4603 HW9

1. Suppose $f : [a,b] \to \mathbb{R}$ is continuous and injective. Prove that f is either strictly increasing or strictly decreasing.

2. Let $f : \mathbb{R} \to \mathbb{R}$ and suppose that for each $c \in \mathbb{R}$, the equation $f(x) = c$ has exactly two solutions. Prove that f is not continuous.

3. Define $f : (0,\infty) \to \mathbb{R}$ by $f(x) = \sqrt{x}$. Use the definition of derivative to show that f is differentiable and $f'(x) = \frac{1}{2\sqrt{x}}$.

4. Let $f : [a,b] \to \mathbb{R}$ be continuous and assume f is differentiable on (a,b). Let $M > 0$. Prove that $|f'(x)| \leq M$ for all $x \in (a,b)$ if and only if $|f(x) - f(y)| \leq M|x-y|$ for all $x, y \in [a,b]$.

5. A function $f : D \to \mathbb{R}$ is called *Lipschitz continuous* if there is $M > 0$ such that for all $x, y \in D$,

$$|f(x) - f(y)| \leq M|x-y|.$$

Prove that if f is Lipschitz continuous, then f is uniformly continuous. Then show that the converse is false. (Hint: Consider $f(x) = \sqrt{x}$ on $[0,1]$ and use Problems 3 and 4.)