CLASS NOTES

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1. Preliminaries

1.1. **What is math?**

Math is truth.

1.2. **Bound and unbound variables.**

First, a word about the English language:
The past participle of "to bound" is "bounded".
So, if you bound something, it becomes bounded.
There is a completely different verb, "to bind",
and its past participle is, confusingly, "bound".
So, if you bind something, it becomes bound, NOT bounded.

At any point in any definition, theorem or proof, every variable is either **bound** or **unbound**. To see how binding and unbinding works exactly, read the beginning of the exposition handout, up to the text "General rules of argument" that appears in the middle of Page 3. Note: A **free** variable is exactly the same thing as an unbound variable.

In class (Lec 01, Slide 18), we went through several examples of binding and freeing of variables. The most common mistake students make on early homework is not being careful about binding of variables. A free variable cannot be used, except in a binding statement. If you use a free variable, it is sometimes a small problem, but often much larger, and can result in no credit being given at all. So: Understanding the "scope" of each variable (where it becomes bound, and where, later, it becomes free) is crucial.

Also, some variables are integers, some variables are sets, some are real numbers, *etc.* Understanding the "type" of each variable is also crucial.

If you have questions about these topics, it's important to come and talk to me. It is hard to explain these topics in written form; typically a conversation is needed.

1.3. **The object ☹.**

AXIOM 1.3.1. $\forall$*set* S, $☹ \notin S$.

AXIOM 1.3.2. $\forall x, \quad x/0 = ☹$.

THEOREM 1.3.3. $1/0 = ☹$.

We wish to set things up so that ☹ is "infective", meaning:

If some expression contains a subexpression is equal to ☹,
then the entire expression equal to ☹.

Toward that end, we make the following axioms:

AXIOM 1.3.4.

$\forall x,\ x + ☹ = ☹ + x = ☹.$
$\forall x,\ x - ☹ = ☹ - x = ☹.$
$\forall x,\ x \cdot ☹ = ☹ \cdot x = ☹.$
$\forall x,\ x/☹ = ☹/x = ☹.$

Also, using $<$ or $>$, ☹ cannot be compared to any object.

AXIOM 1.3.5.

$\forall x,\ \neg(☹ < x).$
$\forall x,\ \neg(☹ > x).$
$\forall x,\ \neg(x < ☹).$
$\forall x,\ \neg(x > ☹).$

Let a and b be strings of characters.
The notation $\boxed{a \quad =^* \quad b}$ is short for $(b = ☹) \vee (a = b)$.
The notation $\boxed{a \quad {}^*\!= \quad b}$ is short for $(a = ☹) \vee (a = b)$.

Note that $0/0 = ☹$, and so $\neg(\forall x \in \mathbb{R},\ x/x = 1)$.
That is, it is NOT correct to say that, for all $x \in \mathbb{R}$, we have $x/x = 1$,
because it doesn't work for $x = 0$.
We could say $\forall x \in \mathbb{R}_0^\times,\ x/x = 1$.

The following theorems illustrate the notation described above:

THEOREM 1.3.6. $\forall x \in \mathbb{R}, \qquad x/x \ {}^*\!= \ 1.$

THEOREM 1.3.7. $\forall x \in \mathbb{R}, \qquad x^2 \ =^* \ x^7/x^5.$

THEOREM 1.3.8. $\forall x \in \mathbb{R}, \qquad x^5/x^3 = x^4/x^2.$

1.4. **Some logic and set theory.**

We use "∀" for "for every".
We use "∃" for "there exists".
We use "&" for "and".
We use "∨" for "or".
We use "¬" for "not".

We use "$\therefore$" for "therefore".
We use "$\Rightarrow$" for "implies".

Let A and B be statements. Then "$A \Leftrightarrow B$" is a statement, and means "$(A \Rightarrow B)\&(B \Rightarrow A)$"; here, the parentheses are crucial.

An **extended real number** or, more succinctly, an **extended real**, is any of the following:

a real number or the symbol ∞ or the two symbol string $-\infty$.

Sometimes "$+\infty$" is used to mean "∞".

DEFINITION 1.4.1. $\boxed{\mathbb{R}^*} := \{-\infty\} \bigcup \mathbb{R} \bigcup \{\infty\}$.

It is our convention that no extended real is considered to be a set:

AXIOM 1.4.2.
$\forall$set A, $\forall x \in \mathbb{R}^*$, $x \neq A$.

We will use ☹ to mean "does not exist".
So, for example, $1/0 =$ ☹. See §1.3 for more information about ☹.
An **object** is any of the following:

an extended real number or a set or ☹.

The notation "$\forall x$," means "for any object x".
The notation "$\exists x$ s.t." means "there exists an object x s.t.".

We use ε for the Greek letter epsilon.
We use $\in$ as an abbreviation for "is an element of".
We use ϕ for the Greek letter phi.
We use $\varnothing$ to mean the empty set. Then $\varnothing = \{\ \}$.
Note that: $\forall x$, $x \notin \varnothing$.
Also, $\forall x \in \varnothing$, $x = 2$, because:
there is no element of $\varnothing$ that is NOT equal to 2.
Also, $\forall x \in \varnothing$, $x \neq 2$, because:
there is no element of $\varnothing$ that is equal to 2.
Also, $\forall x \in \varnothing$, $(x = 2)\,\&\,(x \neq 2)$, because:
there is no element of $\varnothing$ that fails to be
both equal to 2 and not equal to 2 at the same time.

According to some formal systems for writing mathematics

$$\forall x \in \varnothing, \quad x = 2$$

is not a properly formed statement, because: "$\forall$" should always be followed by a variable, then a comma, then "(", and, moreover, the corresponding ")" should appear at the end of the $\forall$ statement. If we believe in such a formatting rule, then

$$\forall x \in \varnothing, \quad x = 2$$

is bad, and would be better written as

$$\forall x, \quad ((x \in \varnothing) \Rightarrow (x = 2)).$$

Note that, no matter which object x is, it is NOT true that $x \in \varnothing$, and so it IS true that $((x \in \varnothing) \Rightarrow (x = 2))$,
because any false assertion DOES imply any other assertion,
and it doesn't matter whether the second assertion is true or false.

An implication that is true because

the assertion on the left of the symbol "$\Rightarrow$" is false

is said to be "null true". So

$$\forall x, ((x \in \varnothing) \Rightarrow (x = 2)).$$

is an example of a null true statement.
More precisely, we should probably say that, for every object x,

$$(x \in \varnothing) \Rightarrow (x = 2)$$

is null true, but we'll allow a certain level of sloppiness here.
Using our more informal way of writing, we would say that

$$\forall x \in \varnothing, \quad x = 2$$

is null true. Following "$\forall x \in \varnothing$,", we could put any assertion about x, and the resulting statement would be true, and, in fact, null true.

By $\boxed{\mathbb{R}}$, we mean the set of all real numbers.
By $\boxed{\mathbb{Q}}$, we mean the set of all rational numbers.
By $\boxed{\mathbb{Z}}$, we mean the set of all integers.
By $\boxed{\mathbb{N}_0}$, we mean the set of all semi-positive (*i.e.* nonnegative integers.
By $\boxed{\mathbb{N}}$, we mean the set of all positive integers.

Then $\quad \mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \quad$ and

$$\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\} \qquad \text{and} \qquad \mathbb{N} = \{1, 2, 3, \ldots\}.$$

Note that $0 \in \mathbb{N}_0$. On the other hand, $0 \notin \mathbb{N}$, or, equivalently, $\neg(0 \in \mathbb{N})$.

DEFINITION 1.4.3. *Let A and B be sets.*
Then $\boxed{A \subseteq B}$ *means:* $\quad \forall x \in A, \quad x \in B$.
Also, $\boxed{B \supseteq A}$ *means the same thing:* $\quad \forall x \in A, \quad x \in B$.

THEOREM 1.4.4. $\mathbb{N} \subseteq \mathbb{N}_0 \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.

THEOREM 1.4.5. $\{1,2,3\} \subseteq \{1,2,3,7\} \supseteq \{2,3,7\} \supseteq \varnothing$.

Note that $\{2,3,7\} \supseteq \varnothing$ is *null* true. That is, because there is NO element of $\varnothing$ that is NOT an element of $\{2,3,7\}$, we conclude that every element of $\varnothing$ is an element of $\{2,3,7\}$, and so $\{2,3,7\} \supseteq \varnothing$. In fact, the same logic shows that $\varnothing$ is a subset of every set:

THEOREM 1.4.6. $\forall$*set* X, $\quad \varnothing \subseteq X$.

The following is called the **Axiom of Extensionality**:

AXIOM 1.4.7. *Let A and B be sets. Then:*

$$(A = B) \Leftrightarrow ((A \subseteq B)\&(B \subseteq A)).$$

DEFINITION 1.4.8. *Let A and B be sets. Then:*

$$\boxed{A \bigcup B} \quad := \quad \{x \,|\, (x \in A) \,\vee\, (x \in B)\} \qquad \text{and}$$

$$\boxed{A \bigcap B} \quad := \quad \{x \,|\, (x \in A) \,\&\, (x \in B)\} \qquad \text{and}$$

$$\boxed{A \backslash B} \quad := \quad \{x \,|\, (x \in A) \,\&\, (x \notin B)\}.$$

THEOREM 1.4.9. *Let $A := \{1,2,3\}$ and $B := \{3,4,5\}$. Then:*

$$A \bigcup B \quad = \quad \{1,2,3,4,5\} \qquad \text{and}$$

$$A \bigcap B \quad = \quad \{3\} \qquad \text{and}$$

$$A \backslash B \quad = \quad \{1,2\}.$$

1.5. **Intervals.**

DEFINITION 1.5.1.

$\forall a, b \in \mathbb{R}^*$,

$$\boxed{(a;b)} := \{x \in \mathbb{R}^* \,|\, a < x < b\} \qquad \text{and}$$

$$\boxed{[a;b)} := \{x \in \mathbb{R}^* \,|\, a \leqslant x < b\} \qquad \text{and}$$

$$\boxed{(a;b]} := \{x \in \mathbb{R}^* \,|\, a < x \leqslant b\} \qquad \text{and}$$

$$\boxed{[a;b]} := \{x \in \mathbb{R}^* \,|\, a \leqslant x \leqslant b\}.$$

Note that $-\infty \notin (-\infty;\infty]$ and that $-7.5, 0, 10^{100}, \infty \in (-\infty;\infty]$.

Note that $\mathbb{R} = (-\infty;\infty)$.

DEFINITION 1.5.2. $\mathbb{Z}^* := \{-\infty\} \bigcup \mathbb{Z} \bigcup \{\infty\}$. *Also:*

$\forall a, b \in \mathbb{R}^*$,

$$\boxed{(a..b)} := \{x \in \mathbb{Z}^* \,|\, a < x < b\} \qquad \text{and}$$

$$\boxed{[a..b)} := \{x \in \mathbb{Z}^* \,|\, a \leqslant x < b\} \qquad \text{and}$$

$$\boxed{(a..b]} := \{x \in \mathbb{Z}^* \,|\, a < x \leqslant b\} \qquad \text{and}$$

$$\boxed{[a..b]} := \{x \in \mathbb{Z}^* \,|\, a \leqslant x \leqslant b\}.$$

THEOREM 1.5.3. $[1..7] = \{1,2,3,4,5,6,7\}$ *and*
$[2..1] = \varnothing$ *and*
$[-\infty..4) = \{-\infty\} \bigcup \{\ldots,-2,-1,0,1,2,3\}$ *and*
$[-\infty..4] = \{-\infty\} \bigcup \{\ldots,-2,-1,0,1,2,3,4\}$.

DEFINITION 1.5.4. *Let A be a set and let z be an object. Then:*
$\boxed{A_z^+} := A \bigcup \{z\}$ *and* $\boxed{A_z^\times} := A \backslash \{z\}$.

In class, we graphed $[1;2)_3^+$ and $(1;3]_2^\times$ on number lines.

1.6. Manipulation of inequalities.

THEOREM 1.6.1. *Let $a,b,A,B \in \mathbb{R}$.*
Assume $(\, a < A \,) \,\&\, (\, b < B \,)$. Then $a + b < A + B$.

THEOREM 1.6.2. *Let $a,b,A,B \in \mathbb{R}$.*
Assume $(\, 0 \leqslant a < A \,) \,\&\, (\, 0 \leqslant b < B \,)$. Then $ab < AB$.

THEOREM 1.6.3. *Let $a,b,A,B \in \mathbb{R}$.*
Assume $(\, a \leqslant A \,) \,\&\, (\, b \leqslant B \,)$. Then $a + b \leqslant A + B$.

THEOREM 1.6.4. *Let $a,b,A,B \in \mathbb{R}$.*
Assume $(\, 0 \leqslant a \leqslant A \,) \,\&\, (\, 0 \leqslant b \leqslant B \,)$. Then $ab \leqslant AB$.

If we have mixed inequalities (strict and semi), then we get strict for addition:

THEOREM 1.6.5. *Let $a,b,A,B \in \mathbb{R}$.*
Assume $(\, a < A \,) \,\&\, (\, b \leqslant B \,)$. Then $a + b < A + B$.

For positive numbers, the product of a the product of a strict inequality with a semi-inequality is a strict inequality:

THEOREM 1.6.6. *Let $a,b,A,B \in \mathbb{R}$.*
Assume $(\, 0 < a < A \,) \,\&\, (\, 0 < b \leqslant B \,)$. Then $ab < AB$.

It is a common mistake to think that, for nonnegative numbers, the product of a strict inequality with a semi-inequality should give a strict inequality. In fact, if we have mixed inequalities (strict and semi), then we get semi for multiplication:

THEOREM 1.6.7. *Let $a,b,A,B \in \mathbb{R}$.*
Assume $(\, 0 < a < A \,) \,\&\, (\, 0 \leqslant b \leqslant B \,)$. Then $ab \leqslant AB$.

Note that, in the conclusion of the preceding theorem, we cannot write $ab < AB$ because of the possibility that $0 = b = B$.

1.7. **Basic algebraic facts.**

The following is called the **Naive Product Rule**:

THEOREM 1.7.1. *Let* $a, b, A, B \in \mathbb{R}$. *Then:*
$$A \cdot B - a \cdot b \;=\; (A - a) \cdot b \;+\; a \cdot (B - b) \;+\; (A - a) \cdot (B - b).$$

1.8. **The Axiom of Choice.**

We imagine that at the beginning of time,
the Grand Oracle has chosen,
from every nonempty set A, an element denoted CH_A.
This is embodied in the **Axiom of Choice**:

AXIOM 1.8.1. $\forall$*nonempty set* S, $\mathrm{CH}_S \in S$.

We also make the convention that $\mathrm{CH}_\varnothing = ☹$.

Alternate notation for $\boxed{\mathrm{CH}_S}$: $\boxed{\mathrm{CH}(S)}$ or $\boxed{\mathrm{CH}\, S}$.

Then: $\forall$set S, we have: $\mathrm{CH}_S \;{}^*\!\!\in\; S$.

THEOREM 1.8.2. $\mathrm{CH}\{4\} = 4$ *and* $\mathrm{CH}\{\,\{1, 2, 3\}\,\} \;=\; \{1, 2, 3\}$.

For sets with more than one element, we do not know which is chosen, but we do know that one of them is:

THEOREM 1.8.3. $(\,\mathrm{CH}\{2, 3\} = 2\,) \vee (\,\mathrm{CH}\{2, 3\} = 3\,)$.

THEOREM 1.8.4.
$$(\,\mathrm{CH}\{2, 3, 5\} = 2\,) \vee (\,\mathrm{CH}\{2, 3, 5\} = 3\,) \vee (\,\mathrm{CH}\{2, 3, 5\} = 5\,).$$

1.9. **Unique element of a set.**

DEFINITION 1.9.1. *Let* A *be an object.*
By A *is a* **singleton** *or* **singleton set**, *we mean:*
A *is a nonempty set* *and* $\forall x, y \in A,\; x = y$.

THEOREM 1.9.2.
$(\,\{3\}$ *is a singleton*$\,)$ *and*
$(\,\{\;\{1, 2, 3\}\;\}$ *is a singleton*$\,)$ *and*
$(\,\neg(\{1, 2, 3\}$ *is a singleton*$)\,)$ *and*
$(\,\neg(\varnothing$ *is a singleton*$)\,)$.

DEFINITION 1.9.3.

For set A, $\boxed{\mathrm{UE}_A} := \begin{cases} \mathrm{CH}_A, & \textit{if A is a singleton} \\ ☹, & \textit{if A is not a singleton.} \end{cases}$

Alternative notations: $\boxed{\mathrm{UE}\,A}$ and $\boxed{\mathrm{UE}(A)}$.

THEOREM 1.9.4.
$\mathrm{UE}\{3\} = 3$ *and*
$\mathrm{UE}\{\ \{1,2,3\}\ \} = \{1,2,3\}$ *and*
$\mathrm{UE}\{1,2\} =$ ☹ *and*
$\mathrm{UE}\{1,2,3\} =$ ☹ *and*
$\mathrm{UE}\,\varnothing =$ ☹.

For any objects a and B, the notation $\boxed{a\ {}^*\!\!\in\ B}$ means:

$(a =$ ☹$) \vee (a \in B)$.

THEOREM 1.9.5.
$\mathrm{UE}\{1\}\ {}^*\!\!\in\ \{1\}$ *and*
$\mathrm{UE}\{\ \{1,2,3\}\ \}\ {}^*\!\!\in\ \{\ \{1,2,3\}\ \}$ *and*
$\mathrm{UE}\{1,2,3\}\ {}^*\!\!\in\ \{1,2,3\}$ *and*
$\mathrm{UE}\,\varnothing\ {}^*\!\!\in\ \varnothing$.

THEOREM 1.9.6. $\forall A, \quad UE_A\ {}^*\!\!\in\ A.$

1.10. **Well-ordering and completeness axioms.**

DEFINITION 1.10.1. *Let* $S \subseteq \mathbb{R}^*$, $a \in \mathbb{R}^*$.
Then $\boxed{S > a}$ *means:* $\forall x \in S,\ x > a$.
Also, $\boxed{S \geqslant a}$ *means:* $\forall x \in S,\ x \geqslant a$.
Also, $\boxed{S < a}$ *means:* $\forall x \in S,\ x < a$.
Also, $\boxed{S \leqslant a}$ *means:* $\forall x \in S,\ xlea$.
Also, $\boxed{a < S}$ *means:* $\forall x \in S,\ a < x$.
Also, $\boxed{a \leqslant S}$ *means:* $\forall x \in S,\ a \leqslant x$.
Also, $\boxed{a > S}$ *means:* $\forall x \in S,\ a > x$.
Also, $\boxed{a \geqslant S}$ *means:* $\forall x \in S,\ a \geqslant x$.

THEOREM 1.10.2.
$(\mathbb{N} > 0)\,\&\,(N_0 \geqslant 0)\,\&\,(1 \leqslant \mathbb{N})\,\&\,(-3 < \mathbb{N}_0)\,\&\,(\neg(0 < \mathbb{N}_0))$.

THEOREM 1.10.3. $\forall x \in \varnothing,\ 5 \leqslant x$.

THEOREM 1.10.4. $5 \leqslant \varnothing$.

THEOREM 1.10.5. $(7 \leqslant \varnothing)\,\&\,(-\infty \leqslant \varnothing)\,\&\,(\infty \leqslant \varnothing)$.

THEOREM 1.10.6. $\forall x \in \mathbb{R}^*,\ x \leqslant \varnothing$.

THEOREM 1.10.7. $\forall x \in \mathbb{R}^*,\ x < \varnothing$.

THEOREM 1.10.8. $\forall x \in \mathbb{R}^*,\ x \geqslant \varnothing$.

THEOREM 1.10.9. $\forall x \in \mathbb{R}^*,\ x > \varnothing$.

In the next definition, LB stands for "Lower Bounds",
and UB stand for "Upper Bounds".

DEFINITION 1.10.10. *Let* $S \subseteq \mathbb{R}^*$. *Then:*
$\boxed{\mathrm{LB}_S} := \{x \in \mathbb{R}^* \,|\, x \leqslant S\}$ *and*
$\boxed{\mathrm{UB}_S} := \{x \in \mathbb{R}^* \,|\, x \geqslant S\}$.

Alternate notations for LB_S are: $\boxed{\mathrm{LB}(S)}$ and $\boxed{\mathrm{LB}\,S}$.
Alternate notations for UB_S are: $\boxed{\mathrm{UB}(S)}$ and $\boxed{\mathrm{UB}\,S}$.

THEOREM 1.10.11. $\mathrm{LB}\{3,4,5\} = [-\infty; 3]$ *and* $\mathrm{LB}\{3,4,5\} = [5; \infty]$.

By Theorem 1.10.6 and Theorem 1.10.8 above, we get:

THEOREM 1.10.12. $(\mathrm{LB}_\varnothing = \mathbb{R}^*)\,\&\,(\mathrm{UB}_\varnothing = \mathbb{R}^*)$.

DEFINITION 1.10.13. *Let* $S \subseteq \mathbb{R}^*$. *Then:*
$\boxed{\min_S} := \mathrm{UE}(S \cap \mathrm{LB}_S)$ *and* $\boxed{\max_S} := \mathrm{UE}(S \cap \mathrm{UB}_S)$.

Alternate notations for $\min_S$ are: $\boxed{\min(S)}$ and $\boxed{\min\ S}$.
Alternate notations for $\max_S$ are: $\boxed{\max(S)}$ and $\boxed{\max\ S}$.

We have $\mathrm{LB}[1;2) = [-\infty;1]$ and $\mathrm{UB}[1;2) = [2;\infty]$. Then:

THEOREM 1.10.14.
$\min[1;2) = \mathrm{UE}([1;2) \bigcap [-\infty;1]) = 1$ *and*
$\max[1;2) = \mathrm{UE}([1;2) \bigcap [2;\infty]) = ☹$.

We have $\mathrm{LB}_\varnothing = \mathbb{R}^*$ and $\mathrm{UB}\varnothing = \mathbb{R}^*$. Then:

THEOREM 1.10.15.
$\min_\varnothing = \mathrm{UE}(\varnothing \bigcap \mathbb{R}^*) = ☹$ *and*
$\max_\varnothing = \mathrm{UE}(\varnothing \bigcap \mathbb{R}^*) = ☹$.

