4603 HW9

1. Suppose $f : [a, b] \to \mathbb{R}$ is continuous and injective. Prove that f is either strictly increasing or strictly decreasing.

Solution. Suppose f is not strictly increasing or strictly decreasing. Then there exist $x, y, z \in [a, b]$ such that $x < y < z$ and either (i) or (ii) below holds:

$$\text{(i) } f(x) < f(y), \quad f(y) > f(z)$$
$$\text{(ii) } f(x) > f(y), \quad f(y) < f(z)$$

Assume (i) holds. If $f(z) = f(x)$ then f is not injective, contradiction. If $f(z) > f(x)$ then $f(x) < f(z) < f(y)$, so by the IVT there exists $u \in (x, y)$ such that $f(u) = f(z)$. As $z > y > u$ this contradicts injectivity of f. If $f(z) < f(x)$ then $f(z) < f(x) < f(y)$, so by the IVT there exists $v \in (y, z)$ such that $f(v) = f(x)$. As $x < y < v$ this contradicts injectivity of f. So we see that (i) leads to a contradiction. Similar arguments show that (ii) also leads to a contradiction. So f must be strictly increasing or strictly decreasing.

2. Let $f : \mathbb{R} \to \mathbb{R}$ and suppose that:

$$(*) \text{ for each } c \in \mathbb{R}, \text{ the equation } f(x) = c \text{ has exactly two solutions.}$$

Prove that f is not continuous.

Solution. Pick $c \in \mathbb{R}$ and pick $x < y$ such that $f(x) = f(y) = c$. Since f is continuous, it attains a minimum value, p, and a maximum value, q, on $[x, y]$. If $p = q = c$ then f is constant on $[x, y]$, a contradiction to $(*)$. If $p < c < q$ then by IVT there is $z \in (x, y)$ such that $f(z) = c$, again a contradiction to $(*)$. So it must be that exactly one of p or q is equal to c. Without loss of generality assume that $p = c$ and $q = f(u) > c$ for some $u \in (x, y)$.

Now by $(*)$ there exists $v \neq u$ such that $f(v) = f(u) = q$. One of the following must hold:

$$\text{(i) } x < u < v < y \qquad \text{(ii) } x < v < u < y$$
$$\text{(iii) } v < x < u < y \qquad \text{(iv) } x < u < y < v$$

In all of cases (i)-(iv) we have

$$f(x) = f(y) = p, \qquad f(u) = f(v) = q.$$

We will consider only cases (i) and (iii), as the proofs in cases (ii) and (iv) are analogous.

Suppose (i) holds. Pick any $c \in (u, v)$. Then $c \in [x, y]$ so we must have $p \leq f(c) \leq q$. Combining this with $(*)$, we see that actually $p < f(c) < q$. Now by IVT, there exists $a \in (x, u)$ and $b \in (v, y)$ such that $f(a) = f(b) = f(c)$. This contradicts $(*)$.

Now suppose (iii) holds. Pick any r such that $p < r < q$. Then by IVT there exists $a \in (v, x)$, $b \in (x, u)$, and $c \in (y, v)$ such that $f(a) = f(b) = f(c) = r$.

3. Define $f : (0, \infty) \to \mathbb{R}$ by $f(x) = \sqrt{x}$. Use the definition of derivative to show that f is differentiable and $f'(x) = \frac{1}{2\sqrt{x}}$.

Solution. Let $x \in (0, \infty)$ and note that

$$\lim_{y \to x} \frac{\sqrt{y} - \sqrt{x}}{y - x} = \lim_{y \to x} \frac{1}{\sqrt{y} + \sqrt{x}} = \frac{1}{2\sqrt{x}},$$

where in the last step above we have used algebraic properties of limits along with the fact that the function $g(y) = \sqrt{y}$ is continuous for $y > 0$.

4. Let $f : [a, b] \to \mathbb{R}$ be continuous and assume f is differentiable on (a, b). Let $M > 0$. Prove that $|f'(x)| \leq M$ for all $x \in (a, b)$ if and only if $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in [a, b]$.

Solution. Assume that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in [a, b]$. Let $x \in [a, b]$. Then for all $y \in [a, b]$ such that $y \neq x$,

$$\left| \frac{f(y) - f(x)}{y - x} \right| \leq M.$$

Taking a limit of the above expression as $y \to x$, we see that $|f'(x)| \leq M$.

Conversely, assume that $|f'(x)| \leq M$ for all $x \in (a, b)$. Suppose that there is $x < y \in [a, b]$ such that

$$|f(x) - f(y)| > M|x - y|.$$

Then by the MVT, there exists $c \in (x, y)$ such that

$$f'(c) = \frac{f(y) - f(x)}{y - x}$$

so that $|f'(c)| > M$, contradiction.

5. A function $f : D \to \mathbb{R}$ is called *Lipschitz continuous* if there is $M > 0$ such that for all $x, y \in D$,

$$|f(x) - f(y)| \leq M|x - y|.$$

Prove that if f is Lipschitz continuous, then f is uniformly continuous. Then show that the converse is false. (Hint: Consider $f(x) = \sqrt{x}$ on $[0, 1]$ and use Problems 3 and 4.)

Solution. Let f be Lipschitz continuous with constant M. Let $\epsilon > 0$ and pick $\delta = \epsilon/M$. Then $x, y \in D$ and $|x - y| < \delta$ imply

$$|f(x) - f(y)| \leq M|x - y| < \delta M = \epsilon.$$

To see that the converse is false, note that $f : [0, 1] \to \mathbb{R}$ is continuous, and f is differentiable on $(0, 1)$, but $f'(x) = 1/(2\sqrt{x})$ is not bounded on $(0, 1)$, so by Problem 4 f is not Lipschitz continuous.