f : A \cup B \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & x \in A \\ 1, & x \in B \end{cases}$$

We first show f is continuous. Let $x \in A \cup B$ and $\epsilon > 0$. If $x \in A$, pick $\delta > 0$ such that $(x - \delta, x + \delta) \subset A$; if $x \in B$ pick $\delta > 0$ such that $(x - \delta, x + \delta) \subset B$. Let $y \in A \cup B$ be such that $|x - y| < \delta$. Then either x and y are both in A, or they are both in B. In either case $|f(x) - f(y)| = 0 < \epsilon$, proving f is continuous.

Now we show f is uniformly continuous on A (the proof that f is uniformly continuous on B is analogous). Let $\epsilon > 0$ and pick any $\delta > 0$. Then $x, y \in A$ and $|x - y| < \delta$ imply $|f(x) - f(y)| = |0 - 0| = 0 < \epsilon$.

To see that f is not uniformly continuous on $A \cup B$, let $\epsilon = 1$ and take any $\delta > 0$. Let $x = 1 - \min\{1/2, \delta/4\}$ and $y = 1 + \min\{1/2, \delta/4\}$. Then $x, y \in A \cup B$ and $|x - y| < \delta$ but $|f(x) - f(y)| = |0 - 1| = 1 \geq \epsilon$.

2. Prove that $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if for each open set $U \subset \mathbb{R}$, the preimage $f^{-1}(U)$ is open. Also prove a similar statement for closed sets.

Solution. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and $U \subset \mathbb{R}$ is open. Let $x \in f^{-1}(U)$. As $f(x) \in U$, there is $\epsilon > 0$ such that $(f(x) - \epsilon, f(x) + \epsilon) \subset U$. Pick $\delta > 0$ such that $y \in (x - \delta, x + \delta)$ implies $f(y) \in (f(x) - \epsilon, f(x) + \epsilon)$. Thus $y \in (x - \delta, x + \delta)$ implies $f(y) \in U$; that is, $f(x - \delta, x + \delta) \subset U$. It follows that $(x - \delta, x + \delta) \subset f^{-1}(U)$, which proves $f^{-1}(U)$ is open.

Conversely, assume that for each open set $U \subset \mathbb{R}$, the preimage $f^{-1}(U)$ is open. Let $x \in \mathbb{R}$ and $\epsilon > 0$. Let $U = (f(x) - \epsilon, f(x) + \epsilon)$. Since U is open, $f^{-1}(U)$ is open. Also $x \in f^{-1}(U)$, so there exists $\delta > 0$ such that $(x - \delta, x + \delta) \subset f^{-1}(U)$. Then $y \in (x - \delta, x + \delta)$ implies $y \in f^{-1}(U)$ and so $f(y) \in U = (f(x) - \epsilon, f(x) + \epsilon)$. This proves f is continuous.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $A \subset \mathbb{R}$ compact. Must it be true that $f^{-1}(A)$ is compact? Prove it or provide a counterexample.

Solution. No. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \frac{1}{1 + x^2}.$$

and observe that $0 \leq f(x) \leq 1$ for all $x \in \mathbb{R}$. Now $[0, 1] \subset \mathbb{R}$ is compact but $f^{-1}([0, 1]) = \mathbb{R}$ is not compact.