THEOREM 1.10.16. *Let* $S \subseteq \mathbb{R}^*$. *Then:*
$(\,\min S \ {}^*\!\!\in\ S\,)\,\&\,(\,\max S \ {}^*\!\!\in\ S\,)$.

Proof. We have $\min S = \mathrm{UE}(S \cap \mathrm{LB}_S)\ {}^*\!\in\ S \cap \mathrm{LB}_S \subseteq S$,
so $\min S\ {}^*\!\in\ S$.

It remains to show: $\max S\ {}^*\!\in\ S$.

We have $\max S = \mathrm{UE}(S \cap \mathrm{UB}_S)\ {}^*\!\in\ S \cap \mathrm{UB}_S \subseteq S$,
so $\max S\ {}^*\!\in\ S$. □

THEOREM 1.10.17. *Let $S \subseteq \mathbb{R}^*$, $x, y \in S \cap \mathrm{LB}_S$. Then $x = y$.*

Proof. Since $x, y \in \mathrm{LB}_S$, we get: $(\, x \leqslant S\,) \,\&\, (\, y \leqslant S\,)$.
Since $x \in S \geqslant y$, we get $x \geqslant y$. Since $y \in S \geqslant x$, we get $y \geqslant x$.
Since $x \geqslant y$ and $y \geqslant x$, we get $x = y$. □

The preceding theorem says that $S \cap \mathrm{LB}_S$ cannot have two unequal elements; equivalently, that set is empty or singleton:

THEOREM 1.10.18. *Let $S \subseteq \mathbb{R}^*$.*
Then $(\, S \cap \mathrm{LB}_S = \varnothing\,) \vee (\, S \cap \mathrm{LB}_S$ is a singleton$\,)$.

THEOREM 1.10.19. *Let $S \subseteq \mathbb{R}^*$, $a \in \mathbb{R}^*$. Then:*
$$(\, a = \min S\,) \quad \Leftrightarrow \quad (\,(\, a \in S\,) \,\&\, (\, a \leqslant S\,)\,).$$

Notes on proof: We leave $\Rightarrow$ as an exercise; it follows from the definitions. For $\Leftarrow$, from $(\, a \in S\,) \,\&\, (\, a \leqslant S\,)$, we get $a \in S \cap \mathrm{LB}_S$, which shows that $S \cap \mathrm{LB}_S \neq \varnothing$. Then, by Theorem 1.10.18, $S \cap \mathrm{LB}_S$ is a singleton. So, since $a \in S \cap \mathrm{LB}_S$, we get $S \cap \mathrm{LB}_S = \{a\}$. Then $\min S = \mathrm{UE}(S \cap \mathrm{LB}_S) = \mathrm{UE}\{a\} = a$, so $a = \min S$.

Similar reasoning gives:

THEOREM 1.10.20. *Let $S \subseteq \mathbb{R}^*$, $a \in \mathbb{R}^*$. Then:*
$$(\, a = \max S\,) \quad \Leftrightarrow \quad (\,(\, a \in S\,) \,\&\, (\, a \geqslant S\,)\,).$$

By $\boxed{a\ {}^*\!\leqslant\ b}$, we mean: $(\, a = ☹\,) \vee (\, a \leqslant b\,)$,
or, equivalently, $(\, a \neq ☹\,) \Rightarrow (\, a \leqslant b\,)$.

By $\boxed{a\ \leqslant^*\ b}$, we mean: $(\, b = ☹\,) \vee (\, a \leqslant b\,)$,
or, equivalently, $(\, b \neq ☹\,) \Rightarrow (\, a \leqslant b\,)$.

By $\boxed{a\ {}^*\!\geqslant\ b}$, we mean: $(\, a = ☹\,) \vee (\, a \geqslant b\,)$,
or, equivalently, $(\, a \neq ☹\,) \Rightarrow (\, a \geqslant b\,)$.

By $\boxed{a\ \geqslant^*\ b}$, we mean: $(\, b = ☹\,) \vee (\, a \geqslant b\,)$,
or, equivalently, $(\, b \neq ☹\,) \Rightarrow (\, a \geqslant b\,)$.

THEOREM 1.10.21. *Let $S \subseteq \mathbb{R}^*$. Then $\min S\ {}^*\!\leqslant\ S$.*

Proof. We wish to show: $(\min S \neq ☹) \Rightarrow (\min S \leqslant S)$.
Assume $\min S \neq ☹$. **Want:** $\min S \leqslant S$.
We have $\min S = \mathrm{UE}(S \cap \mathrm{LB}_S) \overset{*}{\in} S \cap \mathrm{LB}_S \subseteq \mathrm{LB}_S$,
so, contracting, we get $\min S \in \mathrm{LB}_S$.
Then, by definition of LB_S, we conclude: $\min S \leqslant S$. □

A similar proof yields:

THEOREM 1.10.22. *Let* $S \subseteq \mathbb{R}^*$. *Then* $S \overset{*}{\leqslant} \max S$.

DEFINITION 1.10.23. *Let* $S \subseteq \mathbb{R}^*$. *Then:*
$\boxed{\inf_S} := \max(\mathrm{LB}_S)$ *and* $\boxed{\sup_S} := \min(\mathrm{UB}_S)$.

Alternate notations for $\inf_S$ are: $\boxed{\inf(S)}$ and $\boxed{\inf\ S}$.
Alternate notations for $\sup_S$ are: $\boxed{\sup(S)}$ and $\boxed{\sup\ S}$.

The inf is sometimes called the "greatest lower bound".
The sup is sometimes called the "least upper bound".

THEOREM 1.10.24.
$\inf[1;2) = \max[-\infty;1] = 1$ *and* $\sup[1;2) = \min[2,\infty] = 2$.

THEOREM 1.10.25.
$\inf \varnothing = \max \mathbb{R}^* = \infty$ *and* $\sup \varnothing = \min \mathbb{R}^* = -\infty$.

The following is the **Well-Ordering Axiom**.

AXIOM 1.10.26. $\forall$*nonempty* $S \subseteq \mathbb{N}_0$, $\min S \neq ☹$.

The following is the **Completeness Axiom**.

AXIOM 1.10.27. $\forall S \subseteq \mathbb{R}^*$, $\inf_S \neq ☹ \neq \sup_S$.

THEOREM 1.10.28. *Let* $S \subseteq \mathbb{R}^*$.
Then $\inf_S \geqslant \mathrm{LB}_S$ *and* $\sup_S \leqslant \mathrm{UB}_S$.

Proof. By Axiom 1.10.27, $\inf_S \neq ☹ \neq \sup_S/$
We have $\inf_S = \max(\mathrm{LB}_S) \overset{*}{\geqslant} \mathrm{LB}_S$.
So, since $\inf_S \neq ☹$, we get: $\inf_S \geqslant \mathrm{LB}_S$.
It remains to prove: $\sup_S \leqslant \mathrm{UB}_S$.
We have $\sup_S = \min(\mathrm{UB}_S) \overset{*}{\leqslant} \mathrm{UB}_S$.
So, since $\sup_S \neq ☹$, we get: $\sup_S \leqslant \mathrm{UB}_S$. □

THEOREM 1.10.29. *Let* $S \subseteq \mathbb{R}^*$. *Then* $S \geqslant \inf_S$ *and* $S \leqslant \sup_S$.

Proof. By Axiom 1.10.27, $\inf_S \neq ☹ \neq \sup_S$.
We have $\inf_S = \max(\mathrm{LB}_S) \overset{*}{\in} \mathrm{LB}_S$.
So, since $\inf_S \neq ☹$, we get: $\inf_S \in \mathrm{LB}_S$.
Then, by definition of LB_S, we get: $\inf_S \leqslant S$. Then $S \geqslant \inf_S$.
It remains to prove: $S \leqslant \sup_S$.
We have $\sup_S = \min(\mathrm{UB}_S) \overset{*}{\in} \mathrm{UB}_S$.
So, since $\sup_S \neq ☹$, we get: $\sup_S \in \mathrm{UB}_S$.
Then, by definition of UB_S, we get: $\sup_S \geqslant S$. Then $S \leqslant \sup_S$. □

THEOREM 1.10.30. *Let* $A \subseteq \mathbb{R}^*$, $z \in \mathbb{R}^*$.
Assume $A \leqslant z$. *Then* $\sup_A \leqslant z$.

Proof. Since $z \geqslant A$, we get: $z \in \mathrm{UB}_A$.
Then $z \in \mathrm{UB}_A \geqslant \sup_A$, so $z \geqslant \sup_A$, so $\sup_A \leqslant z$. □

THEOREM 1.10.31. *Let* $A \subseteq \mathbb{R}^*$, $z \in \mathbb{R}^*$.
Assume $A \geqslant z$. *Then* $\inf_A \geqslant z$.

Proof. Since $z \leqslant A$, we get: $z \in \mathrm{LB}_A$.
Then $z \in \mathrm{LB}_A \leqslant \inf_A$, so $z \leqslant \inf_A$, so $\inf_A \geqslant z$. □

THEOREM 1.10.32. *Let* $S \subseteq \mathbb{R}^*$. *Then* $\inf_S \overset{*}{=} \min_S$.

Proof. Know: $\min_S = \mathrm{UE}(S \bigcap \mathrm{LB}_S)$.
Want: $(\min_S \neq ☹) \Rightarrow (\inf_S = \min_S)$.
Assume $\min_S \neq ☹$. **Want:** $\inf_S = \min_S$.
Since $\min_S \neq ☹$ and $\min_S = \mathrm{UE}(S \bigcap \mathrm{LB}_S)$,
we conclude: $\min_S = \mathrm{UE}(S \bigcap \mathrm{LB}_S)$.
Since $\min_S = \mathrm{UE}(S \bigcap \mathrm{LB}_S) \overset{*}{\in} S \bigcap \mathrm{LB}_S$,
we get $\min_S \in S \bigcap \mathrm{LB}_S$, and so $\min_S \in S$ and $\min_S \in \mathrm{LB}_S$.
We have $\min_S \in S \geqslant \inf_S$ and $\min_S \in \mathrm{LB}_S \leqslant \inf_S$,
so $\min_S \geqslant \inf_S$ and $\min_S \leqslant \inf_S$, and so $\inf_S = \min_S$. □

THEOREM 1.10.33. *Let* $S \subseteq \mathbb{R}^*$, $z \in \mathrm{LB}_S$, $a \in [-\infty; z]$.
Then $a \in \mathrm{LB}_S$.

Proof. Since $a \leqslant z \leqslant S$, we get $a \leqslant S$. Then $a \in \mathrm{LB}_S$. □

1.11. Mathematical induction.

The following theorem is called the **Principle of Mathematical Induction** or **PMI**:

THEOREM 1.11.1. *Let $S \subseteq \mathbb{N}$. Assume $1 \in S$. Assume: $\forall k \in S$, $k+1 \in S$. Then: $S = \mathbb{N}$.*

The intuitive idea that S is closed under "successor", meaning that whenever a positive integer k is in S, then its successor $k+1$ is also in S. So, since $1 \in S$, we see that $2 \in S$. Then, since $2 \in S$, we see that $3 \in S$. Then, since $3 \in S$, we see that $4 \in S$. And so on. For any integer, we can eventually show that that integer is in S. Then $\mathbb{N} \subseteq S$. So since $S \subseteq \mathbb{N}$, we conclude, from the Axiom of Extensionality, that $S = \mathbb{N}$.

We omit a formal proof for Theorem 1.11.1, but it would involve the Well-Ordering Axiom, described earlier. We focus instead on how to use Theorem 1.11.1, using the **PMI template**, see EH (20).

THEOREM 1.11.2. $\forall k \in \mathbb{N}, \quad 1+2+3+\cdots+k = \dfrac{k(k+1)}{2}$.

Proof. Let $S := \left\{ k \in \mathbb{N} \,\middle|\, 1+2+3+\cdots+k = \dfrac{k(k+1)}{2} \right\}$.

Want: $S = \mathbb{N}$. Since $1 = \dfrac{1\cdot(1+1)}{2}$, we see that $1 \in S$.
By the PMI, **it suffices to prove:** $\forall k \in S$, $k+1 \in S$.
Given $k \in S$. **Want:** $k+1 \in S$.

Know: $1+2+3+\cdots+k = \dfrac{k(k+1)}{2}$.

Want: $1+2+3+\cdots+k+(k+1) = \dfrac{(k+1)((k+1)+1)}{2}$.

We have: $1+2+3+\cdots+k+(k+1) \;=\; \dfrac{k(k+1)}{2} + (k+1)$

$$= \frac{k}{2}\cdot(k+1) + 1\cdot(k+1) = \left(\frac{k}{2}+1\right)\cdot(k+1)$$
$$= \left(\frac{k+2}{2}\right)\cdot(k+1) = \frac{(k+2)(k+1)}{2}$$
$$= \frac{(k+1)(k+2)}{2} = \frac{(k+1)((k+1)+1)}{2}.$$ □

The following theorem is called the **0-PMI**:

THEOREM 1.11.3. *Let $S \subseteq \mathbb{N}_0$. Assume $0 \in S$. Assume: $\forall k \in S$, $k+1 \in S$. Then: $S = \mathbb{N}_0$.*

Idea of proof: The set S is closed under succesor. So, since $0 \in S$, it follows that $1 \in S$, and then that $2 \in S$ and then that $3 \in S$, *etc.*

Here is an example of how to use the 0-PMI:

THEOREM 1.11.4. $\forall k \in \mathbb{N}_0, \quad 2^k \geqslant k+1.$

Proof. Let $S := \{k \in \mathbb{N}_0 \,|\, 2^k \geqslant k+1\}$. **Want:** $S = \mathbb{N}_0$.
Since $2^0 = 1 \geqslant 0+1$, we see that $0 \in S$.
By the 0-PMI, **it suffices to show:** $\forall k \in S$, $k+1 \in S$.
Given $k \in S$. **Want:** $k+1 \in S$.
Know: $2^k \geqslant k+1$. Want $2^{k+1} \geqslant (k+1)+1$.
Since $k \in S \subseteq \mathbb{N}_0 \geqslant 0$, we get $k \geqslant 0$, so $(k+1)+(k+1) \geqslant (k+1)+(0+1)$.
Then $2^{k+1} \;=\; 2^k \cdot 2 \;=\; 2^k \cdot (1+1) = 2^k \cdot 1 + 2^k \cdot 1 = 2^k + 2^k$
$\geqslant (k+1)+(k+1) \geqslant (k+1)+(0+1) = (k+1)+1.$ □

1.12. **The Archimedean Principle.**

The following is **The Archimedean Axiom**:

AXIOM 1.12.1. $\sup \mathbb{N} = \infty$.

The following is **The Archimedean Principle** or **AP**:

THEOREM 1.12.2. $\forall x \in \mathbb{R}$, $\exists j \in \mathbb{N}$ *s.t.* $j > x$.

Proof. Given $x \in \mathbb{R}$. **Want:** $\exists j \in \mathbb{N}$ s.t. $j > x$.
Assume $\neg(\,\exists j \in \mathbb{N}$ s.t. $j > x\,)$. **Want:** Contradiction.
Then $\forall j \in \mathbb{N}$, $j \leqslant x$, so $\mathbb{N} \leqslant x$.
Then $\sup \mathbb{N} \leqslant x$.
Since $\sup \mathbb{N} \leqslant x \in \mathbb{R} < \infty$, we get $\sup \mathbb{N} < \infty$, so $\sup \mathbb{N} \neq \infty$.
However, by Axiom 1.12.1, we have: $\sup \mathbb{N} = \infty$. Contradiction. □

It is a theorem in propositional logic that, for any mathematical statements P and Q,

$$(\,P \vee Q\,) \Leftrightarrow (\,(\,\neg P\,) \Rightarrow Q\,).$$

It follows, for any two objects a and b, that

$$(\,a \;\;=^*\;\; b\,) \Leftrightarrow (\,(\,b \neq ☹\,) \Rightarrow (\,a = b\,)\,).$$

Next is **The Reciprocal Archimedean Principle** or **RAP**:

THEOREM 1.12.3. $\forall \varepsilon > 0$, $\exists j \in \mathbb{N}$ *s.t.* $1/j < \varepsilon$.

Proof. Given $\varepsilon > 0$. **Want:** $\exists j \in \mathbb{N}$ s.t. $1/j < \varepsilon$.
Since $\varepsilon > 0$, we see that $(\,1/\varepsilon \in \mathbb{R}\,)\,\&\,(\,1/\varepsilon > 0\,)\,\&\,(\,1/(1/\varepsilon) = \varepsilon\,)$.

By the AP, choose $j \in \mathbb{N}$ s.t. $j > 1/\varepsilon$. Then $j \in \mathbb{N}$. **Want:** $1/j < \varepsilon$. Since $j > 1/\varepsilon > 0$, we get $1/j < 1/(1/\varepsilon)$. Then $1/j < 1/(1/\varepsilon) = \varepsilon$. □

We can restate the preceding theorem as:

$$\forall \varepsilon > 0,\ \neg(\,\forall j \in \mathbb{N},\ \ 1/j < \varepsilon\,).$$

Equivalently,

$$\forall \varepsilon > 0,\ \neg(\,\{1, 1/2, 1/3, \ldots\} \geqslant \varepsilon\,).$$

Equivalently,

$$\forall \varepsilon > 0,\ \neg(\,\varepsilon \in \mathrm{LB}\{1, 1/2, 1/3, \ldots\}\,).$$

The following expresses the same thing:

THEOREM 1.12.4. $\forall \varepsilon > 0,\ \varepsilon \notin \mathrm{LB}\{1, 1/2, 1/3, \ldots\}$.

In the preceding theorem, ε is a real variable, by convention. However the theorem would even be true if we use ∞ for ε:

THEOREM 1.12.5. $\infty \notin \mathrm{LB}\{1, 1/2, 1/3, \ldots\}$.

THEOREM 1.12.6. $\mathrm{LB}\{1, 1/2, 1/3, \ldots\} = [-\infty; 0]$.

Proof. We have $0 \leqslant \{1, 1/2, 1/3, \ldots\}$, so $0 \in \mathrm{LB}\{1, 1/2, 1/3, \ldots\}$.
Then, by Theorem 1.10.33, $[-\infty; 0] \subseteq \mathrm{LB}\{1, 1/2, 1/3, \ldots\}$.
It remains to show: $\mathrm{LB}\{1, 1/2, 1/3, \ldots\} \subseteq [-\infty; 0]$.
Want: $\forall \varepsilon \in \mathrm{LB}\{1, 1/2, 1/3, \ldots\}, \quad \varepsilon \in [-\infty; 0]$.
Given $\varepsilon \in \mathrm{LB}\{1, 1/2, 1/3, \ldots\}$. **Want:** $\varepsilon \in [-\infty; 0]$.
By Theorem 1.12.4 and Theorem 1.12.5, $\varepsilon \notin [0; \infty]$, so $\varepsilon \in \mathbb{R}^* \backslash [0; \infty]$.
Then $\varepsilon \in \mathbb{R}^* \backslash [0; \infty] = [-\infty; 0]$. □

Unassigned HW: Show that $\mathrm{LB}[-\infty; 0] = [0; \infty]$.
We use that unassigned HW in the following proof.

THEOREM 1.12.7.

$$\min\{1, 1/2, 1/3, \ldots\} = ☹ \quad \textit{and} \quad \inf\{1, 1/2, 1/3, \ldots\} = 0.$$

Proof. By Theorem 1.12.6, $\mathrm{LB}\{1, 1/2, 1/3, \ldots\} = [-\infty; 0]$.

$$\text{Then}\quad \min\{1, 1/2, 1/3, \ldots\} = \mathrm{UE}(\,\{1, 1/2, 1/3, \ldots\} \bigcap [-\infty; 0]\,) = \mathrm{UE}(\,\varnothing\,),$$

so, since $\mathrm{UE}(\varnothing) = ☹$, we get $\min\{1, 1/2, 1/3, \ldots\} = ☹$.
It remains to show: $\inf\{1, 1/2, 1/3, \ldots\} = 0$.

$$\text{We have}\quad \max[-\infty; 0] = \mathrm{UE}(\,[-\infty; 0] \bigcap [0; \infty]\,) = \mathrm{UE}(\,\{0\}\,) = 0,$$

so $\max[-\infty; 0] = 0$.

$$\text{We have } \inf\{1, 1/2, 1/3, \ldots\} = \max(\,\mathrm{LB}\{1, 1/2, 1/3, \ldots\}\,) = \max[-\infty; 0] = 0.$$

□

The preceding theorems show the process by which we can prove computations of

LB, UB, min, max, inf, sup.

As you can see, these proofs can be laborious, and, generally, we will omit them. If you understand the definitions, then they become straightforward, even if they can be, at first, somewhat intimidating. In any case, they belong in a course on the foundations of the real number system, a prerequisite to real analysis.

1.13. **Translating and reflecting sets of real numbers.**

DEFINITION 1.13.1. *Let $S \subseteq \mathbb{R}$. Then:*
$$\boxed{-S} := \{-y \,|\, y \in S\}.$$

THEOREM 1.13.2. $-\{2,5,9\} = \{-2,-5,-9\}$.

DEFINITION 1.13.3. *Let $S \subseteq \mathbb{R}$, $x \in \mathbb{R}$. Then:*
$$\boxed{x+S} := \{x+y \,|\, y \in S\} \text{ and } \boxed{S+x} := \{y+x \,|\, y \in S\}.$$

THEOREM 1.13.4. $4 + \{1,2,5\} = \{5,6,9\} = \{1,2,5\} + 4$.

THEOREM 1.13.5. $\forall S \subseteq \mathbb{R}$, $\forall x \in \mathbb{R}$, $\quad x + S = S + x$.

DEFINITION 1.13.6. *Let $S \subseteq \mathbb{R}$, $x \in \mathbb{R}$. Then:*
$$\boxed{x-S} := \{x-y \,|\, y \in S\} \text{ and } \boxed{S-x} := \{y-x \,|\, y \in S\}.$$

THEOREM 1.13.7.
$$8 - \{2,9\} \;=\; \{6,-1\} \;=\; -\{-6,1\} \;=\; -(\{2,9\}-8).$$

THEOREM 1.13.8. $\forall S \subseteq \mathbb{R}$, $\forall x \in \mathbb{R}$, $\quad x - S = -(S - x)$.

DEFINITION 1.13.9. *Let $S \subseteq \mathbb{R}$, $x \in \mathbb{R}$. Then:*
$$\boxed{x \cdot S} := \{x \cdot y \,|\, y \in S\} \text{ and } \boxed{S \cdot x} := \{y \cdot x \,|\, y \in S\}.$$

Note that, by general sloppiness, $\cdot$ is often omitted in multiplication, and we might write: $\forall S \subseteq \mathbb{R}$, $\forall x \in \mathbb{R}$,

$xS := \{xy \,|\, y \in S\}$ and $Sx := \{yx \,|\, y \in S\}$.

THEOREM 1.13.10. $2 \cdot \{1,3,4\} = \{2,6,8\} = \{1,3,4\} \cdot 2$.

THEOREM 1.13.11. $\forall S \subseteq \mathbb{R}$, $\forall x \in \mathbb{R}$, $\quad x \cdot S = S \cdot x$.

DEFINITION 1.13.12. *Let $S \subseteq \mathbb{R}$, $x \in \mathbb{R}_0^\times$. Then:*
$$\boxed{S/x} := \{y \cdot x \,|\, y \in S\}.$$

DEFINITION 1.13.13. *Let* $S \subseteq \mathbb{R}_0^\times$, $x \in \mathbb{R}$. *Then:*
$\boxed{x/S} := \{x/y \,|\, y \in S\}$.

THEOREM 1.13.14. *Let* $S \subseteq \mathbb{R}_0^\times$, $x \in \mathbb{R}_0^\times$. *Then:*
$x/S = 1/(S/x)$ *and* $S/x = 1/(x/S)$.

THEOREM 1.13.15. $2\mathbb{N} := \{2, 4, 6, 8, \ldots\}$ *and* $2\mathbb{N}-1 = \{1, 3, 5, 7, \ldots\}$.

1.14. **Roots and powers of real numbers.**

It is our convention that $0^0 = 1$. In fact:

THEOREM 1.14.1. $\forall x \in \mathbb{R}, \quad x^0 = 1$.
Also, $\forall x \in \mathbb{R}, \forall j \in \mathbb{N}_0, \quad x^{j+1} = x^j \cdot x$.

DEFINITION 1.14.2. *Let* $x \in \mathbb{R}$. *Then* $\boxed{\sqrt{x}} := \max\{w \in \mathbb{R} \,|\, w^2 = x\}$.

THEOREM 1.14.3. $\sqrt{25} = \max[-5; 5] = 5$.

THEOREM 1.14.4. $\sqrt{2} \notin \mathbb{Q}$.

We have $\{w \in \mathbb{R} \,|\, w^2 \leqslant -1\} = \varnothing$, so $\sqrt{-1} = \max \varnothing$.
So, since $\max \varnothing = ☹$, we get:

THEOREM 1.14.5. $\sqrt{-1} = ☹$.

THEOREM 1.14.6. $\forall x \geqslant 0, \quad \sqrt{x} \neq ☹$.

THEOREM 1.14.7. $\forall x < 0, \quad \sqrt{x} = ☹$.

THEOREM 1.14.8. $\forall x \geqslant 0, \quad (\sqrt{x})^2 = x = \sqrt{x^2}$.

DEFINITION 1.14.9. $\forall x \in \mathbb{R}, \quad \boxed{|x|} := \sqrt{x^2}$.

THEOREM 1.14.10. $|-5| = \sqrt{(-5)^2} = \sqrt{25} = 5$.

DEFINITION 1.14.11. *Let* $k \in \mathbb{N}$, $x \in \mathbb{R}$. *Then:*
$\boxed{\sqrt[k]{x}} := \max\{w \in \mathbb{R} \,|\, w^k \leqslant x\}$.

THEOREM 1.14.12. $\forall k \in 2\mathbb{N}, \forall x \geqslant 0, \quad \sqrt[k]{x} \neq ☹$.

THEOREM 1.14.13. $\forall k \in 2\mathbb{N} - 1, \forall x \in \mathbb{R}, \quad \sqrt[k]{x} \neq ☹$.

THEOREM 1.14.14. $\forall k \in 2\mathbb{N}, \forall x < 0, \quad \sqrt[k]{x} = ☹$.

THEOREM 1.14.15. $\forall k \in 2\mathbb{N}, \forall x \geqslant 0, \quad (\sqrt[k]{x})^k = x = \sqrt[k]{x^k}$.

THEOREM 1.14.16. $\forall k \in 2\mathbb{N} - 1$, $\forall x \in \mathbb{R}$, $\quad (\sqrt[k]{x})^k = x = \sqrt[k]{x^k}$.

THEOREM 1.14.17. $\forall x \in \mathbb{R}$, $\quad \sqrt[1]{x} = x$.

THEOREM 1.14.18. $\forall x \geqslant 0$, $\quad \sqrt[2]{x} = \sqrt{x}$.

THEOREM 1.14.19. $\forall x \in \mathbb{R}$, $\quad \sqrt[2]{x} = \sqrt{x}$.

THEOREM 1.14.20. $\forall x \in \mathbb{R}$, $\quad \sqrt[3]{x} = \max\{w \in \mathbb{R} \,|\, w^3 \leqslant x\}$.

We have $\{w \in \mathbb{R} \,|\, w^3 \leqslant 8\} = (-\infty; 2]$. Then:

THEOREM 1.14.21. $\sqrt[3]{-8} = \max(-\infty; 2] = 2$.

THEOREM 1.14.22. $\forall x \in \mathbb{R}$, $\quad (\sqrt[3]{x})^3 = x = \sqrt[3]{x^3}$.

1.15. **Properties of absolute value.**

THEOREM 1.15.1. $\forall a, b \in \mathbb{R}$, $\quad |a - b| = |b - a|$.

It is crucial to us to take a statement like
When x is close to 2, x^2 is close to 4
and give it a rigorous meaning.
This requires us to find a rigorous way of talking about "closeness".
To say
a is close to b
is to say
the distance from a to b is close to zero.
Making this rigorous requires us to rigorize both
distance and close to zero.

We do not formally define distance in this course,
but we have an intuitive sense of distance, and:
the distance from 2 to 5 is $5 - 2$.
Also,
the distance from 9 to 1 is $9 - 1$.
The general rule is: $\forall a, b \in \mathbb{R}$,
the distance from a to b is $|b - a|$.
(NOTE: $\forall a, b \in \mathbb{R}$, $|a - b| = |b - a|$.)
So, when you see an expression of the form $|b - a|$,
you can interpret it, geometrically as a statement about
the distance from a to b.
This geometric intuition is indispensable.

Since absolute value plays such a big role,
we record a number of its properties, like:

THEOREM 1.15.2. *$\forall a, b \in \mathbb{R}$, we have:* $|a \cdot b| = |a| \cdot |b|$
and $|a + b| \leqslant |a| + |b|$.

The next theorem is called the **Triangle Inequality**.

THEOREM 1.15.3. *Let $a, b, c \in \mathbb{R}$. Then:* $|a - c| \leqslant |a - b| + |b - c|$.

Proof. $|a - c| = |(a - b) + (b - c)| \leqslant |a - b| + |b - c|$. □

THEOREM 1.15.4. $\forall a, b, c \in \mathbb{R}$, $|abc| = |a| \cdot |b| \cdot |c|$ *and*
$|a + b + c| \leqslant |a| + |b| + |c|$.

In the conclusion of the following theorem, we cannot write
$|x - 2| \cdot |x^3 - 8x^2 + 7x| \ < \ \delta \cdot (|x|^3 + 8 \cdot |x|^2 + 7 \cdot |x|)$
because of the possibility that $x = 0$.

THEOREM 1.15.5. *Let $x \in \mathbb{R}$, $\delta > 0$. Assume $|x - 2| < \delta$.
Then $|x - 2| \cdot |x^3 - 8x^2 + 7x| \ \leqslant \ \delta \cdot (|x|^3 + 8 \cdot |x|^2 + 7 \cdot |x|)$.*

Proof. We have:

$$\begin{aligned} |x^3 - 8x^2 + 7x| &= |x^3 + (-8x^2) + 7x| \\ &\leqslant |x^3| + |-8x^2| + |7x| \\ &= |x|^3 + |-8| \cdot |x|^2 + |7| \cdot |x| \\ &= |x|^3 + 8 \cdot |x|^2 + 7 \cdot |x|. \end{aligned}$$

So, since $0 \leqslant |x - 2| \leqslant \delta$, we get:
$|x - 2| \cdot |x^3 - 8x^2 + 7x| \leqslant \delta \cdot (|x|^3 + 8 \cdot |x|^2 + 7 \cdot |x|)$. □

THEOREM 1.15.6. *Let $a, b \in \mathbb{R}$, $\varepsilon > 0$. Then:*

$$\begin{aligned} (\,|b - a| < \varepsilon\,) \ &\Leftrightarrow \ (\,a - \varepsilon < b < a + \varepsilon\,) \qquad \textit{and} \\ (\,|b - a| < \varepsilon\,) \ &\Leftrightarrow \ (\,b - \varepsilon < a < b + \varepsilon\,) \qquad \textit{and} \\ (\,|b - a| \leqslant \varepsilon\,) \ &\Leftrightarrow \ (\,a - \varepsilon \leqslant b \leqslant a + \varepsilon\,) \qquad \textit{and} \\ (\,|b - a| \leqslant \varepsilon\,) \ &\Leftrightarrow \ (\,b - \varepsilon \leqslant a \leqslant b + \varepsilon\,). \end{aligned}$$

1.16. **A doubly quantified theorem.**

In this course, there are exactly two symbols that are called **quantifier**s. The first is "$\forall$", the second "$\exists$". They both appear in the following theorem.

THEOREM 1.16.1. *$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\delta^6 + 5\delta^4 + \delta \leqslant \varepsilon$.*

The stream of characters above is an example of a "mathematical statement". We will often just say "statement" to mean "mathematical statement". This has a technical definition which we will not go into here, but the intuition is that a statement is a stream of characters that has a mathematical meaning.

A character stream that is not a statement, like

gyre and gimbel in the wabe

cannot be analyzed mathematically, and so will be ignored in this course. While you are not expected to know the technical definition of a statement, there are some rules you should know for how to build complex statements out of simpler ones. For example, for any two statements A and B, the character stream "$(A) \Rightarrow (B)$" is also a statement, though, in practice, we are often sloppy and leave off those parenthesis and simply write "$A \Rightarrow B$". So, not only will we not give any technical definition of a statement, we will not even be purists about following that technical definition exactly.

Incidentally, similar remarks hold for "$A \& B$" and "$A \vee B$".

Let A and B be statements. Then the character stream "$A \Rightarrow B$" is a statement, and is considered equivalent to saying "if A, then B". There is a difference of usage between "$A \therefore B$" and "$A \Rightarrow B$": The statement "$A \Rightarrow B$" means, intuitively, "I am unsure of whether A is true, but, if it is, then B is also true". The statement "$A \therefore B$" means, intuitively, "I am completely sure that A is true, and it follows that B is true as well".

As mentioned above, Theorem 1.16.1 above involves two quantifiers. First is the **universal quantifier** "$\forall$", which means "for all" or, sometimes, "for any". Second is the **existential quantifier** "$\exists$" which means "there exists".

Because it has two quantifiers, Theorem 1.16.1 is "doubly quantified".

We turned Theorem 1.16.1 into a game: You give me $\varepsilon > 0$. I give you $\delta > 0$. We check to see if $\delta^6 + 5\delta^4 + \delta \leqslant \varepsilon$ is true. If it is, then I win. If not, then you win.

We played the game, and it was clear that I would win every time.

We developed a strategy: Once you give me $\varepsilon > 0$, I could find $\delta > 0$ such that all three of the following hold:

$$\delta^6 \leqslant \varepsilon/3 \text{ and } 5\delta^4 \leqslant \varepsilon/3 \text{ and } \delta \leqslant \varepsilon/3.$$

This suggests setting

$$\delta := \min\{\ \sqrt[6]{\varepsilon/3}\ ,\ \sqrt[4]{\varepsilon/15}\ ,\ \varepsilon/3\ \}.$$

We can structure the proof of Theorem 1.16.1:

Proof. Given $\varepsilon > 0$.
Want: $\exists \delta > 0$ s.t. $\delta^6 + 5\delta^4 + \delta \leqslant \varepsilon$.

Want: $\delta^6 + 5\delta^4 + \delta \leqslant \varepsilon$.

□

It remains to fill in the δ-strategy and finish:

Proof. Given $\varepsilon > 0$.
Want: $\exists \delta > 0$ s.t. $\delta^6 + 5\delta^4 + \delta \leqslant \varepsilon$.
Let $\delta := \min\{\ \sqrt[6]{\varepsilon/3}\ ,\ \sqrt[4]{\varepsilon/15}\ ,\ \varepsilon/3\ \}$.
Then $\delta > 0$.
Want: $\delta^6 + 5\delta^4 + \delta \leqslant \varepsilon$.
We know $0 \leqslant \delta \leqslant \sqrt[6]{\varepsilon/3}$, so $\delta^6 \leqslant \varepsilon/3$.
We also know $0 \leqslant \delta \leqslant \sqrt[4]{\varepsilon/15}$, so $\delta^4 \leqslant \varepsilon/15$, so $5\delta^4 \leqslant \varepsilon/3$.
Finally, we know $\delta \leqslant \varepsilon/3$.
Since $\delta^6 \leqslant \varepsilon/3$ and $5\delta^4 \leqslant \varepsilon/3$ and $\delta \leqslant \varepsilon/3$,
we conclude: $\delta^6 + 5\delta^4 + \delta \leqslant (\varepsilon/3) + (\varepsilon/3) + (\varepsilon/3)$.
Then $\delta^6 + 5\delta^4 + \delta \leqslant \varepsilon$. □

We can state Theorem 1.16.1 in a slightly different format, and the change makes the proof a little simpler because ε is bound within the statement of the theorem in a way that keeps the binding valid until the end of the proof:

THEOREM 1.16.2. *Let $\varepsilon > 0$. Then $\exists \delta > 0$ s.t. $\delta^6 + 5\delta^4 + \delta \leqslant \varepsilon$.*

Proof. Let $\delta := \min\{\ \sqrt[6]{\varepsilon/3}\ ,\ \sqrt[4]{\varepsilon/15}\ ,\ \varepsilon/3\ \}$.
Then $\delta > 0$.
Want: $\delta^6 + 5\delta^4 + \delta \leqslant \varepsilon$.
We know $0 \leqslant \delta \leqslant \sqrt[6]{\varepsilon/3}$, so $\delta^6 \leqslant \varepsilon/3$.
We also know $0 \leqslant \delta \leqslant \sqrt[4]{\varepsilon/15}$, so $\delta^4 \leqslant \varepsilon/15$. Then $5\delta^4 \leqslant \varepsilon/3$.
Finally, we know $\delta \leqslant \varepsilon/3$.
Since $\delta^6 \leqslant \varepsilon/3$ and $5\delta^4 \leqslant \varepsilon/3$ and $\delta \leqslant \varepsilon/3$,

we conclude: $\delta^6 + 5\delta^4 + \delta \leqslant (\varepsilon/3) + (\varepsilon/3) + (\varepsilon/3)$.
Then $\delta^6 + 5\delta^4 + \delta \leqslant \varepsilon$. □

Another similar theorem, and similar proof:

THEOREM 1.16.3. *$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $4\delta^8 + 7\delta^2 + 5\delta \leqslant 9\varepsilon$.*

Proof. Given $\varepsilon > 0$.
Want: $\exists \delta > 0$ s.t. $4\delta^8 + 7\delta^2 + 5\delta \leqslant 9\varepsilon$.
Let $\delta := \min\{\ \sqrt[8]{3\varepsilon/4}\ ,\ \sqrt{3\varepsilon/7}\ ,\ 3\varepsilon/5\ \}$.
Then $\delta > 0$.
Want: $4\delta^8 + 7\delta^2 + 5\delta \leqslant 9\varepsilon$.
We know $0 \leqslant \delta \leqslant \sqrt[8]{3\varepsilon/4}$, so $\delta^8 \leqslant 3\varepsilon/4$. Then $4\delta^8 \leqslant 3\varepsilon$.
We also know $0 \leqslant \delta \leqslant \sqrt{3\varepsilon/7}$, so $\delta^2 \leqslant 3\varepsilon/7$. Then $7\delta^2 \leqslant 3\varepsilon$.
Finally, we know $\delta \leqslant 3\varepsilon/5$. Then $5\delta \leqslant 3\varepsilon$.
Since $4\delta^8 \leqslant 3\varepsilon$ and $7\delta^2 \leqslant 3\varepsilon$ and $5\delta \leqslant 3\varepsilon$,
we conclude: $4\delta^8 + 7\delta^2 + 5\delta \leqslant 3\varepsilon/3 + 3\varepsilon + 3\varepsilon$.
Then $4\delta^8 + 7\delta^2 + 5\delta \leqslant 9\varepsilon$. □

1.17. **Triply quantified theorems with implication.**

THEOREM 1.17.1.
$\forall M \in \mathbb{R}$, $\exists \delta > 0$ s.t., $\forall x \in \mathbb{R}$,
$(\,0 < x < \delta\,) \Rightarrow (\,1/x > M\,)$.

Proof. Given $M \in \mathbb{R}$.
Want: $\exists \delta > 0$ s.t., $\forall x \in \mathbb{R}$,
$(\,0 < x < \delta\,) \Rightarrow (\,1/x > M\,)$.
Let $\delta := 1/(\max\{M, 1\})$. Then $\delta > 0$.
Want: $\forall x \in \mathbb{R}$, $(\,0 < x < \delta\,) \Rightarrow (\,1/x > M\,)$.
Given $x \in \mathbb{R}$. Assume $0 < x < \delta$. **Want:** $1/x > M$.
Since $0 < x < \delta$, it follows that $1/x > 1/\delta$.
Then $1/x > 1/\delta = \max\{M, 1\} \geqslant M$. □

THEOREM 1.17.2. *$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall x \in \mathbb{R}$,*
$(\ |x - 2| < \delta\ ) \Rightarrow (\ |x^4 - 5x^2 + 2x| < \varepsilon\ )$.

Proof. Given $\varepsilon > 0$.
Want: $\exists \delta > 0$ s.t. $\forall x \in \mathbb{R}$,
$(\ |x - 2| < \delta\ ) \Rightarrow (\ |x^4 - 5x^2 + 2x| < \varepsilon\ )$.
Let $\delta := \min\{1, \varepsilon/49\}$. Then $\delta < 1$ and $\delta < \varepsilon/49$ and $\delta > 0$.
Want: $\forall x \in \mathbb{R}$, $(\ |x - 2| < \delta\ ) \Rightarrow (\ |x^4 - 5x^2 + 2x| < \varepsilon\ )$.

Given $x \in \mathbb{R}$. Assume $|x-2| < \delta$. **Want:** $|x^4 - 5x^2 + 2x| < \varepsilon$.
We have $|x| \leqslant |(x-2)+2| \leqslant |x-2| + |2| = |x-2| + 2 < \delta + 2$,
so, since $\delta < 1$, we conclude that $|x| < 3$.
Since $48/49 < 1$ and $\varepsilon > 0$, we get $48 \cdot \varepsilon/49 < \varepsilon$.
Since $x^4 - 5x^2 + 2x = (x-2) \cdot (x^3 + 2x^2 - x)$,
we get $|x^4 - 5x^2 + 2x| = |x-2| \cdot |x^3 + 2x^2 - x|$.
Then $|x^4 - 5x^2 + 2x| = |x-2| \cdot |x^3 + 2x^2 - x| \leqslant \delta \cdot (|x|^3 + 2 \cdot |x|^2 + |x|)$.
So, since $|x| < 3$, this gives $|x^4 - 5x^2 + 2x| \leqslant \delta \cdot (3^3 + 2 \cdot 3^2 + 3)$.
So, since $3^3 + 2 \cdot 3^2 + 3 = 48$, this gives $|x^4 - 5x^2 + 2x| \leqslant 48 \cdot \delta$.
So, since $\delta < \varepsilon/49$, this gives $|x^4 - 5x^2 + 2x| \leqslant 48 \cdot \varepsilon/49$.
Then $|x^4 - 5x^2 + 2x| \leqslant 48 \cdot \varepsilon/49 < \varepsilon$. □

1.18. **Primitive ordered pairs.**

THEOREM 1.18.1. $\{1,2\} = \{2,1\} = \{1,1,2,2,2\}$.

THEOREM 1.18.2. $\{5,5\} = \{5\}$ *and* $\{\ \{5\}\ ,\ \{5\}\ \} = \{\ \{5\}\ \}$.

THEOREM 1.18.3. $\{\ \{5\}\ ,\ \{5,5\}\ \} = \{\ \{5\}\ \}$.

DEFINITION 1.18.4. $\forall x, y,$ $\boxed{\langle\langle x,y\rangle\rangle} := \begin{cases} \{\ \{x\}\ ,\ \{x,y\}\ \}, & \textit{if } x \neq ☹ \neq y \\ ☹, & \textit{if } ((x = ☹) \vee (y = ☹)). \end{cases}$

The notation $\langle\langle x,y\rangle\rangle$ is read "the **primitive ordered pair** x, y".

THEOREM 1.18.5. $\langle\langle 1,2\rangle\rangle = \{\ \{1\}\ ,\ \{1,2\}\ \}$ *and*
$\langle\langle 6,\{7,8\}\rangle\rangle = \{\ \{6\}\ ,\ \{6,\{7,8\}\}\ \}$ *and*
$\langle\langle 5,5\rangle\rangle = \{\ \{5\}\ ,\ \{5,5\}\ \} = \{\ \{5\}\ \}$ *and*
$\langle\langle ☹,5\rangle\rangle = \langle\langle ☹,☹\rangle\rangle = \langle\langle \{2,1\},☹\rangle\rangle = ☹$.

THEOREM 1.18.6. $\langle\langle 1,2\rangle\rangle = \{\ \{1\}\ ,\ \{1,2\}\ \}$ *and*
$\langle\langle 2,1\rangle\rangle = \{\ \{2\}\ ,\ \{2,1\}\ \}$.

THEOREM 1.18.7. $\langle\langle 1,2\rangle\rangle \neq \langle\langle 2,1\rangle\rangle$.

THEOREM 1.18.8. $\forall$*non-*☹ $a, b, x, y,$
$(\ \langle\langle a,b\rangle\rangle = \langle\langle x,y\rangle\rangle\ )\ \Leftrightarrow\ (\ (a = x)\ \&\ (b = y)\ )$.

1.19. **Relations.**

DEFINITION 1.19.1. *Let R be a set. Then R is a* **relation** *means:*
$\forall z \in R,\ \exists x, y$ *s.t.* $z = \langle\langle x,y\rangle\rangle$.

In other words, a relation is a set of primitive ordered pairs.

Since ☹ = $\langle\langle$☹, ☹$\rangle\rangle$, we see that, technically, ☹ is a primitive ordered pair. However, ☹ is never an element of any set, and so it cannot be an element of a relation. So, even if it's somewhat redundant, it's possibly clearer to say that a relation is a set of non-☹ primitive ordered pairs.

More generally, any set is a set of non-☹ objects.

THEOREM 1.19.2. $\{\ \langle\langle 1,2\rangle\rangle\ ,\ \langle\langle 1,3\rangle\rangle\ ,\ \langle\langle 4,5\rangle\rangle\ \}$ *is a relation.*

THEOREM 1.19.3.
$\{\ \langle\langle 1,\{2,3\}\rangle\rangle\ ,\ \langle\langle 1,9\rangle\rangle\ ,\ \langle\langle 7,6\rangle\rangle\ \}$ *is a relation.*

Since $\qquad \forall x,y, \quad \langle\langle x,y\rangle\rangle \neq \{1,2,3\}, \qquad$ we conclude:

THEOREM 1.19.4.
$\{\ \{1,2,3\}\ ,\ \langle\langle 1,9\rangle\rangle\ ,\ \langle\langle 7,6\rangle\rangle\ \}$ *is NOT a relation.*

The following is null true:

THEOREM 1.19.5. $\varnothing$ *is a relation.*

DEFINITION 1.19.6. *Let R be a relation. Then:*

$$\boxed{\mathbb{D}_R} \quad := \quad \{\ x \mid \exists y \text{ s.t. } \langle\langle x,y\rangle\rangle \in R\ \} \quad \text{and}$$
$$\boxed{\mathbb{I}_R} \quad := \quad \{\ y \mid \exists x \text{ s.t. } \langle\langle x,y\rangle\rangle \in R\ \}.$$

We call $\mathbb{D}_R$ the **domain** of R.
We call $\mathbb{I}_R$ the **image** of R.
Unassigned HW: Show that $\mathbb{D}_\varnothing = \varnothing$ and $\mathbb{R}_\varnothing = \varnothing$.

DEFINITION 1.19.7. *Let R be a relation, x an object. Then:*

$$\boxed{\mathrm{VL}_x^R} \quad := \quad \{\ y \mid \langle\langle x,y\rangle\rangle \in R\ \}.$$

We call VL_x^R the **vertical line** through x in R.

We justified the following theorem by graphing R.

THEOREM 1.19.8. *Let $R := \{\langle\langle 1,2\rangle\rangle, \langle\langle 1,3\rangle\rangle, \langle\langle 4,5\rangle\rangle\}$.*
Then $\mathbb{D}_R = \{1,4\}$ and $\mathbb{I}_R = \{2,3,5\}$ and
$\mathrm{VL}_1^R = \{2,3\}$ *and* $\mathrm{VL}_4^R = \{5\}$ *and* $\mathrm{VL}_3^R = \varnothing$.

We justified the following two theorems by looking at the graph of the relation of the preceding theorem.

THEOREM 1.19.9. *Let R be a relation and let x be an object.*
Then: $\quad (x \in \mathbb{D}_R) \Leftrightarrow (\mathrm{VL}_x^R \neq \varnothing)$.

THEOREM 1.19.10. *Let R be a relation and let y be an object.*
Then: $\quad (y \in I_R) \Leftrightarrow (\exists x \in \mathbb{D}_R \text{ s.t. } \langle\langle x, y \rangle\rangle \in R)$.

Some relations cannot be graphed, and yet we can still study domain, range and vertical lines for them. For example:

THEOREM 1.19.11. *Let* $R := \{ \langle\langle 1, \{2,3\} \rangle\rangle, \langle\langle 1, 9 \rangle\rangle, \langle\langle 6, 7 \rangle\rangle \}$.
Then $\mathbb{D}_R = \{1, 6\}$ *and* $I_R = \{ \{2,3\}, 7, 9 \}$ *and*
$\mathrm{VL}_1^R = \{ \{2,3\}, 9\}$ *and* $\mathrm{VL}_6^R = \{7\}$ *and* $\mathrm{VL}_3^R = \varnothing$.

THEOREM 1.19.12. *Let R be a relation. Then* $\mathrm{VL}_{☹}^R = \varnothing$.

Proof. Assume $\mathrm{VL}_{☹}^R \neq \varnothing$. **Want:** Contradiction.
Choose $y \in \mathrm{VL}_{☹}^R$. Then $\langle\langle ☹, y \rangle\rangle \in R$.
By Definition 1.18.4, we have: $\langle\langle ☹, y \rangle\rangle = ☹$.
Since $☹ = \langle\langle ☹, y \rangle\rangle \in R$, we get: $☹ \in R$.
Since R is a set, by Axiom 1.3.1, we have: $☹ \notin R$.
Contradiction. □

1.20. **Functions.**

DEFINITION 1.20.1. *For any object f, by f is a* **function** *we mean:* $(f$ *is a relation*$)$ & $(\forall x \in \mathbb{D}_f,\ \mathrm{VL}_x^f$ *is a singleton* $)$.

In other words, a function is a relation for which each of its vertical lines, through points in its domain, is a singleton.

THEOREM 1.20.2. $\{ \langle\langle 1, 2 \rangle\rangle, \langle\langle 1, 3 \rangle\rangle, \langle\langle 4, 5 \rangle\rangle \}$ *is NOT a function.*

THEOREM 1.20.3. $\{ \langle\langle 3, 7 \rangle\rangle, \langle\langle 2, 7 \rangle\rangle, \langle\langle 1, 8 \rangle\rangle \}$ *IS a function.*

The function $\{ \langle\langle 3, 7 \rangle\rangle, \langle\langle 2, 7 \rangle\rangle, \langle\langle 1, 8 \rangle\rangle \}$ will typically be written:

$$\begin{pmatrix} 3 \mapsto 7 \\ 2 \mapsto 7 \\ 1 \mapsto 8 \end{pmatrix}.$$

Generally, for any $n \in \mathbb{N}$, for any objects $x_1, \ldots, x_n$, for any objects $y_1, \ldots, y_n$, we define

$$\begin{pmatrix} x_1 \mapsto y_1 \\ \vdots \\ x_n \mapsto y_n \end{pmatrix} \quad := \quad \{ \, \langle\!\langle x_1, y_1 \rangle\!\rangle \, , \, \ldots \, , \, \langle\!\langle x_n, y_n \rangle\!\rangle \, \}.$$

We are eventually going to cease use of $\langle\!\langle \bullet, \bullet \rangle\!\rangle$ in favor of higher-level notation, and this is an example.

THEOREM 1.20.4.

$$\begin{pmatrix} 2 \mapsto \{3,5\} \\ 8 \mapsto 9 \\ 6 \mapsto -1 \\ 4 \mapsto \{6\} \end{pmatrix} = \begin{pmatrix} 2 \mapsto \{3,5\} \\ 4 \mapsto \{6\} \\ 6 \mapsto -1 \\ 8 \mapsto 9 \end{pmatrix}.$$

THEOREM 1.20.5. *Let f be a relation. Then:* $(\, f \textit{ is a function}) \Leftrightarrow (\, \forall x, y, z, \, (\, \langle\!\langle x, y \rangle\!\rangle, \langle\!\langle x, z \rangle\!\rangle \in f \,) \Rightarrow (\, y = z \,)\,).$

THEOREM 1.20.6. *$\varnothing$ is a function.*

THEOREM 1.20.7. $(\, \mathbb{D}_\varnothing = \varnothing = \mathbb{I}_\varnothing \,) \,\&\, (\, \forall x, \ \mathrm{VL}_x^\varnothing = \varnothing \,).$

DEFINITION 1.20.8. *Let f be a function and let x be an object. Then we define:*

$\boxed{f_x} := \mathrm{UE}(\mathrm{VL}_x^f).$

Alternate notation for f_x is $\boxed{f(x)}$.

THEOREM 1.20.9.

Let $f := \begin{pmatrix} 2 \mapsto 3 \\ 5 \mapsto 8 \\ 7 \mapsto 6 \end{pmatrix}$. *Then:* $(\, f(5) = 8 \,) \,\&\, (\, f(2) = 3 \,) \,\&\, (\, f(9) = ☹ \,).$

THEOREM 1.20.10.

Let $f := \begin{pmatrix} 2 \mapsto \{3,5\} \\ 4 \mapsto \{6\} \\ 6 \mapsto -1 \\ 8 \mapsto 9 \end{pmatrix}$. *Then:* $(f_6 = -1) \& (f_2 = \{3,5\}) \& (f_3 = ☹).$

THEOREM 1.20.11. $\forall x, \quad \varnothing_x = ☹.$

THEOREM 1.20.12. *Let f be a function and let x be an object. Then:* $(\, x \in \mathbb{D}_f \,) \Leftrightarrow (\, \mathrm{VL}_x^f \textit{ is a singleton} \,) \Leftrightarrow (\, f_x \neq ☹ \,).$

THEOREM 1.20.13. *Let f be a function and let x, y be objects.*
Then: $(\langle\langle x,y\rangle\rangle \in f) \Leftrightarrow (y \in \mathrm{VL}_x^f) \Leftrightarrow (f_x = y)$.

The next axiom is part of the general philosophy that
"☹ is infective."

AXIOM 1.20.14. *For all q, $☹_q = ☹$ and $q_{☹} = ☹$.*

THEOREM 1.20.15. *Let* $S := \left\{ \begin{pmatrix} 1 \mapsto 1 \\ 2 \mapsto 2 \end{pmatrix} \right\}$.
Let $f := \mathrm{UE}_S$. *Then:* $f_1 = 1$ *and* $f_2 = 2$ *and* $f_3 = ☹$ *and* $f_{☹} = ☹$.

THEOREM 1.20.16. *Let* $S := \left\{ \begin{pmatrix} 1 \mapsto 1 \\ 2 \mapsto 2 \end{pmatrix} , \begin{pmatrix} 1 \mapsto 2 \\ 2 \mapsto 1 \end{pmatrix} \right\}$.
Let $f := \mathrm{UE}_S$. *Then:* $f_1 = ☹$ *and* $f_2 = ☹$.

DEFINITION 1.20.17. *Let A and B be sets. Then:*
$\boxed{f : A \dashrightarrow B}$ *means* $(f$ *is a function*$)\,\&\,(\mathbb{D}_f \subseteq A)\,\&\,(\mathbb{I}_f \subseteq B)$ *and*
$\boxed{f : A \to B}$ *means* $(f$ *is a function*$)\,\&\,(\mathbb{D}_f = A)\,\&\,(\mathbb{I}_f \subseteq B)$ *and*
$\boxed{f : A \to> B}$ *means* $(f$ *is a function*$)\,\&\,(\mathbb{D}_f = A)\,\&\,(\mathbb{I}_f = B)$.

THEOREM 1.20.18. *Let* $f := \begin{pmatrix} 2 \mapsto 3 \\ 5 \mapsto 8 \\ 7 \mapsto 6 \end{pmatrix}$. *Then:*

$f : \{2,5,7\} \to \{3,4,6,8\}$ *and*
$f : \{2,5,7\} \to \{1,3,4,6,8,9\}$ *and*
$f : \{2,4,5,7\} \dashrightarrow \{3,4,6,8\}$ *and*
$f : \{1,2,4,5,7\} \dashrightarrow \{1,3,4,6,8\}$ *and*
$f : \{2,5,7\} \to> \{3,6,8\}$.

DEFINITION 1.20.19. *Let f be a function. Then:*
by f is **one-to-one** *or* **1-1**, *we mean:*
$\forall w, x \in \mathbb{D}_f, \quad (f_w = f_x) \Rightarrow (w = x)$.

THEOREM 1.20.20. *Let* $f := \begin{pmatrix} 2 \mapsto 3 \\ 5 \mapsto 8 \\ 7 \mapsto 6 \end{pmatrix}$. *Then f is 1-1.*

THEOREM 1.20.21. *Let* $f := \begin{pmatrix} 2 \mapsto 5 \\ 5 \mapsto 8 \\ 7 \mapsto 5 \end{pmatrix}$. *Then f is NOT 1-1.*

We graphed the function f from the next theorem.
It was the graph of $y = x^3$.

THEOREM 1.20.22. *Let* $f := \{\langle\langle x, y\rangle\rangle \,|\, (x, y \in \mathbb{R}) \,\&\, (y = x^3)\}$.
Then f *is 1-1.*

We graphed the function f from the next theorem.
It was the parabola given by $y = x^2$.

THEOREM 1.20.23. *Let* $f := \{\langle\langle x, y\rangle\rangle \,|\, (x, y \in \mathbb{R}) \,\&\, (y = x^2)\}$.
Then f *is NOT 1-1.*

DEFINITION 1.20.24. *Let* A *and* B *be sets. Then:*
$\boxed{f : A \hookrightarrow B}$ *means* $(f : A \to B) \,\&\, (f$ *is 1-1*$)$ *and*
$\boxed{f : A \hookrightarrow> B}$ *means* $(f : A \to> B) \,\&\, (f$ *is 1-1*$)$.

THEOREM 1.20.25. *Let* $f := \begin{pmatrix} 2 \mapsto 3 \\ 5 \mapsto 8 \\ 7 \mapsto 6 \end{pmatrix}$. *Then:*

$f : \{2, 5, 7\} \hookrightarrow \{3, 4, 6, 8\}$ *and*
$f : \{2, 5, 7\} \hookrightarrow \{1, 3, 4, 6, 8, 9\}$ *and*
$f : \{2, 5, 7\} \hookrightarrow> \{3, 6, 8\}$.

We have now introduced enough notation that, going forward,
we can avoid writing $\langle\langle \bullet, \bullet \rangle\rangle$.
For example, instead of writing $\langle\langle x, y\rangle\rangle \in f$,
we will write $f_x = y$ or $f(x) = y$.

Instead of, for example, writing
Let $f := \{\langle\langle x, y\rangle\rangle \,|\, x, y \in \mathbb{R}\,,\, y = x^3\}$,
please write
Let $f : \mathbb{R} \to \mathbb{R}$ be defined by: $\forall x \in \mathbb{R}$, $f(x) = x^3$
or Define $f : \mathbb{R} \to \mathbb{R}$ by: $\forall x \in \mathbb{R}$, $f(x) = x^3$.
One of the advantages of this is that,
because $f(x)$ is given by a formula,
it is clear that f is a function.
That is, each x corresponds to exactly one y, namely x^3.
We will see this, in the proof of (a) in the next theorem.

We graphed the function f from the next theorem.
It was the parabola given by $y = x^2$.

THEOREM 1.20.26. *Define* $f : \mathbb{R} \to \mathbb{R}$ *by:* $\forall x \in \mathbb{R},\ f_x = x^2$.
Then $f : \mathbb{R} \to> [0; \infty)$.

We graphed the function f from the next theorem.
It was the graph of $y = x^3$.

THEOREM 1.20.27. *Define* $f : \mathbb{R} \to \mathbb{R}$ *by:* $\forall x \in \mathbb{R},\ f_x = x^3$.
Then $f : \mathbb{R} \hookrightarrow> \mathbb{R}$.

Proof. **Want:** (α) $f : \mathbb{R} \to \mathbb{R}$ and (β) f is 1-1.

Proof of (α):
Want: (a) f is a function and (b) $\mathbb{D}_f = \mathbb{R}$ and (c) $\mathbb{I}_f = \mathbb{R}$.

Proof of (a):
Want:$\forall x, y, z,\quad ((f_x = y) \,\&\, (f_x = z)) \Rightarrow (y = z)$.
Given x, y, z. Assume: $(f_x = y) \,\&\, (f_x = z)$. **Want:** $y = z$.
We have $y = f_x = z$.
End of proof of (a).

Proof of (b):
Since $f : \mathbb{R} \to \mathbb{R}$, we get $\mathbb{D}_f = \mathbb{R}$.
End of proof of (b).

Proof of (c):
Since $f : \mathbb{R} \to \mathbb{R}$, we get $\mathbb{I}_f \subseteq \mathbb{R}$.
It remains to show that $\mathbb{R} \subseteq \mathbb{I}_f$.
Want: $\forall y \in \mathbb{R},\quad y \in \mathbb{I}_f$.
Given $y \in \mathbb{R}$. **Want:** $y \in \mathbb{I}_f$.
Want:$\exists x \in \mathbb{D}_f$ s.t. $y = f_x$.
Since $f : \mathbb{R} \to \mathbb{R}$, we get $\mathbb{D}_f = \mathbb{R}$.
Let $x := \sqrt[3]{y}$. Then $x \in \mathbb{R} = \mathbb{D}_f$. **Want:** $y = f_x$.
We have $y = (\sqrt[3]{y})^3 = x^3 = f_x$.
End of proof of (c).

End of proof of (α).

Proof of (β):
Want:$\forall w, x \in \mathbb{D}_f,\quad (f_w = f_x) \Rightarrow (w = x)$.
Given $w, x \in \mathbb{D}_f$. Assume $f_w = f_x$. **Want:** $w = x$.

We have $w = \sqrt[3]{w^3} = \sqrt[3]{f_w} = \sqrt[3]{f_x} = \sqrt[3]{x^3} = x$.
End of proof of (β). □

The next two theorems are quantified equivalences for equality of functions. In the first, we assume we have a common domain. In the second, we only assume we have a common superdomain.

THEOREM 1.20.28. *Let A be a set. Let ϕ and ψ be functions.*
Assume $\mathbb{D}_\phi = A$ and $\mathbb{D}_\psi = A$.
Then $(\ \phi = \psi\ ) \Leftrightarrow (\ \forall x \in A,\ \ \phi_x = \psi_x\ )$.

THEOREM 1.20.29. *Let S be a set. Let ϕ and ψ be functions.*
Assume $\mathbb{D}_\phi \subseteq S$ and $\mathbb{D}_\psi \subseteq S$.
Then $(\ \phi = \psi\ ) \Leftrightarrow (\ \forall x \in S,\ \ \phi_x \overset{*}{=} \psi_x\ )$.

It a basic property of the real numbers that:
$\forall x \in \mathbb{R}, \qquad x^2/x = x^3/x^2.$
This property is used in the proof of the following theorem:

THEOREM 1.20.30. *Define $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$ by:*
$\forall x \in \mathbb{R}, \qquad f(x) = x^2/x \quad$ *and* $\quad g(x) = x^3/x^2.$
Then $f = g$.

Proof. Since $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, we get $\mathbb{D}_f \subseteq \mathbb{R}$ and $\mathbb{D}_g \subseteq \mathbb{R}$.
Want: $\quad \forall x \in \mathbb{R}, \quad f_x = g_x$.
Given $x \in \mathbb{R}$. **Want:** $f_x = g_x$.
We have $f_x = x^2/x = x^3/x^2 = g_x$. □

We talked about various ways of picturing functions.

DEFINITION 1.20.31. *Let f be a function, A a set. Then:*
$$\boxed{f_*A} := \{f_x \,|\, x \in A \cap \mathbb{D}_f\} \qquad \textit{and}$$
$$\boxed{f^*A} := \{x \in \mathbb{D}_f \,|\, f_x \in A\}.$$

Alternate notation: $\boxed{f_*(A)}$ and $\boxed{f^*(A)}$.
We do NOT use $f(A)$ and $f^{-1}(A)$.

We discussed how to picture f_*A and f^*A.
The set f_*A is called the f-**forward-image of** A.
The set f^*A is called the f-**pre-image of** A

THEOREM 1.20.32. *Let* $f := \begin{pmatrix} 1 \mapsto 4 \\ 2 \mapsto 6 \\ 3 \mapsto 8 \\ 9 \mapsto 8 \end{pmatrix}$.

Then $f_*\{0,1,2\} = \{4,6\}$ *and* $f^*\{6,7,8\} = \{2,3,9\}$.

DEFINITION 1.20.33. $\forall a, b$, *by* $\boxed{a \equiv b}$, *we mean:* $a = b \neq ☹$.

THEOREM 1.20.34. $\forall a, b$, $(a \equiv b) \Rightarrow (a \neq ☹ \neq b)$.

We have quantified equivalences for $y \in f_*S$:

THEOREM 1.20.35. *Let f be a function, S a set, y an object. Then:*
$$(y \in f_*S) \Leftrightarrow (\exists x \in S \cap \mathbb{D}_f \text{ s.t. } f_x = y) \Leftrightarrow (\exists x \in S \text{ s.t. } f_x \equiv y).$$

We have an equivalence for $y \in f^*S$:

THEOREM 1.20.36. *Let f be a function, S a set, x an object. Then:*
$$(x \in f^*S) \Leftrightarrow (f_x \in S).$$

We omitted formal proofs of the preceding two theorems, but used pictures to motivate them.

In the following, the set $f^*\{y\}$ is called the f**-fiber over** y.
We justified that terminology with a picture.

THEOREM 1.20.37. *Let f be a function, x, y objects. Then:*
$$(x \in f^*\{y\}) \Leftrightarrow (f_x = y).$$

Proof. We have $(x \in f^*\{y\}) \Leftrightarrow (f_x \in \{y\}) \Leftrightarrow (f_x = y)$. □

We define **agreement, on a set, of two functions**:

DEFINITION 1.20.38. *Let f and g be functions and let S be a set. By* **on** S**,** $f = g$, *we mean:* $\forall x \in S$, $f_x = g_x$.

Note that, for any two functions f and g, for any set S,
if on S, $f = g$, then $S \subseteq \mathbb{D}_f \cap \mathbb{D}_g$.

THEOREM 1.20.39. *Let f be a function. Then* $f_{☹} = ☹$.

Proof. By definition, we have $f_{☹} = \mathrm{UE}(\mathrm{VL}^f_{☹})$.
By Theorem 1.19.12, $\mathrm{VL}^f_{☹} = \varnothing$. Then $f_{☹} = \mathrm{UE}(\varnothing) = ☹$. □

1.21. **Restriction of functions.**

We next define the **restriction of f to a subset of $\mathbb{D}_f$**:

DEFINITION 1.21.1. *Let f be a function, $S \subseteq \mathbb{D}_f$.*
Then $\boxed{f|S} : S \to \mathbb{I}_f$ *is defined by:* $\forall x \in S$, $(f|S)_x = f_x$.

We defined $f : \mathbb{R} \to \mathbb{R}$ by $\forall x \in \mathbb{R}$, $f_x = x^2$.
We graphed f and noted that f is NOT 1-1.
We graphed $f|[0;\infty)$ and noted that $f|[0;\infty)$ IS 1-1.
We next define **restriction,** $f \subseteq g$:

DEFINITION 1.21.2. *Let f and g be functions.*
By $\boxed{f \subseteq g}$, *we mean:* $\quad (\mathbb{D}_f \subseteq \mathbb{D}_g) \,\&\, (g|\mathbb{D}_f = f)$.

We next define **extension,** $g \supseteq f$:

DEFINITION 1.21.3. *Let f and g be functions.*
By $\boxed{g \supseteq f}$, *we mean:* $\quad (\mathbb{D}_f \subseteq \mathbb{D}_g) \,\&\, (g|\mathbb{D}_f = f)$.

Note: $(f \subseteq g) \Leftrightarrow (g \supseteq f)$.

Unassigned HW:

THEOREM 1.21.4. *$\forall$function g, $\forall S \subseteq \mathbb{D}_g$, $g|S \subseteq g$.*

1.22. **Composition of functions.**

DEFINITION 1.22.1. *Let f and g be functions. Then:*
$\boxed{g \circ f}$ *is the function defined by:*

$$\forall x, \quad (g \circ f)_x := g_{f_x}$$

We read $g \circ f$ as "f then g" or g **compose** f".
The function $g \circ f$ is called the **composition** of g and f.

THEOREM 1.22.2. *Let* $f := \begin{pmatrix} 1 \mapsto 7 \\ 3 \mapsto 3 \\ 8 \mapsto 6 \\ 9 \mapsto 2 \end{pmatrix}$ *and* $g := \begin{pmatrix} 2 \mapsto 0 \\ 3 \mapsto 9 \\ 4 \mapsto 5 \end{pmatrix}$.

$$\text{Then } g \circ f = \begin{pmatrix} 3 \mapsto 9 \\ 9 \mapsto 0 \end{pmatrix}.$$

THEOREM 1.22.3. *Let f and g be functions. Then:*

$$\mathbb{D}_{g\circ f} = f^*(\mathbb{D}_g) \quad \text{and} \quad \mathbb{I}_{g\circ f} = g_*(\mathbb{I}_f).$$

THEOREM 1.22.4. *Let A, B, C be sets, f, g be functions. Then:*
(1) $[\,(\,(\,f : A \dashrightarrow B\,)\,\&\,(\,g : B \dashrightarrow C\,)\,) \Rightarrow (\,g \circ f : A \dashrightarrow C\,)\,]$ *and*
(2) $[\,(\,(\,f : A \to B\,)\,\&\,(\,g : B \to C\,)\,) \Rightarrow (\,g \circ f : A \to C\,)\,]$ *and*
(3) $[\,(\,(\,f : A \hookrightarrow B\,)\,\&\,(\,g : B \hookrightarrow C\,)\,) \Rightarrow (\,g \circ f : A \hookrightarrow C\,)\,]$ *and*
(4) $[\,(\,(\,f : A \twoheadrightarrow B\,)\,\&\,(\,g : B \twoheadrightarrow C\,)\,) \Rightarrow (\,g \circ f : A \twoheadrightarrow C\,)\,]$ *and*
(5) $[\,(\,(\,f : A \hookrightarrow\!\!\!\!\!\twoheadrightarrow B\,)\,\&\,(\,g : B \hookrightarrow\!\!\!\!\!\twoheadrightarrow C\,)\,) \Rightarrow (\,g \circ f : A \hookrightarrow\!\!\!\!\!\twoheadrightarrow C\,)\,]$.

1.23. **Identity and inverse and characteristic functions.** The function id^A in the next definition is called the **identity function** on A.

DEFINITION 1.23.1. *Let A be a set.*
Then $\boxed{\mathrm{id}^A} : A \to A$ *is defined by:* $\forall x \in A,\quad \mathrm{id}^A_x = x$.

THEOREM 1.23.2. $$\mathrm{id}^{\{2,4,6\}} = \begin{pmatrix} 2 \mapsto 2 \\ 4 \mapsto 4 \\ 6 \mapsto 6 \end{pmatrix}.$$

THEOREM 1.23.3. $\mathrm{id}^{\{2,4,6\}}_7 = ☹$ *and* $\mathrm{id}^{\{2,4,6\}}_4 = 4$.

THEOREM 1.23.4. $\mathrm{id}^{\mathbb{R}}_4 = 4$ *and* $\mathrm{id}^{\mathbb{R}}_{\{4\}} = ☹$.

THEOREM 1.23.5. $\mathrm{id}^{\mathbb{R}_0^\times}_4 = 4$ *and* $\mathrm{id}^{\mathbb{R}_0^\times}_0 = ☹$.

THEOREM 1.23.6. *Let* $f := \begin{pmatrix} 1 \mapsto 4 \\ 2 \mapsto 6 \\ 3 \mapsto 8 \end{pmatrix}$ *and* $g := \begin{pmatrix} 4 \mapsto 1 \\ 6 \mapsto 2 \\ 8 \mapsto 3 \end{pmatrix}$.

$$\textit{Then: } g \circ f = \begin{pmatrix} 1 \mapsto 1 \\ 2 \mapsto 2 \\ 3 \mapsto 3 \end{pmatrix} = \mathrm{id}^{\{1,2,3\}} = \mathrm{id}^{\mathbb{D}_f} \qquad \textit{and}$$
$$f \circ g = \begin{pmatrix} 4 \mapsto 4 \\ 6 \mapsto 6 \\ 8 \mapsto 8 \end{pmatrix} = \mathrm{id}^{\{4,6,8\}} = \mathrm{id}^{\mathbb{D}_g}.$$

THEOREM 1.23.7. *Let A, B be sets. Let $f : A \to B$, $g : B \to A$. Assume $g \circ f = \mathrm{id}^A$. Then: (1) $f : A \hookrightarrow B$ and (2) $g : B \twoheadrightarrow A$.*

Proof. Proof of (1):
Want: $\forall w, x \in \mathbb{D}_f,\ (f_w = f_x) \Rightarrow (w = x)$.
Given $w, x \in \mathbb{D}_f$. Assume $f_w = f_x$. **Want:** $w = x$.
Since $w, x \in \mathbb{D}_f = A$, we get $(g \circ f)_w = \mathrm{id}^A_w$ and $(g \circ f)_x = \mathrm{id}^A_x$.
Then $w = \mathrm{id}^A_w = (g \circ f)_w = g_{f_w} = g_{f_x} = (g \circ f)_x = \mathrm{id}^A_x = x$.
End of proof of (1).

Proof of (2):
Want:$\mathbb{I}_g = A$. Since $g : B \to A$, we know: $\mathbb{I}_g \subseteq A$. **Want:** $A \subseteq \mathbb{I}_g$.
Want:$\forall x \in A$, $x \in \mathbb{I}_g$. Given $x \in A$. **Want:** $x \in \mathbb{I}_g$.
Since $x \in A = \mathbb{D}_f$, we get $f_x \in \mathbb{I}_f$.
Let $y := f_x$. Then $y \in \mathbb{I}_f$.
Then $y \in \mathbb{I}_f \subseteq B = \mathbb{D}_g$. Then $g_y \in \mathbb{I}_g$.
We have $g_y = g_{f_x} = (g \circ f)_x = \mathrm{id}^A_x = x$. Then $g_y = x$.
Then $x = g_y \in \mathbb{I}_g$.
End of proof of (2). □

THEOREM 1.23.8. *Let A, B be sets, $f : A \to B$.*
Assume: $\exists g : B \to A$ s.t. ($g \circ f = \mathrm{id}^A$) & ($f \circ g = \mathrm{id}^B$).
Then $f : A \hookrightarrow\!\!> B$.

Proof. Since $g \circ f = \mathrm{id}^A$, by (1) of Theorem 1.23.7, $f : A \hookrightarrow B$.
Want: $f : A \to\!\!> B$.
Since $f \circ g = \mathrm{id}^B$, by (2) of Theorem 1.23.7, $f : A \to\!\!> B$. □

The function f^{-1} below is called the **inverse function** of f.

DEFINITION 1.23.9. *Let f be a 1-1 function.*
Then $\boxed{f^{-1}} : \mathbb{I}_f \to \mathbb{D}_f$ *is defined by:* $\forall y \in \mathbb{I}_f$, $f^{-1}_y = \mathrm{UE}(f^*\{y\})$.

THEOREM 1.23.10. *Let* $f := \begin{pmatrix} 1 \mapsto 4 \\ 2 \mapsto 6 \\ 3 \mapsto 8 \end{pmatrix}$.

Then: $f^{-1}_4 = \mathrm{UE}(f^*\{4\}) = \mathrm{UE}\{1\} = 1$.

THEOREM 1.23.11. *Let* $f := \begin{pmatrix} 1 \mapsto 4 \\ 2 \mapsto 6 \\ 3 \mapsto 8 \end{pmatrix}$ *and* $g := \begin{pmatrix} 4 \mapsto 1 \\ 6 \mapsto 2 \\ 8 \mapsto 3 \end{pmatrix}$.

Then: $g = f^{-1}$ *and* $f = g^{-1}$.

THEOREM 1.23.12. *Let f be a 1-1 function.*
Then: $\forall x \in \mathbb{D}_f$, $f^{-1}_{f_x} = x$.
Also: $\forall y \in \mathbb{I}_f$, $f_{f^{-1}_y} = y$.

DEFINITION 1.23.13. *Let S be a set and let $A \subseteq S$.*
Then $\boxed{\chi^S_A} : S \to \{0, 1\}$ *is defined by:*

$$\forall q \in S, \qquad \chi_A^S(q) = \begin{cases} 1, & \textit{if } q \in A \\ 0, & \textit{if } q \notin A. \end{cases}$$

The function χ_A^S, from the preceding definition, is called the **characteristic function** of A in S.

1.24. **The axiom of choice.**

We imagine that at the beginning of time,
the Grand Oracle has chosen,
from every nonempty set A, an element denoted CH_A.
This is embodied in the **Axiom of Choice**:

AXIOM 1.24.1. *$\forall$nonempty set S,* $\mathrm{CH}_S \in S$.

We also make the convention that $\mathrm{CH}_\varnothing = ☹$.

Alternate notation for CH_S: $\mathrm{CH}(S)$ or $\mathrm{CH}S$.

Then: $\forall$set S, we have: $\mathrm{CH}_S \ {}^*\!\!\in\ S$.

THEOREM 1.24.2. $\mathrm{CH}\{4\} = 4$ *and* $\mathrm{CH}\{\{1,2,3\}\} = \{1,2,3\}$.

The chosen element of any singleton set is its unique element:

THEOREM 1.24.3. $\forall x$, $\mathrm{CH}\{x\} = x = \mathrm{UE}\{x\}$.

For sets with more than one element, we do not know which is chosen, but we do know that one of them is:

THEOREM 1.24.4. $(\mathrm{CH}\{2,3\} = 2) \vee (\mathrm{CH}\{2,3\} = 3)$.

THEOREM 1.24.5.
$$(\mathrm{CH}\{2,3,5\} = 2) \vee (\mathrm{CH}\{2,3,5\} = 3) \vee (\mathrm{CH}\{2,3,5\} = 5).$$

1.25. **The world of sets - part 1.**

THEOREM 1.25.1. *Let A be a set.*
Then $\mathrm{id}^A : A \hookrightarrow\!\!> A$.

THEOREM 1.25.2. *Let A, B be sets, $f : A \hookrightarrow\!\!> B$.*
Then $f^{-1} : B \hookrightarrow\!\!> A$.

By (5) of Theorem 1.22.4, we have:

THEOREM 1.25.3. *Let A, B, C be sets, $f : A \hookrightarrow\!\!> B$, $g : B \hookrightarrow\!\!> C$. Then $g \circ f : A \hookrightarrow\!\!> C$.*

DEFINITION 1.25.4. *Let A, B be sets. Then:*

$\boxed{\exists A \hookrightarrow B}$ *means: $\exists$a function f s.t. $f : A \hookrightarrow B$ and*

$\boxed{\exists A \to\!\!> B}$ *means: $\exists$a function f s.t. $f : A \to\!\!> B$ and*

$\boxed{\exists A \hookrightarrow\!\!> B}$ *means: $\exists$a function f s.t. $f : A \hookrightarrow\!\!> B$.*

THEOREM 1.25.5.

$\forall$*set A,* $\exists A \hookrightarrow\!\!> A$.

$\forall$*sets A, B,* $(\, \exists A \hookrightarrow\!\!> B\,) \Rightarrow (\, \exists B \hookrightarrow\!\!> A\,)$.

$\forall$*sets A, B, C,* $(\,(\, \exists A \hookrightarrow\!\!> B\,) \& (\, \exists B \hookrightarrow\!\!> C\,)\,) \Rightarrow (\, \exists A \hookrightarrow\!\!> C\,)$.

THEOREM 1.25.6. *Let A, B be sets, $g : B \to\!\!> A$. Then $\exists f : A \to B$ s.t. $g \circ f = \mathrm{id}^A$.*

Proof.
Claim: $\forall x \in A$, $g^*\{x\} \neq \varnothing$.
Proof of claim:
Given $x \in A$. **Want:** $g^*\{x\} \neq \varnothing$.
Since $g : B \to A$, we get $\mathbb{I}_g = A$.
Since $x \in A = \mathbb{I}_g$, we get $x \in \mathbb{I}_g$.
Since $x \in \mathbb{I}_g$, choose $y \in \mathbb{D}_g$ s.t. $g_y = x$.
Since $g_y = x$, we get $y \in g^*\{x\}$. Then $g^*\{x\} \neq \varnothing$.
End of proof of claim.

Since $g : B \to A$, we get $\mathbb{D}_g = B$.
Then: $\forall$set S, $g^*(S) \subseteq B$.
So, from the claim and the axiom of choice, we conclude:

$\forall x \in A$, $\mathrm{CH}_{g^*\{x\}} \in g^*\{x\}$.

Then: $\forall x \in A$, $\mathrm{CH}_{g^*\{x\}} \in g^*\{x\} \subseteq B$.
Define $f : A \to B$ by: $\forall x \in A$, $f_x = \mathrm{CH}_{g^*\{x\}}$.
Then $f : A \to B$. **Want:** $g \circ f = \mathrm{id}^A$.
Since $f : A \to B$ and $g : B \to A$, we get $g \circ f : A \to A$.
Then $\mathbb{D}_{g \circ f} = A$. Also, $\mathbb{D}_{\mathrm{id}^A} = A$. **Want:** $\forall x \in A$, $(g \circ f)_x = \mathrm{id}^A_x$.
Given $x \in A$. **Want:** $(g \circ f)_x = \mathrm{id}^A_x$. **Want:** $g_{f_x} = x$.
Let $y := f_x$. **Want:** $g_y = x$.
Since $y = f_x = \mathrm{CH}_{g^*\{x\}} \in g^*\{x\}$, we get $y \in g^*\{x\}$.
Since $y \in g^*\{x\}$, we get $g_y \in \{x\}$. Then $g_{f_x} = x$. □

THEOREM 1.25.7. *Let A, B be sets.* *Assume:* $\exists B \twoheadrightarrow A$.
Then: $\exists A \hookrightarrow B$.

Proof. **Want:**$\exists$function f s.t. $f : A \hookrightarrow B$.
Since $\exists B \twoheadrightarrow A$, choose a function g s.t. $g : B \twoheadrightarrow A$.
By Theorem 1.25.6, choose $f : A \to B$ s.t. $g \circ f = \mathrm{id}^A$.
Then f is a function. **Want:** $f : A \hookrightarrow B$.
By (1) of Theorem 1.23.7, $f : A \hookrightarrow B$. □

THEOREM 1.25.8. *Let A, B be sets, $f : A \hookrightarrow B$.*
Assume $A \neq \varnothing$. Then $\exists g : B \to A$ s.t. $g \circ f = \mathrm{id}^A$.

Proof. Since f is 1-1, we know: $\forall x \in \mathbb{D}_f$, $f^{-1}_{f_x} = x$.
Since $A \neq \varnothing$, choose w s.t. $w \in A$.

Define $g : B \to A$ by: $\forall y \in B$, $g_y = \begin{cases} f^{-1}_y, & \text{if } y \in \mathbb{I}_f \\ w, & \text{if } y \notin \mathbb{I}_f. \end{cases}$

Then $g : B \to A$. **Want:** $g \circ f = \mathrm{id}^A$.
Since $f : A \to B$ and $g : B \to A$, we get $g \circ f : A \to A$.
Then $\mathbb{D}_{g \circ f} = A$. Also, $\mathbb{D}_{\mathrm{id}^A} = A$. **Want:** $\forall x \in A$, $(g \circ f)_x = \mathrm{id}^A_x$.
Given $x \in A$. **Want:** $(g \circ f)_x = \mathrm{id}^A_x$. **Want:** $g_{f_x} = x$.
Let $y := f_x$. **Want:** $g_y = x$.
Since $x \in A = \mathbb{D}_f$, we get $f_x \in \mathbb{I}_f$ and $f^{-1}_{f_x} = x$.
Since $y = f_x \in \mathbb{I}_f$, by definition of g, we get $g_y = f^{-1}_y$.
Then $g_y = f^{-1}_y = f^{-1}_{f_x} = x$. □

THEOREM 1.25.9. *Let A, B be sets.* *Assume:* $\exists A \hookrightarrow B$.
Assume $A \neq \varnothing$. *Then:* $\exists B \twoheadrightarrow A$.

Proof. **Want:**$\exists$function g s.t. $g : B \twoheadrightarrow A$.
Since $\exists A \hookrightarrow B$, choose a function f s.t. $f : A \hookrightarrow B$.
By Theorem 1.25.8, choose $g : B \to A$ s.t. $g \circ f = \mathrm{id}^A$.
Then g is a function. **Want:** $g : B \twoheadrightarrow A$.
By (2) of Theorem 1.23.7, $g : B \twoheadrightarrow A$. □

1.26. **Sequences and zero-sequences.**

DEFINITION 1.26.1. *Let s be an object.*
By s is a **sequence**, *we mean:*
s is a function and $\mathbb{D}_s = \mathbb{N}$.
By s is a **zero-sequence**, *we mean:*
s is a function and $\mathbb{D}_s = \mathbb{N}_0$.

Let s be a sequence. Then s is denoted $\boxed{(s_1, s_2, s_3, \ldots)}$.

To say "Let $s := (1, 1/2, 1/3, 1/4, \ldots)$"
is equivalent to saying "Define $s : \mathbb{N} \to \mathbb{R}$ by: $\forall j \in \mathbb{N}$, $s_j = 1/j$".

Let s be a zero-sequence. Then s is denoted $\boxed{{}_0(s_0, s_1, s_2, s_3, \ldots)}$.

To say "Let $s := {}_0(1, 2, 4, 8, 16, 32, 64, \ldots)$"
is equivalent to saying "Define $s : \mathbb{N}_0 \to \mathbb{R}$ by: $\forall j \in \mathbb{N}_0$, $s_j = 2^j$".

DEFINITION 1.26.2. *Let f be a function, $j \in \mathbb{N}_0$. Then:*

$$\boxed{f_\circ^j} \quad := \quad \begin{cases} \mathrm{id}^{\mathbb{D}_f}, & \text{if } j = 0 \\ f, & \text{if } j = 1 \\ f \circ f, & \text{if } j = 2 \\ f \circ f \circ f, & \text{if } j = 3 \\ f \circ f \circ f \circ f, & \text{if } j = 4 \\ \quad \vdots & \end{cases}.$$

DEFINITION 1.26.3. *Let A be a set, $f : A \to A$, $x \in A$.*
Define $s : \mathbb{N}_0 \to A$ by: $\forall j \in \mathbb{N}_0$, $s_j = f_\circ^j(x)$.
Then s is called the **semi-forward-orbit** *of x under f.*

THEOREM 1.26.4. *Define $f : \mathbb{R} \to \mathbb{R}$ by: $\forall x \in \mathbb{R}$, $f_x = 2x$.*
Let s be the semi-forward-orbit of 1 under f.
Then $s = {}_0(\,1\,,\,2\,,\,4\,,\,8\,,\,16\,,\,32\,,\,\ldots)$.

THEOREM 1.26.5. *Let A be a set, $f : A \to A$, $x \in A$.*
Let s be the semi-forward-orbit of x under f.
Then: $(\ s_0 = x\ )$ & $(\ \forall j \in \mathbb{N}_0,\ s_{j+1} = f_{s_j}\ )$.

DEFINITION 1.26.6. *Let A be a set, $f : A \to A$, $x \in A$.*
Let s be the strict-forward-orbit of $\sqrt{5}$ under f.
Then s is called the **strict-forward-orbit** *of x under f.*

THEOREM 1.26.7. *Define $f : \mathbb{R} \to \mathbb{R}$ by: $\forall x \in \mathbb{R}$, $f_x = x + 1$.*
Define $s \in \mathbb{R}^{\mathbb{N}}$ by: $\forall j \in \mathbb{N}$, $s_j = f_\circ^j(\sqrt{5})$.
Then $s = (\,\sqrt{5}{+}1\,,\,\sqrt{5}{+}2\,,\,\sqrt{5}{+}3\,,\,\sqrt{5}{+}4\,,\,\sqrt{5}{+}5\,,\,\sqrt{5}{+}6\,,\,\ldots)$.

THEOREM 1.26.8. *Let A be a set, $f : A \to A$, $x \in A$.*
Let s be the semi-forward-orbit of x under f.
Then: $(s_0 = x) \,\&\, (\forall j \in \mathbb{N}_0,\ s_{j+1} = f_{s_j})$.

1.27. **Size of a set.**

DEFINITION 1.27.1. *Let A be a set. Then:*

$$\boxed{\#A} \quad := \quad \sup\{k \in \mathbb{N}_0 \,|\, \exists [1..k] \hookrightarrow A\}.$$

We have:
$[1..0] = \varnothing$. $\quad \forall k \in \mathbb{N}$, $[1..k] = \{1, 2, 3, \ldots, k\}$.
$\exists [1..0] \hookrightarrow \varnothing$.
$\forall k \in \mathbb{N}$, $\nexists [1..k] \hookrightarrow \varnothing$.

THEOREM 1.27.2. $\#\varnothing = 0$.

We have: $\forall k \in \mathbb{N}$, $[1..k] = \{1, 2, 3, \ldots, k\}$.
$2 \cdot \mathbb{N} = \{2, 4, 6, 8, \ldots\}$.
$\forall k \in [0..4], \exists [1..k] \hookrightarrow \{1, 5, 8, 9\}$.
$\forall k \in [5..\infty)$, $\nexists [1..k] \hookrightarrow \{1, 5, 8, 9\}$.
$\forall k \in [0..50]$, $\exists [1..k] \hookrightarrow \{2, 4, 6, 8, \ldots, 100\}$.
$\forall k \in [51..\infty)$, $\nexists [1..k] \hookrightarrow \{2, 4, 6, 8, \ldots, 100\}$.
$\forall k \in \mathbb{N}_0$, $\exists [1..k] \hookrightarrow 2 \cdot \mathbb{N}$.

THEOREM 1.27.3.

$\#\{1, 5, 8, 9\} = 4$ *and*
$\#\{2, 4, 6, 8, \ldots, 100\} = 50$ *and*
$\#2 \cdot \mathbb{N} = \infty$.

THEOREM 1.27.4. $\#\mathbb{N} = \#\mathbb{N}_0 = \#\mathbb{Z} = \#\mathbb{Q} = \#\mathbb{R} = \infty$.

DEFINITION 1.27.5. *Let A be a set.*
By A is **finite***, we mean: $\#A < \infty$.*
By A is **infinite***, we mean: $\#A = \infty$.*

1.28. **The world of sets - part 2.**

THEOREM 1.28.1. *Let S and T be sets.*
Then: $\exists S \hookrightarrow T$ *or* $\exists T \hookrightarrow S$.

We omit the proof of the preceding theorem.

We organize "The World of Sets" in such a way that
for any two sets S and T,

$$(S \text{ appears above } T) \text{ iff } (\exists S \hookrightarrow T \text{ but not } \exists T \hookrightarrow S)$$

and

$$(T \text{ appears above } S) \text{ iff } (\exists T \hookrightarrow S \text{ but not } \exists S \hookrightarrow T)$$

and

$$(S \text{ appears side-by-side with } T) \text{ iff } (\exists S \hookrightarrow T \text{ and } \exists T \hookrightarrow S).$$

According to our next result, that last condition

$$\exists S \hookrightarrow T \quad \text{and} \quad \exists T \hookrightarrow S$$

is equivalent to

$$\exists S \hookrightarrow> T.$$

So two sets appear side-by-side iff they are bijective.

The following is called the **Schroeder-Bernstein Theorem**,

THEOREM 1.28.2. *Let S and T be sets.*
Assume: $(\exists S \hookrightarrow T) \;\&\; (\exists T \hookrightarrow S)$.
Then: $\exists S \hookrightarrow> T$.

We omit the proof of the preceding theorem.

We picture The World of Sets, starting with $\varnothing$ at the bottom.
By itself, it occupies the lowest level in The World of Sets.
Then the singleton sets are all side-by-side, just above the lowest level.
On the next level up are all sets with two elements.
On the next level up are all sets with three elements.
Next is an ellipsis, $\vdots$ indicating all the levels of finite sets.
Next is a horizontal line dividing finite sets from infinite sets.
Somewhere above that line appears $\mathbb{N}$.
First question: Are there any infinite sets that are strictly below $\mathbb{N}$.
The next theorem answers that in the negative:

THEOREM 1.28.3. $\forall$*set* S, $\quad$ (S *is infinite*) $\Leftrightarrow (\exists \mathbb{N} \hookrightarrow S)$.

We omit the proof of the preceding theorem.

So, in The World of Sets, the level with $\mathbb{N}$ is the lowest level that is above the dividing line between finite and infinite sets. We draw a line just above that level. Any set below that line is referred to as a countable set, and any set above that line is said to be uncountable. Any set on the same level with $\mathbb{N}$ is called countably infinite. Formally:

DEFINITION 1.28.4. *Let S be a set. Then:*

S *is* **countable** *means:* $\exists S \hookrightarrow \mathbb{N}$ *and*
S *is* **uncountable** *means:* $\nexists S \hookrightarrow \mathbb{N}$ *and*
S *is* **countably infinite** *means:* $\exists S \hookrightarrow\!> \mathbb{N}$.

Note that, by Schroeder-Bernstein, a set is countably infinite iff it is both countable and infinite.

Next question: Where do we place $\mathbb{Q}$?

We first look at the set $\mathbb{Q} \cap (0; \infty)$ of positive rational numbers:

THEOREM 1.28.5. $\exists \mathbb{N} \to\!> \mathbb{Q} \cap (0; \infty)$.

Proof. The sequence
$(\; 1/1 \; , \; 2/1, 1/2 \; , \; 3/1, 2/2, 1/3 \; , \; 4/1, 3/2, 2/3, 1/4, \; , \; \ldots \;)$
is a surjection $\mathbb{N} \to\!> S$. Then $\exists \mathbb{N} \to\!> \mathbb{Q} \cap (0; \infty)$. □

By Theorem 1.25.7, in The World of Sets,
there are no surjections from a set to a set on a higher level.
Then Theorem 1.28.5 says that
$\mathbb{Q} \cap (0; \infty)$ is either at the countable level with $\mathbb{N}$, or else below.
Unassigned HW: Show that $\mathbb{Q} \cap (0; \infty)$ is countably infinite.

We next show that we place $\mathbb{Q}$ on the same level with $\mathbb{N}$,
that is, $\mathbb{Q}$ belongs on the level of countably infinite sets.

THEOREM 1.28.6. *The set $\mathbb{Q}$ is countable.*

Proof. By Theorem 1.28.5, choose a function s s.t. $s : \mathbb{N} \to\!> \mathbb{Q} \cap (0; \infty)$.
Then the sequence
$(\; 0 \; , \; s_1, -s_1 \; , \; s_2, -s_2 \; , \; s_3, -s_3 \; , \; \ldots \;)$
is a surjection $\mathbb{N} \to\!> \mathbb{Q}$. Then, by Theorem 1.25.7, $\exists \mathbb{Q} \hookrightarrow \mathbb{N}$.
Since $\mathrm{id}^{\mathbb{N}} : \mathbb{N} \hookrightarrow \mathbb{Q}$, we get $\exists \mathbb{N} \hookrightarrow \mathbb{Q}$.
Then, by Schroeder-Bernstein, $\exists \mathbb{N} \hookrightarrow\!> \mathbb{Q}$. □

Next question: Where do we place $\mathbb{N}_0$ and $\mathbb{Z}$?

The next theorem says that if three sets admit a cycle of injections, then they admit a cycle of bijections:

THEOREM 1.28.7. *Let A, B, C be sets.*
Assume ($\exists A \hookrightarrow B$) & ($\exists B \hookrightarrow C$) & ($\exists C \hookrightarrow A$).
Then ($\exists A \hookrightarrow> B$) & ($\exists B \hookrightarrow> C$) & ($\exists C \hookrightarrow> A$).

Proof. By composition, since ($\exists B \hookrightarrow C$) & ($\exists C \hookrightarrow A$),
we see that $\exists B \hookrightarrow A$.
So, since $\exists A \hookrightarrow B$, by Schroeder-Bernstein, we get: $\exists A \hookrightarrow> B$.
By composition, since ($\exists C \hookrightarrow A$) & ($\exists A \hookrightarrow B$),
we see that $\exists C \hookrightarrow B$.
So, since $\exists B \hookrightarrow C$, by Schroeder-Bernstein, we get: $\exists B \hookrightarrow> C$.
It remains to show: $\exists C \hookrightarrow> A$.
By composition, since ($\exists A \hookrightarrow B$) & ($\exists B \hookrightarrow C$),
we see that $\exists A \hookrightarrow C$.
So, since $\exists C \hookrightarrow A$, by Schroeder-Bernstein, we get: $\exists C \hookrightarrow> A$. □

THEOREM 1.28.8. *Let A, B, C be sets.*
Assume ($A \subseteq B \subseteq C$) & ($\exists A \hookrightarrow> C$).
Then ($\exists A \hookrightarrow> B$) & ($\exists B \hookrightarrow> C$).

Proof. Since $A \subseteq B \subseteq C$, we see that

$$\mathrm{id}^A : A \hookrightarrow B \quad \text{and} \quad \mathrm{id}^B : B \hookrightarrow C.$$

Then $\exists A \hookrightarrow B$ and $\exists B \hookrightarrow C$.
Since $\exists A \hookrightarrow> C$, by inversion, we get $\exists C \hookrightarrow> A$. Then $\exists C \hookrightarrow A$.
Then, by Theorem 1.28.7, ($\exists A \hookrightarrow> B$) & ($\exists B \hookrightarrow> C$). □

THEOREM 1.28.9. *The sets $\mathbb{N}_0$ and $\mathbb{Z}$ are both countably infinite.*

Proof. By Theorem 1.28.6, $\exists \mathbb{N} \hookrightarrow> \mathbb{Q}$.
Since $\mathbb{N} \subseteq \mathbb{N}_0 \subseteq \mathbb{Q}$ and $\exists \mathbb{N} \hookrightarrow> \mathbb{Q}$,
we conclude, from Theorem 1.28.8 that $\exists \mathbb{N} \hookrightarrow> \mathbb{N}_0$.
Then $\mathbb{N}_0$ is countably ininite. **Want:** $\mathbb{Z}$ is countably infinite.
Since $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$ and $\exists \mathbb{N} \hookrightarrow> \mathbb{Q}$,
we conclude, from Theorem 1.28.8 that $\exists \mathbb{N} \hookrightarrow> \mathbb{Z}$.
Then $\mathbb{Z}$ is countably infinite. □

In The World of Sets, we now see that
$\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{Q}$ are all on the countably infinite level.
Next question: Where do we place $\mathbb{R}$?

DEFINITION 1.28.10. *Let A, B be sets. Then:*

$$\boxed{B^A} \quad := \quad \{\textit{functions } f \mid f : A \to B\}.$$

DEFINITION 1.28.11. *Let T be a set. Then:*

$$\boxed{2^T} \quad := \quad \{sets\ S \,|\, S \subseteq T\}.$$

THEOREM 1.28.12. $\{3,4,5\}^{\{1,2\}} \qquad =$

$$\left\{ \begin{array}{cccccc} \begin{pmatrix} 1 \mapsto 3 \\ 2 \mapsto 3 \end{pmatrix} & , & \begin{pmatrix} 1 \mapsto 3 \\ 2 \mapsto 4 \end{pmatrix} & , & \begin{pmatrix} 1 \mapsto 3 \\ 2 \mapsto 5 \end{pmatrix} & , \\ \begin{pmatrix} 1 \mapsto 4 \\ 2 \mapsto 3 \end{pmatrix} & , & \begin{pmatrix} 1 \mapsto 4 \\ 2 \mapsto 4 \end{pmatrix} & , & \begin{pmatrix} 1 \mapsto 4 \\ 2 \mapsto 5 \end{pmatrix} & , \\ \begin{pmatrix} 1 \mapsto 5 \\ 2 \mapsto 3 \end{pmatrix} & , & \begin{pmatrix} 1 \mapsto 5 \\ 2 \mapsto 4 \end{pmatrix} & , & \begin{pmatrix} 1 \mapsto 5 \\ 2 \mapsto 5 \end{pmatrix} & \end{array} \right\}.$$

THEOREM 1.28.13. *Let A, B be finite sets. Then $\#(B^A) = (\#B)^{\#A}$.*

THEOREM 1.28.14. $\{0,1\}^{\{7,8,9\}} \qquad =$

$$\left\{ \begin{array}{c} \begin{pmatrix} 7 \mapsto 0 \\ 8 \mapsto 0 \\ 9 \mapsto 0 \end{pmatrix}, \begin{pmatrix} 7 \mapsto 0 \\ 8 \mapsto 0 \\ 9 \mapsto 1 \end{pmatrix}, \begin{pmatrix} 7 \mapsto 0 \\ 8 \mapsto 1 \\ 9 \mapsto 0 \end{pmatrix}, \begin{pmatrix} 7 \mapsto 0 \\ 8 \mapsto 1 \\ 9 \mapsto 1 \end{pmatrix}, \\ \begin{pmatrix} 7 \mapsto 1 \\ 8 \mapsto 0 \\ 9 \mapsto 0 \end{pmatrix}, \begin{pmatrix} 7 \mapsto 1 \\ 8 \mapsto 0 \\ 9 \mapsto 1 \end{pmatrix}, \begin{pmatrix} 7 \mapsto 1 \\ 8 \mapsto 1 \\ 9 \mapsto 0 \end{pmatrix}, \begin{pmatrix} 7 \mapsto 1 \\ 8 \mapsto 1 \\ 9 \mapsto 1 \end{pmatrix} \end{array} \right\}.$$

THEOREM 1.28.15. *Let S be a set.*
Then $f \mapsto f^*\{1\} \,:\, \{0,1\}^S \hookrightarrow\!> 2^S$.

We omit the proof.
Applying Theorem 1.28.15 to the case $S = \{7,8,9\}$, we see:
$f \mapsto f^*\{1\} \,:\, \{0,1\}^{\{7,8,9\}} \hookrightarrow\!> 2^{\{7,8,9\}}$.
Since, in Theorem 1.28.14, we calculated $\{0,1\}^{\{7,8,9\}}$,
in order to calculate $2^{\{7,8,9\}}$, we can simply, for each $f \in \{0,1\}^{\{7,8,9\}}$,
calculate $f^*\{1\}$, and assemble the resulting sets into a set of sets:

THEOREM 1.28.16. $2^{\{7,8,9\}} \qquad =$

$$\left\{ \begin{array}{cccccccc} \emptyset & , & \{9\} & , & \{8\} & , & \{8,9\} & , \\ \{7\} & , & \{7,9\} & , & \{7,8\} & , & \{7,8,9\} & \end{array} \right\}.$$

The following is a consequence of Theorem 1.28.15:

THEOREM 1.28.17. *Let S be a set.*
Then $\exists\ \{0,1\}^S \hookrightarrow\!> 2^S$.

THEOREM 1.28.18. *Let S be a set.*
Then $x \mapsto \{x\} : S \hookrightarrow 2^S$.

The preceding theorem is left as Unassigned HW.
The following is a consequence of the preceding theorem.

THEOREM 1.28.19. *Let S be a set.*
Then $\exists S \hookrightarrow 2^S$.

THEOREM 1.28.20. *Let S be a set.*
Then $\nexists 2^S \hookrightarrow S$.

The preceding theorem is proved by "Cantor diagonalization".
That proof is omitted.
The preceding two theorems tell us that, in The World of Sets,
for any set S, 2^S must be placed strictly higher than S.
As a consequence, while $\varnothing$ is at the bottom of the World of Sets,
there is no top to The World of Sets; that is,
for any set S, the set 2^S is higher; moreover,
by Theorem 1.28.17, 2^S is side-by-side with $\{0,1\}^S$.

For any two bijective sets A and B,
2^A and 2^B are bijective as well:

THEOREM 1.28.21. *Let A, B be sets.*
Assume $\exists A \hookrightarrow\!\!> B$.
Then $\exists 2^A \hookrightarrow\!\!> 2^B$.

The proof is an Unassigned HW.

In The World of Sets, we create a level
called the "continuum cardinality" level,
into which we place the sets $2^{\mathbb{N}}$, $2^{\mathbb{N}_0}$, $2^{\mathbb{Z}}$, $2^{\mathbb{Q}}$.
This level also has the sets $\{0,1\}^{\mathbb{N}}$, $\{0,1\}^{\mathbb{N}_0}$, $\{0,1\}^{\mathbb{Z}}$, $\{0,1\}^{\mathbb{Q}}$.

NOTE: There is an Axiom of Set Theorem called the "Continuum Hypothesis", which states that there are no sets strictly between countably infinite and continuum cardinality. Some set-theorists may adopt this axiom, while others adopt its negation as an axiom. In this course, we are agnostic about this question.

We now turn to proving that $\mathbb{R}$ has continuum cardinality.

THEOREM 1.28.22. $\exists \{0,1\}^{\mathbb{N}} \hookrightarrow \mathbb{R}$.

Idea of proof: The mapping which sends
$f \in \{0,1\}^{\mathbb{N}}$ to the base ten number $0.f_1 f_2 f_3 \cdots \in \mathbb{R}$
is an injection.

NOTE: It is not surjective because 0.2 is not in the image. The only digits allowed are 0 and 1.

NOTE: Because we do not allow the digit 9, the map is 1-1.

THEOREM 1.28.23. $\exists \{0,1\}^{\mathbb{N}} \twoheadrightarrow [0;1]$.

Idea of proof: The mapping which sends
$f \in \{0,1\}^{\mathbb{N}}$ to the base two number $0.f_1 f_2 f_3 \cdots \in [0;1]$
is a surjection.

NOTE: It is not injective because, in base two, $0.01111\cdots = 0.10000\cdots$.

NOTE: The number 1 is in the image because $1 = 0.1111\cdots$.

THEOREM 1.28.24. $\exists [-1;1] \hookrightarrow \{0,1\}^{\mathbb{N}}$.

Proof. By Theorem 1.28.23, $\exists \{0,1\}^{\mathbb{N}} \twoheadrightarrow [0;1]$.
So, since $x \mapsto 2x-1 : [0;1] \twoheadrightarrow [-1;1]$,
by composing, we get $\exists \{0,1\}^{\mathbb{N}} \twoheadrightarrow [-1;1]$.
Then, by Theorem 1.25.7, we get $\exists [-1;1] \hookrightarrow \{0,1\}^{\mathbb{N}}$. □

THEOREM 1.28.25. $\exists \mathbb{R} \hookrightarrow [-1;1]$.

Proof. Define $f : \mathbb{R} \to (-1;1)$ by: $\forall x \in \mathbb{R}$, $f_x = x/\sqrt{1+x^2}$.
Define $g : (-1;1) \to \mathbb{R}$ by: $\forall y \in \mathbb{R}$, $g_y = y/\sqrt{1-y^2}$.
Then $g \circ f = \mathrm{id}^{\mathbb{R}}$ and $f \circ g = \mathrm{id}^{(-1;1)}$, so, by Theorem 1.23.8,
we see that $f : \mathbb{R} \hookrightarrow\!\!\!\!\twoheadrightarrow (-1;1)$.
Then $f : \mathbb{R} \hookrightarrow (-1;1)$. Then $\exists \mathbb{R} \hookrightarrow (-1;1)$. □

The next theorem asserts that, in The World of Sets,
$\mathbb{R}$ belongs on the continuum cardinality level, with $2^{\mathbb{N}}$.

THEOREM 1.28.26. $\exists 2^{\mathbb{N}} \hookrightarrow\!\!\!\!\twoheadrightarrow \mathbb{R}$.

Proof. By Theorem 1.28.17, **want:** $\exists \{0,1\}^{\mathbb{N}} \hookrightarrow\!\!\!\!\twoheadrightarrow \mathbb{R}$.
By Theorem 1.28.22, $\exists \{0,1\}^{\mathbb{N}} \hookrightarrow \mathbb{R}$.

By Theorem 1.28.25, $\exists \mathbb{R} \hookrightarrow [-1;1]$.
By Theorem 1.28.24, $\exists [-1;1] \hookrightarrow \{0,1\}^{\mathbb{N}}$.
Then, by Theorem 1.28.7, $\exists \{0,1\}^{\mathbb{N}} \hookrightarrow\!\!> \mathbb{R}$. $\square$

1.29. **Partitions.**

DEFINITION 1.29.1. *Let $\mathcal{S}$ be a set of sets.*
By $\mathcal{S}$ is **pairwise-disjoint***, we mean:*
$$\forall A, B \in \mathcal{S}, \quad (\, A \neq B \,) \Rightarrow (\, A \cap B = \varnothing \,).$$

We have $\{\ [1;3]\ ,\ [3;5]\ \}$ is NOT pairwise-disjoint,
because $[1;3] \cap [3;5] = \{3\} \neq \varnothing$.
By contrast, $\{\ [1;3)\ ,\ [3;5)\ \}$ IS pairwise-disjoint.
Note also that we may put in the empty set:
$\{\ [1;3)\ ,\ [3;5)\ ,\ \varnothing\ \}$ IS pairwise-disjoint.

DEFINITION 1.29.2. *Let X be a set and let $\mathcal{S}$ be a set of sets.*
By $\mathcal{S}$ is a **partition** *of X, we mean:*
$\mathcal{S}$ is pairwise-disjoint and $\bigcup \mathcal{S} = X$.

Let $X := [1;5)$, $\mathcal{S} := \{\, [1;3)\, ,\, [3;5)\, ,\, \varnothing\, \}$.
Then $\mathcal{S}$ is a partition of X.
However, the empty set really plays very little role here,
so we can remove it, as follows:
We have $\mathcal{S}^{\times}_{\varnothing} = \mathcal{S}\backslash\{\varnothing\} = \{\, [1;3)\, ,\, [3;5)\, \}$,
and $\mathcal{S}^{\times}_{\varnothing}$ is also a partition of X.
More generally, we have:

THEOREM 1.29.3. *Let X be a set and let $\mathcal{S}$ be a partition of S.*
Then $\mathcal{S}^{\times}_{\varnothing}$ is a partition of X.

DEFINITION 1.29.4. *Let X be a set.*
Let $\mathcal{P}$ and $\mathcal{Q}$ be two partitions of X.
By $\mathcal{P}$ is a **refinement** *of $\mathcal{Q}$, we mean:*
$\forall P \in \mathcal{P}$, $\exists Q \in \mathcal{Q}$ s.t. $P \subseteq Q$.

Let $X := [1;5)$, $\mathcal{Q} := \{\, [1;3)\, ,\, [3;5)\, \}$, $\mathcal{P} := \{\, [1;2)\, ,\, [2;3)\, ,\, [3;4)\, ,\, [4;5)\, \}$.
Then $\mathcal{P}$ is a refinement of $\mathcal{Q}$.
Note that $\{\, [1;2)\, ,\, [2;3)\, \}$ is a partition of $[1;3)$
and that $\{\, [3;4)\, ,\, [4;5)\, \}$ is a partition of $[3;5)$,
so each element of $\mathcal{Q}$ is partitioned by a subset of $\mathcal{P}$.
More generally, we have:

THEOREM 1.29.5. *Let X be a set.*
Let $\mathcal{P}$ and $\mathcal{Q}$ be two partitions of X.
Assume that $\mathcal{P}$ is a refinement of $\mathcal{Q}$.
Let $Q \in \mathcal{Q}$. Let $\mathcal{S} := \{P \in \mathcal{P} \,|\, P \subseteq Q\}$.
Then $\mathcal{S}$ is a partition of Q.

DEFINITION 1.29.6. *Let X be a set.*
Let $\mathcal{P}$ and $\mathcal{Q}$ be two partitions of X.
By $\mathcal{P}$ and $\mathcal{Q}$ are **comparable***, we mean:*
$\mathcal{P}$ is a refinement of $\mathcal{Q}$ or $\mathcal{Q}$ is a refinement of $\mathcal{P}$.
By $\mathcal{P}$ and $\mathcal{Q}$ are **incomparable***, we mean:*
$\mathcal{P}$ and $\mathcal{Q}$ are not comparable.

Let $X := [1;7)$, $\mathcal{P} := \{\,[1;4)\,,\,[4;7)\,\}$, $\mathcal{Q} := \{\,[1;3)\,,\,[3;5)\,,\,[5,7)\,\}$.
Then $\mathcal{P}$ and $\mathcal{Q}$ are incomparable.
However, by intersecting each element of $\mathcal{P}$ with each element of $\mathcal{Q}$, we can find a partition of X that is
simultaneously a refinement of $\mathcal{P}$ and a refinement of $\mathcal{Q}$,
as follows. We compute:
$$[1;4) \cap [1;3) = [1;3), \quad [1;4] \cap [3;5) = [3;4), \quad [1;4) \cap [5;7) = \varnothing,$$
$$[4;7) \cap [1;3) = \varnothing, \quad [4;7] \cap [3;5) = [4;5), \quad [4;7) \cap [5;7) = [5;7).$$
Let $\mathcal{S} := \{\,[1;3)\,,\,[3;4)\,,\,\varnothing\,,\,\varnothing\,,\,[4;5)\,,\,[5;7)\,\}$.
Then $\mathcal{S}$ is a partition of X and, also,
$\mathcal{S}$ is a common refinement of $\mathcal{P}$ and $\mathcal{Q}$.
More generally, we have:

THEOREM 1.29.7. *Let X be a set.*
Let $\mathcal{P}$ and $\mathcal{Q}$ be two partitions of X.
Let $\mathcal{S} := \{P \cap Q \,|\, P \in \mathcal{P}\,,\, Q \in \mathcal{Q}\}$.
Then: $\mathcal{S}$ is a partition of X and
$\mathcal{S}$ is a refinement of $\mathcal{P}$ and $\mathcal{S}$ is a refinement of $\mathcal{Q}$.

1.30. **Algebra of functionals.**

DEFINITION 1.30.1. *Let f be an object.*
By f is a **functional***, we mean: $(\,f$ is a function$)\,\&\,(\,\mathbb{I}_f \subseteq \mathbb{R}\,)$.*

That is, a functional is a real-valued function.

DEFINITION 1.30.2. *Let f be a functional.*
Then $\boxed{-f}$ is the functional defined by:
$$\forall x, \quad (-f)_x \;=\; -f_x.$$

THEOREM 1.30.3. *$\forall$functional f, $\mathbb{D}_{-f} = \mathbb{D}_f$.*

THEOREM 1.30.4. $-(1, 2, 3, \ldots) = (-1, -2, -3, \ldots)$.

DEFINITION 1.30.5. *Let f be afunctional, $c \in \mathbb{R}$.*
Then $\boxed{c \cdot f}$ and $f \cdot c$ are the functionals defined by:

$$\forall x, \quad (c \cdot f)_x = c \cdot f_x \qquad \textit{and}$$
$$\forall x, \quad (f \cdot c)_x = f_x \cdot c.$$

The "·" is often omitted.

THEOREM 1.30.6. $2 \cdot (1, 2, 3, \ldots) = (2, 4, 6, \ldots) = (1, 2, 3, \ldots) \cdot 2$.

THEOREM 1.30.7. *$\forall$ functional f, $\forall c \in \mathbb{R}$, $c \cdot f = f \cdot c$ and $\mathbb{D}_{c \cdot f} = \mathbb{D}_f = \mathbb{D}_{f \cdot c}$.*

THEOREM 1.30.8. *$\forall$ functional f, $1 \cdot f = f$ and $(-1) \cdot f = -f$.*

DEFINITION 1.30.9. *Let f and g be functionals.*
Then $\boxed{f + g}$ and $\boxed{f - g}$ and $\boxed{f \cdot g}$ and $\boxed{f/g}$ are the functionals defined by:

$$\forall x, \quad (f + g)_x = f_x + g_x \qquad \textit{and}$$
$$\forall x, \quad (f - g)_x = f_x - g_x \qquad \textit{and}$$
$$\forall x, \quad (f \cdot g)_x = f_x \cdot g_x \qquad \textit{and}$$
$$\forall x, \quad (f/g)_x = f_x/g_x.$$

The "·" is often omitted. We sometimes write $\dfrac{f}{g}$ instead of f/g.

THEOREM 1.30.10. *Let f and g be functionals. Then:*

$$f + g = g + x \quad \textit{and} \quad f - g = -(g - f) \quad \textit{and} \quad f \cdot g = g \cdot f \quad \textit{and}$$
$$\mathbb{D}_{f+g} = \mathbb{D}_{f-g} = \mathbb{D}_{f+g} = \mathbb{D}_f \cap \mathbb{D}_g \quad \textit{and} \quad \mathbb{D}_{f/g} = \mathbb{D}_f \cap [g^*(\mathbb{R}_0^*)].$$

DEFINITION 1.30.11. *Let f be a functional, $c \in \mathbb{R}$. Then:*
$\boxed{c/f}$ and $\boxed{f/c}$ are the functionals defined by:

$$\forall x, \ (c/f)_x = c/f_x \qquad \textit{and}$$
$$\forall x, \ (f/c)_x = f_x/c.$$

We sometimes write $\dfrac{c}{f}$ instead of c/f and $\dfrac{f}{c}$ instead of f/c.

THEOREM 1.30.12. *Let $s \in \mathbb{R}^{\mathbb{N}}$, $t \in \mathbb{R}^{\mathbb{N}}$. Then $s \cdot (1/t) = s/t$.*

Proof. We have $\mathbb{D}_{s \cdot (1/t)} \subseteq \mathbb{N}$ and $\mathbb{D}_{s/t} \subseteq \mathbb{N}$.
Want: $\forall j \in \mathbb{N}, \quad (s \cdot (1/t))_j = (s/t)_j$.

Given $j \in \mathbb{N}$. **Want:** $(s \cdot (1/t))_j = (s/t)_j$.
We have $(s \cdot (1/t))_j = s_j \cdot (1/t)_j = s_j \cdot (1/t_j)$
$\stackrel{*}{=} s_j/t_j = (s/t)_j$. □

1.31. **Balls in $\mathbb{R}$.**

In the next definition, $B(a, \varepsilon)$ is called
the **open ball** about a of radius ε.
We are sometimes sloppy and forget to say "open".
By default, in this course, a "ball" is an open ball.

DEFINITION 1.31.1. *Let $a \in \mathbb{R}$, $\varepsilon \in \mathbb{R}$.*
Then $\boxed{B(a, \varepsilon)} := \{x \in \mathbb{R} \text{ s.t. } |x - a| < \varepsilon\}$.

THEOREM 1.31.2. *Let $a \in \mathbb{R}$, $\varepsilon > 0$.*
Then $B(a, \varepsilon) = (a - \varepsilon; a + \varepsilon)$.

THEOREM 1.31.3. $B(0, 1/6) = (-1/6; 1/6)$.

DEFINITION 1.31.4.
$\forall a \in \mathbb{R}$, $\boxed{\mathcal{B}(a)} := \{B(a, r) \,|\, r > 0\}$.
$\mathcal{B}_{\mathbb{R}} := \{B(a, r) \,|\, a \in \mathbb{R}, r > 0\}$.

The next theorem is the **Subset Recentering Theorem**:

THEOREM 1.31.5. *Let $C \in \mathcal{B}_{\mathbb{R}}$, $x \in C$.*
Then $\exists B \in \mathcal{B}(x)$ s.t. $B \subseteq C$.

Proof. Choose $a \in \mathbb{R}$, $r > 0$ s.t. $C = B(a, r)$.
Since $x \in C = B(a, r)$, we get $|x - a| < r$.
Let $\varepsilon := r - |x - a|$. Then $\varepsilon > 0$. Let $B := B(x, \varepsilon)$.
Then $B \in \mathcal{B}(x)$. **Want:** $B \subseteq C$.
Want: $\forall z \in B$, $z \in C$.
Given $z \in B$. **Want:** $z \in C$.
Since $z \in B = B(x, \varepsilon)$, we get $|z - x| < \varepsilon$.
Then $|z - x| < \varepsilon = r - |a - x|$, so $|z - x| + |x - a| < r$.
Then $|z - a| \leqslant |z - x| + |x - a| < r$.
Then $|z - a| < r$, so $z \in B(a, r)$.
Then $z \in B(a, r) = C$. □

THEOREM 1.31.6. *Let $b \in \mathbb{R}$, $a < b$.*
Let $q \in (a; b)$. Then: $\exists \varepsilon > 0$ s.t. $B(q, \varepsilon) \subseteq (a; b)$.

Proof. Let $c := (a+b)/2$, $r := (b-a)/2$.
Then $B(c,r) = (c-r;c+r) = (a;b)$.
Let $C := B(c,r)$. Then $C = (a;b)$ and $C \in \mathcal{B}_{\mathbb{R}}$.
Since $q \in (a;b) = C$, by Theorem 1.31.5,
choose $B \in \mathcal{B}(q)$ s.t. $B \subseteq C$.
Since $B \in \mathcal{B}(q)$, choose $\varepsilon > 0$ s.t. $B = B(q,\varepsilon)$.
Then $\varepsilon > 0$. **Want:** $B(q,\varepsilon) \subseteq (a;b)$.
We have: $B(q,\varepsilon) = B \subseteq C = (a;b)$. □

The next theorem is the **Superset Recentering Theorem**:

THEOREM 1.31.7. *Let $B \in \mathcal{B}_{\mathbb{R}}$, $a \in \mathbb{R}$.*
Then $\exists C \in \mathcal{B}(a)$ s.t. $B \subseteq C$.

Proof. Choose $\alpha \in \mathbb{R}$, $\rho > 0$ s.t. $B = B(\alpha,\rho)$.
Let $s := |\alpha - a|$. Let $C := B(a, \rho + s)$.
Then $C \in \mathcal{B}(a)$. **Want:** $B \subseteq C$.
Want: $\forall x \in B$, $x \in C$.
Given $x \in B$. **Want:** $x \in C$.
Since $x \in B = B(\alpha,\rho)$, we get $|x-\alpha| < \rho$.
We have $|x-a| \leqslant |x-\alpha| + |\alpha - a| < \rho + s$, so $|x-a| < \rho + s$.
Then $x \in B(a,\rho+s) = C$. □

1.32. Bounded sets in $\mathbb{R}$.

DEFINITION 1.32.1. *Let $S \subseteq \mathbb{R}$.*
By S is **bounded**, *we mean: $\exists B \in \mathcal{B}_{\mathbb{R}}$ s.t. $S \subseteq B$.*

DEFINITION 1.32.2. *Let $S \subseteq \mathbb{R}$.*
By S is **unbounded**, *we mean: S is not bounded.*

THEOREM 1.32.3.
$[1;2]$ *is bounded.*
$[1;\infty)$ *is unbounded.*
$\{1\,,\,1/2\,,\,1/3\,,\,\ldots\}$ *is bounded.*
$\{2\,,\,4\,,\,6\,,\,8\,,\,\ldots\}$ *is unbounded.*

DEFINITION 1.32.4. *Let $X \subseteq \mathbb{R}$.*
By X is **bounded above**, *we mean: $\exists z \in \mathbb{R}$ s.t. $X \leqslant z$.*
By X is **bounded below**, *we mean: $\exists z \in \mathbb{R}$ s.t. $z \leqslant X$.*

THEOREM 1.32.5. *Let* $t \in \mathbb{R}^{\mathbb{N}}$.

Assume t *is convergent.* *Then* $\mathbb{I}_t$ *is bounded above.*

Proof. **Want:** $\exists z \in \mathbb{R}$ s.t. $\mathbb{I}_t \leqslant z$.
Since t is convergent, choose $a \in \mathbb{R}$ s.t. $t \to a$.
Since $t \to a$, choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\, j \geqslant K\,) \Rightarrow (\, |t_j - a| < 1\,).$$
Then: $\forall j \in [K..\infty)$, $a - 1 < t_j < a + 1$.
Let $b := \max\{t_1, \ldots, t_K\}$.
Then: $\forall j \in [1..K]$, $t_j \leqslant b$ and $\forall j \in [K..\infty)$, $t_j < a + 1$.
Let $z := \max\{a + 1, b\}$. Then $z \in \mathbb{R}$. **Want:** $\mathbb{I}_t \leqslant z$.
Want: $\forall y \in \mathbb{I}_t$, $y \leqslant z$.
Given $y \in \mathbb{I}_t$. **Want:** $y \leqslant z$.
Since $y \in \mathbb{I}_t$, choose $j \in \mathbb{D}_t$ s.t. $y = t_j$. **Want:** $t_j \leqslant z$.
At least one of the following is true:

(α) $j \in [1..K]$ or (β) $j \in [K..\infty)$.

Case (α): Since $j \in [1..K]$, we get $t_j \leqslant b$.
Then $t_j \leqslant b \leqslant \max\{a + 1, b\} = z$, so $t_j \leqslant z$.
End of Case (α).

Case (β): Since $j \in [K..\infty)$, we get $t_j < a + 1$.
Then $t_j < a + 1 \leqslant \max\{a + 1, b\} = z$, so $t_j \leqslant z$.
End of Case (β). □

THEOREM 1.32.6. *Let* $S \subseteq \mathbb{R}$. *Then:*
$[\, S$ *is bounded* $] \Leftrightarrow [\, (\, S$ *is bounded above*$\,)\,\&\,(\, S$ *is bounded below*$\,)\,]$.

THEOREM 1.32.7. *Let* $S, T \subseteq \mathbb{R}$.
Assume T *is bounded* *and* $S \subseteq T$.

Then S *is bounded.*

THEOREM 1.32.8. *Let* s *and* t *be sequences.*
Assume t *is a subsequence of* s *and* $\mathbb{I}_s$ *is bounded.*

Then $\mathbb{I}_t$ *is bounded.*

Proof. Since $\mathbb{I}_s$ is bounded and $\mathbb{I}_t \subseteq \mathbb{I}_s$, it follow that $\mathbb{I}_t$ is bounded. □

THEOREM 1.32.9. $\forall$*finite* $F \subseteq \mathbb{R}$, F *is bounded.*

Proof. Since $\min F \leqslant F \leqslant \max F$,
we see that F is bounded below and above.
Then f is bounded. □

DEFINITION 1.32.10. *Let* $S \subseteq \mathbb{R}$, $t \in \mathbb{R}$.
By S *is* **strictly-t-bounded**, *we mean:*
$$\forall a, b \in S, \quad |a - b| < t.$$
By S *is* **semi-t-bounded**, *we mean:*
$$\forall a, b \in S, \quad |a - b| \leqslant t.$$

THEOREM 1.32.11.
$[-2; 5)$ *is strictly-7-bounded.*
$[-2; 5]$ *is NOT strictly-7-bounded, but IS semi-7-bounded.*
$\forall t \in \mathbb{R}$, $\mathbb{N}$ *is NOT strictly-t-bounded.*
$\forall t \in \mathbb{R}$, $\varnothing$ *IS strictly-t-bounded.*

THEOREM 1.32.12. $\forall a \in \mathbb{R}$, $\forall r > 0$, $B(a, r)$ *is strictly-$2r$-bounded.*

THEOREM 1.32.13. *Let* $S \subseteq \mathbb{R}$, $t > 0$.
Assume s *is strictly-t-bounded.*
Then: $\forall a \in S$, $S \subseteq B(a, t)$.

The next theorem is UnHW:

THEOREM 1.32.14. *Let* $S \subseteq \mathbb{R}$. *Then:*
$(S$ *is bounded*$) \Leftrightarrow (\exists t > 0$ *s.t.* S *is strictly-t-bounded*$)$.

THEOREM 1.32.15. *Let* $a \in \mathbb{R}$, $C, D \in \mathcal{B}(a)$.
Then $C \cap D, C \cup D \in \{C, D\}$.

Proof. Choose $r, s > 0$ s.t. $C = B(a, r)$ and $D = B(a, s)$.
Let $t := \min\{r, s\}$.
Then $C \cap D = B(a, t)$.
Also, $t \in \{r, s\}$, so $B(a, t) \in \{B(a, r), B(a, s)\}$.
Then $C \cap D = B(a, t) \in \{B(a, r), B(a, s)\} = \{C, D\}$.
Want: $C \cup D \in \{C, D\}$.
Let $u := \max\{r, s\}$.
Then $C \cup D = B(a, u)$.
Also, $u \in \{r, s\}$, so $B(a, u) \in \{B(a, r), B(a, s)\}$.
Then $C \cup D = B(a, u) \in \{B(a, r), B(a, s)\} = \{C, D\}$. □

THEOREM 1.32.16. *Let* $X, Y \subseteq \mathbb{R}$.
Assume X *and* Y *are both bounded.* *Then* $X \cup Y$ *is bounded.*

Proof. Since X and Y are both bounded,
choose $A, B \in \mathcal{B}_{\mathbb{R}}$ s.t. $X \subseteq A$ and $Y \subseteq B$.
By Theorem 1.31.7, choose $C, D \in \mathcal{B}(0)$ s.t. $A \subseteq C$ and $B \subseteq D$.

By Theorem 1.32.15, $C \cup D \in \{C, D\}$.
Then $C \cup D \in \{C, D\} \subseteq \mathcal{B}(0) \subseteq \mathcal{B}_\mathbb{R}$, so $C \cup D \in \mathcal{B}_\mathbb{R}$.
It therefore suffices to show: $X \cup Y \subseteq C \cup D$.
We have $X \cup Y \subseteq A \cup B \subseteq C \cup D$. □

THEOREM 1.32.17. *Let $A \subseteq \mathbb{R}$. Assume A is bounded.*
Then $\exists r > 0$ s.t. $A \subseteq B(0, r)$.

Proof. Since A is bounded, choose $B \in \mathcal{B}_R$ s.t. $A \subseteq B$.
By Theorem 1.31.7, choose $C \in \mathcal{B}(0)$ s.t. $B \subseteq C$.
Since $C \in \mathcal{B}(0)$, choose $r > 0$ s.t. $C = B(0, r)$. Then $r > 0$.
Want: $A \subseteq B(0, r)$. We have $A \subseteq B \subseteq C = B(0, r)$. □

1.33. Hausdorff property of the real numbers.

The next theorem is called the **Hausdorff property** of $\mathbb{R}$.

THEOREM 1.33.1. *Let $a, b \in \mathbb{R}$. Assume $a \neq b$.*
Then $\exists \varepsilon > 0$ s.t. $(B(a, \varepsilon)) \cap (B(b, \varepsilon)) = \varnothing$.

Proof. Since $a \neq b$, we get $b - a \neq 0$, so $|b - a| > 0$.
Let $\varepsilon := |b - a|/2$. Then $\varepsilon > 0$.
Want: $(B(a, \varepsilon)) \cap (B(b, \varepsilon)) = \varnothing$.
Assume $(B(a, \varepsilon)) \cap (B(b, \varepsilon)) \neq \varnothing$. **Want:** Contradiction.
Choose x s.t. $x \in (B(a, \varepsilon)) \cap (B(b, \varepsilon))$.
Since $x \in B(a, \varepsilon)$, we get $|x - a| < \varepsilon$.
Since $x \in B(b, \varepsilon)$, we get $|x - b| < \varepsilon$. Then $|b - x| < \varepsilon$.
By the Triangle Inequality, $|b - a| \leqslant |b - x| + |x - a|$.
Then $$|b - a| \leqslant |b - x| + |x - a| < \varepsilon + \varepsilon = 2\varepsilon = |b - a|,$$
so $|b - a| < |b - a|$. Contradiction. □

1.34. Density of $\mathbb{Q}$ in $\mathbb{R}$.

THEOREM 1.34.1. *Let $a, b \in \mathbb{R}$.*
Assume $b - a > 1$. Then $\exists k \in \mathbb{Z}$ s.t. $a < k < b$.

Proof. By the AP, choose $j \in \mathbb{N}$ s.t. $j > -a$.
Let $\alpha := j + a$. Then $\alpha > 0$. Let $\beta := j + b$.
By the AP, choose $\lambda \in \mathbb{N}$ s.t. $\lambda > a$.
Then $\lambda \in (\alpha; \infty)$ and $\lambda \in \mathbb{N} \subseteq \mathbb{Z}$.
Then $\lambda \in (\alpha; \infty) \cap \mathbb{Z}$ Then $\lambda \in (\alpha; \infty) \cap \mathbb{Z} \neq \varnothing$.
Since $\alpha > 0$, we get $(\alpha; \infty) \subseteq (0; \infty)$.

Then $(\alpha;\infty) \cap \mathbb{Z} \subseteq (0;\infty) \cap \mathbb{Z}$.
So, since $(0;\infty) \cap \mathbb{Z} = \mathbb{N}$, we get $(\alpha;\infty) \cap \mathbb{Z} \subseteq \mathbb{N}$.
Then $\varnothing \neq (\alpha;\infty) \cap \mathbb{Z} \subseteq \mathbb{N}$.
Then, by the Well-Ordering Axiom, $\min(\ (\alpha;\infty)\ \cap\ \mathbb{Z}\ ) \neq$ ☹.
Let $\kappa := \min(\ (\alpha;\infty)\ \cap\ \mathbb{Z}\ )$.
Then $\kappa \in (\alpha;\infty)\ \cap\ \mathbb{Z}$. Then $\kappa \in \mathbb{Z}$. Then $\kappa - 1 \in \mathbb{Z}$.
Moreover, since $\kappa - 1 < \kappa = \min(\ (\alpha;\infty)\ \cap\ \mathbb{Z}\ )$,
we get $\kappa - 1 \neq (\alpha;\infty)\ \cap\ \mathbb{Z}$.
So, since $\kappa - 1 \in \mathbb{Z}$, it follows that $\kappa - 1 \neq (\alpha;\infty)$.
So, since $\kappa - 1 \in \mathbb{Z} \subseteq \mathbb{R}$, we get $\kappa - 1 \in \mathbb{R}\backslash(\alpha;\infty)$.
Then $\kappa - 1 \in \mathbb{R}\backslash(\alpha;\infty) = (-\infty;\alpha] \leqslant \alpha$, so $\kappa - 1 \leqslant \alpha$, so $\kappa \leqslant \alpha + 1$.
So, since $1 < \beta - \alpha$, we get $\kappa \leqslant \alpha + (\beta - \alpha)$, and so $\kappa < \beta$.
Also, $\kappa \in (\alpha;\infty)\ \cap\ \mathbb{Z} \subseteq (\alpha;\infty) > \alpha$, so $\kappa > \alpha$. Then $\alpha < \kappa < \beta$.
Let $k := \kappa - j$. Then $k \in \mathbb{Z} - j \subseteq \mathbb{Z}$. **Want:** $a < k < b$.
We have $\alpha - j < \kappa - j < \beta - j$, so $a < k < b$. □

The following is HW#7-2:

THEOREM 1.34.2. *Let $s, t \in \mathbb{R}$.*
Assume $s < t$. Then $\exists x \in \mathbb{Q}$ s.t. $s < x < t$.

1.35. **Some topology on $\mathbb{R}$.**

The **boundary** of a set X is denoted ∂X, ad is defined as follows:

DEFINITION 1.35.1. *Let $X \subseteq \mathbb{R}$.*
Then $\boxed{\partial X}$ $:= \{\, q \in \mathbb{R} \mid (\,\exists s \in X^{\mathbb{N}} \text{ s.t. } s \to q\,)$
$\&\ (\,\exists t \in (\mathbb{R}\backslash X)^{\mathbb{N}} \text{ s.t. } t \to q\,)\,\}$.

Thinking of X as "we", of $\mathbb{R}\backslash X$ is "they" and ∂X as "the wall", then the wall consists of the points that both we and they can approach.

THEOREM 1.35.2. *Let $X := (0;1)$. Then $\partial X = \{0,1\}$.*

Proof. Define $s, t, u, v \in \mathbb{R}^{\mathbb{N}}$ by: $\forall j \in \mathbb{N}$,

$$s_j = \frac{1}{j+1}, \quad t_j = -\frac{1}{j}, \quad u_j = 1 - \frac{1}{j+1}, \quad v_j = 1 + \frac{1}{jh}.$$

Then $s \in X^{\mathbb{N}}$ and $t \in (\mathbb{R}\backslash X)^{\mathbb{N}}$ and $s \to 0$ and $t \to 0$, so $0 \in \partial X$.
Also, $u \in X^{\mathbb{N}}$ and $v \in (\mathbb{R}\backslash X)^{\mathbb{N}}$ and $u \to 1$ and $v \to 1$, so $1 \in \partial X$.
Then $\{0,1\} \subseteq \partial X$. **Want:** $\partial X \subseteq \{0,1\}$.
Want: $\forall q \in \partial X,\ \ q \in \{0,1\}$.

Given $q \in \partial X$. **Want:** $q \in \{0, 1\}$. **Want:** $q \in [0; 1]\backslash(0; 1)$.
Want: $q \in [0; 1]$ and $q \notin (0; 1)$.
Since $q \in \partial X$, choose $y \in X^{\mathbb{N}}$ s.t. $y \to q$
and choose $z \in (\mathbb{R}\backslash X)^{\mathbb{N}}$ s.t. $z \to q$.
We have: $\forall j \in \mathbb{N}$, $y_j \in \mathbb{I}_y \subseteq X = (0; 1) \subseteq [0; 1]$, so $0 \leqslant y_j \leqslant 1$.
So, as $y \to q$, we get: $0 \leqslant q \leqslant 1$. Then $q \in [0; 1]$. **Want:** $q \notin (0; 1)$.
Assume $q \in (0; 1)$. **Want:** Contradiction.
Let $C := B(1/2, 1/2)$. Then $C = (0, 1)$.
Since $q \in (0; 1) = C$, by the Subset Recentering Theorem,
choose $B \in \mathcal{B}(q)$ s.t. $B \subseteq C$.
Since $B \in \mathcal{B}(q)$, choose $\varepsilon > 0$ s.t. $B = B(q, \varepsilon)$.
Since $z \to q$, choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$(j \geqslant K) \Rightarrow (|z_j - q| < \varepsilon)$.
Since $z \in (\mathbb{R}\backslash X)^{\mathbb{N}}$, we have $\mathbb{I}_z \subseteq \mathbb{R}\backslash X$.
Since $K \geqslant K$, by choice of K, we get $|z_K - q| < \varepsilon$, so $z \in B(q, \varepsilon)$.
Then $z_K \in B(q, \varepsilon) = B \subseteq C = (0; 1) = X$, so $z_K \in X$.
Also $z_K \in \mathbb{I}_z \subseteq \mathbb{R}\backslash X$, and so $z_K \notin X$. Contradiction. □

The following is an unassigned HW:

THEOREM 1.35.3.
$\partial[0; 1] = \partial[0; 1) = \partial(0; 1] = \{0, 1\}$.

THEOREM 1.35.4. *$\forall X \subseteq \mathbb{R}$, $\partial X = \partial(\mathbb{R}\backslash X)$.*

Thinking of X as "we", of $\mathbb{R}\backslash X$ is "they" and ∂X as "the wall", then our wall is their wall.

Idea of proof: Keep in mind: $\mathbb{R}\backslash(\mathbb{R}\backslash X) = X$.
If a point in $\mathbb{R}$ can be approached
both by a sequence in X and by a sequence in $\mathbb{R}\backslash X$,
then it can be approached
both by a sequence in $\mathbb{R}\backslash X$ and by a sequence in $\mathbb{R}\backslash(\mathbb{R}\backslash X)$.
Thus any point in ∂X is a point in $\partial(\mathbb{R}\backslash X)$.
Then $\partial X \subseteq \partial(\mathbb{R}\backslash X)$. The reverse inclusion is similar. QED

The **closure** and **interior** of a set X are denoted $\mathrm{Cl}\, X$ and $\mathrm{Int}\, X$, and defined as follows:

DEFINITION 1.35.5. *Let* $X \subseteq \mathbb{R}$.

Then $\boxed{\operatorname{Cl} X} := X \cup \partial X$ *and*

$\boxed{\operatorname{Int} X} := X \setminus \partial X$.

We also use Cl_X and $\operatorname{Cl}(X)$ to denote $\operatorname{Cl} X$.
We also use Int_X and $\operatorname{Int}(X)$ to denote $\operatorname{Int} X$.

Any set is between its interior and closure:

THEOREM 1.35.6. *Let* $S \subseteq \mathbb{R}$. *Then* $\operatorname{Int} S \subseteq S \subseteq \operatorname{Cl} S$.

Unassigned HW:

$\operatorname{Int}(0;1) = \operatorname{Int}[0;1] = \operatorname{Int}[0;1) = \operatorname{Int}(0,1] = (0;1)$ and

$\operatorname{Cl}(0;1) = \operatorname{Cl}[0;1] = \operatorname{Cl}[0;1) = \operatorname{Cl}(0,1] = [0;1]$.

In fact:

THEOREM 1.35.7. *Let* $b \in \mathbb{R}$ *and let* $a < b$. *Then:*

$\forall S \in \{ (a;b), [a;b], [a;b), (a;b] \}$,

$\operatorname{Int}_S = (a;b)$ *and* $\operatorname{Cl}_S = [a;b]$.

DEFINITION 1.35.8. *Let* $X \subseteq \mathbb{R}$.

By X *is* **closed**, *we mean:* $\operatorname{Cl} X = X$.

By X *is* **open**, *we mean:* $\operatorname{Int} X = X$.

Note that:

$[0;1]$ is closed and
$(0;1)$ is open and
$[0;1)$ is neither and
$(0;1]$ is neither.

THEOREM 1.35.9. *Let* $X \subseteq \mathbb{R}$. *Then:*

$[\,(X \text{ is open}) \Leftrightarrow (\partial X \subseteq \mathbb{R} \backslash X)\,]$ *and*

$[\,(X \text{ is closed}) \Leftrightarrow (\partial X \subseteq X)\,]$.

Part of proof:

X is closed $\Leftrightarrow \operatorname{Cl}X = X \Leftrightarrow X \cup \partial X = X \Leftrightarrow \partial X \subseteq X$.

Thinking of X as "we", of $\mathbb{R} \backslash X$ is "they" and ∂X as "the wall", one could say:

we and they both have the same wall.

However,

we might own the wall OR

they might own the wall OR
we both might own part of it.
In the first case, we are closed and they are open.
In the second case, we are open and they are closed.
In the third case, we are neither open nor closed
and they are also neither open nor closed.

Keep in mind that
showing that a set fails to be open
is NOT the same as
showing that it is closed.
Similarly
showing that a set fails to be closed
is NOT the same as
showing that it is open.
Many sets are neither open nor closed.
Closed an open are not opposites.
However they ARE complementary, in the following sense:

THEOREM 1.35.10. *Let $X \subseteq \mathbb{R}$. Then*
$[(\,X \textit{ is open}\,) \Leftrightarrow (\mathbb{R}\backslash X \textit{ is closed}\,)]$ &
$[(\,X \textit{ is closed}\,) \Leftrightarrow (\mathbb{R}\backslash X \textit{ is open}\,)]$.

Thinking of X as "we", of $\mathbb{R}\backslash X$ is "they" and ∂X as "the wall",
then saying that we none of the wall (we are "open")
is the same as saying they own all of it (they are "closed").
Also, saying that we own all of the wall (we are "closed")
is the same as saying they own none of it (they are "open").

We know that some are closed and some are open,
but many sets are neither.
Question: Are any sets BOTH closed AND open.

DEFINITION 1.35.11. *Let $X \subseteq \mathbb{R}$.*
By X is **clopen**, *we mean: X is closed and open.*

For us to be clopen, it would have to be true that
both we and they own all of the wall,
but, since we $\cap$ they $= \varnothing$, this would mean that
the wall simply doesn't exist.

That is,

$$\forall X \subseteq \mathbb{R}, \qquad (\ X \text{ is clopen }) \quad \text{iff} \quad (\ \partial X = \varnothing\ ).$$

THEOREM 1.35.12. *Let* $X \subseteq \mathbb{R}$. *Then:*

$$(\,X \textit{ is clopen}\,) \ \Leftrightarrow\ (\,(\,X = \mathbb{R}\,) \vee (\,X = \varnothing\,)\,).$$

Idea of proof:
Since X is clopen, $\partial X = \varnothing$.
Assume that $X \neq \mathbb{R}$ and $X \neq \varnothing$. **Want:** Contradiction.
Since $X \neq \varnothing$, choose p s.t. $p \in X$.
Since $X \subseteq \mathbb{R}$ and $X \neq \mathbb{R}$, choose $q \in \mathbb{R}$ s.t. $q \notin X$.
Since $p \in X$ and $q \notin X$, we get $p \neq q$, so either $p < q$ or $q < p$.
Then either $p \in X \cap (-\infty; q)$ or $p \in X \cap (q; \infty)$.
In the first case, let $r := \sup(X \cap (-\infty; q))$.
In the second case, let $r := \inf(X \cap (q; \infty))$.
In either case, one can show (with work) that $r \in \partial X$.
Since $r \in \partial X$, we get: $\partial X \neq \varnothing$.
Then $\varnothing \neq \partial X = \varnothing$. Contradiction. QED

THEOREM 1.35.13. *Let* $S \subseteq \mathbb{R}$, $a \in \mathbb{R}$.
Then: $a \in \mathrm{Int}S \quad \Leftrightarrow \quad \exists \delta > 0$ *s.t.* $B(a, \delta) \subseteq S$.

Proof. Proof of $\Rightarrow$*:*
Assume $a \in \mathrm{Int}S$. **Want:** $\exists \delta > 0$ s.t. $B(a, \delta) \subseteq S$.
Assume $\neg(\,\exists \delta > 0$ s.t. $B(a, \delta) \subseteq S\,)$. **Want:** Contradiction.
We have: $\forall \delta > 0$, $B(a, \delta) \nsubseteq S$.
Then: $\forall \delta > 0$, $(B(a, \delta))\backslash S \neq \varnothing$.
For all $j \in \mathbb{N}$, let $Q_j := (B(a, 1/j))\backslash S$.
Then: $\forall j \in \mathbb{N}$, we have: $Q_j \neq \varnothing$ and $Q_j \subseteq \mathbb{R}\backslash S$.
Define $z \in (\mathbb{R}\backslash S)^{\mathbb{N}}$ by: $\forall j \in \mathbb{N}$, $z_j = \mathrm{CH}_{Q_j}$.
We have: $\forall j \in \mathbb{N}$, $z_j \in Q_j = (B(a, 1/j))\backslash S \subseteq B(a, 1/j)$.
Then: $\forall j \in \mathbb{N}$, $|z_j - a| < 1/j$. Then $z \to a$.
Let $y := (a, a, a, a, \ldots)$. Then $y \to a$.
Since $a \in \mathrm{Int}S = S\backslash\partial S \subseteq S$, we get: $y \in S^{\mathbb{N}}$.
Since $y \in S^{\mathbb{N}}$ and $y \to a$ and $z \in (\mathbb{R}\backslash S)^{\mathbb{N}}$ and $z \to a$,
we conclude that: $a \in \partial S$.
Since $a \in \mathrm{Int}S = S\backslash\partial S$, we get: $a \notin \partial S$. Contradiction.
End of proof of $\Rightarrow$.

Proof of $\Leftarrow$:
Assume $\exists\delta > 0$ s.t. $B(a,\delta) \subseteq S$. **Want:** $a \in \mathrm{Int} S$.
Choose $\delta > 0$ s.t. $B(a,\delta) \subseteq S$. **Want:** $a \in S \backslash \partial S$.
We have $|a-a| = 0 < \delta$, so $a \in B(a,\delta)$.
Since $a \in B(a,\delta) \subseteq S$, **it remains to show:** $a \notin \partial S$.
Assume $a \in \partial S$. **Want:** Contradiction.
Since $a \in \partial S$, choose $z \in (\mathbb{R}\backslash S)^{\mathbb{N}}$ s.t. $z \to a$.
Since $z \to a$, choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$(\, j \geqslant K \,) \Rightarrow (\, |z_j - a| < \delta \,).$$

Since $z \in (\mathbb{R}\backslash S)^{\mathbb{N}}$, we get: $z_K \in \mathbb{R}\backslash S$.
Since $K \geqslant K$, by choice of K, we get $|z_K - a| < \delta$, and so $z_K \in B(a,\delta)$.
Then $z_K \in B(a,\delta) \subseteq S$, so $z_K \in S$.
Since $z_K \in \mathbb{R}\backslash S$, we get $z_K \notin S$. Contradiction.
End of proof of $\Leftarrow$. □

Proof. Unassigned HW. □

The following is called **monotonicity of interior**:

THEOREM 1.35.14. *Let* $S, T \subseteq \mathbb{R}$.
Assume $S \subseteq T$. *Then* $\mathrm{Int}\, S \subseteq \mathrm{Int}\, T$.

Proof. Unassigned HW. □

The following is called **monotonicity of closure**:

THEOREM 1.35.15. *Let* $S, T \subseteq \mathbb{R}$.
Assume $S \subseteq T$. *Then* $\mathrm{Cl}\, S \subseteq \mathrm{Cl}\, T$.

Proof. Unassigned HW. □

For any $U \subseteq \mathbb{R}$, we have $\mathrm{Int}\, U \subseteq U$, and so:
$U = \mathrm{Int}\, U$ iff $U \subseteq \mathrm{Int}\, U$.

For any $U \subseteq \mathbb{R}$, for any $a \in \mathbb{R}$, we have:
$a \in \mathrm{Int}\, U$ iff $\exists\delta > 0$ s.t. $B(a,\delta) \subseteq U$
iff $B \in \mathcal{B}(a)$ s.t. $B \subseteq U$.

For any $U \subseteq \mathbb{R}$, we have:
U is open iff $U = \mathrm{Int}\, U$
iff $U \subseteq \mathrm{Int}\, U$
iff $\forall a \in U$, $a \in \mathrm{Int}\, U$

$$\begin{array}{ll} \text{iff} & \forall a \in U,\ \exists \delta > 0 \text{ s.t. } B(a,\delta) \subseteq U \\ \text{iff} & \forall a \in U,\ B \in \mathcal{B}(a) \text{ s.t. } B \subseteq U. \end{array}$$

It follows that the singleton set $\{0\}$ is not open.

THEOREM 1.35.16. *Let $U \in \mathcal{B}_{\mathbb{R}}$. Then U is open.*

Proof. **Want:** $\forall x \in U$, $\exists B \in \mathcal{B}(x)$ s.t. $B \subseteq U$.
Given $x \in U$. **Want:** $\exists B \in \mathcal{B}(x)$ s.t. $B \subseteq U$.
By the Subset Recentering Theorem, $\exists B \in \mathcal{B}(x)$ s.t. $B \subseteq U$. □

THEOREM 1.35.17. *Let $U, V \subseteq \mathbb{R}$. Assume U, V are both open.*
Then: $U \cup V$ and $U \cap V$ are both open.

Proof. This is HW#11-4. □

THEOREM 1.35.18. *Let $C, D \subseteq \mathbb{R}$. Assume C, D are both closed.*
Then: $C \cap D$ and $C \cup D$ are both closed.

Proof. Let $U := \mathbb{R}\backslash C$ and $V := \mathbb{R}\backslash D$.
Then U and V are both open.
Then $U \cup V$ and $U \cap V$ are both open.
Then $R\backslash(U \cup V)$ and $\mathbb{R}\backslash(U \cap V)$ are both closed.
We have $R\backslash(U \cup V) = (\mathbb{R}\backslash U) \cap (\mathbb{R}\backslash V) = C \cap D$
and $\mathbb{R}\backslash(U \cap V) = (\mathbb{R}\backslash U) \cup (\mathbb{R}\backslash V) = C \cup D$.
Then $C \cap D$ and $C \cup D$ are both closed. □

2. Sequences in $\mathbb{R}$

2.1. Limit of a sequence in $\mathbb{R}$.

DEFINITION 2.1.1. *Let s be a sequence, $K \in \mathbb{N}$.*
*Then the K-**tail** of s is $(s_K, s_{K+1}, s_{K+2}, \ldots)$.*

THEOREM 2.1.2. *The* 7 *tail of* $(1, 1/2, 1/3, \ldots)$ *is* $(1/7, 1/8, 1/9, \ldots)$.

We next define **limit of a sequence in $\mathbb{R}$**:

DEFINITION 2.1.3. *Let $s \in \mathbb{R}^{\mathbb{N}}$, $a \in \mathbb{R}$. Then $\boxed{s \to a}$ means:*

$$\forall \varepsilon > 0,\ \exists K \in \mathbb{N} \text{ s.t., } \forall j \in \mathbb{N},$$
$$(\ j \geqslant K\ ) \ \Rightarrow\ (\ |s_j - a| < \varepsilon\ ).$$

We next define the **constant function on S with value a**:

DEFINITION 2.1.4. *Let S be a set, a an object. Then:*
$\boxed{C_a^S} : S \to \{a\}$ *is defined by:*
$$\forall x \in S, \quad C_a^S(x) = a.$$

THEOREM 2.1.5. $\forall a, \quad C_a^{\mathbb{N}} = (\, a\, ,\, a\, ,\, a\, ,\, a\, ,\, a\, ,\, a\, ,\, \ldots).$

THEOREM 2.1.6. *Let $a \in \mathbb{R}$. Then $C_a^{\mathbb{N}} \to a$.*

Proof. **Want:** $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\, j \geqslant K\,) \Rightarrow (\, |(C_a^{\mathbb{N}})_j - a| < \varepsilon\,).$$
Given $\varepsilon > 0$. **Want:** $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\, j \geqslant K\,) \Rightarrow (\, |(C_a^{\mathbb{N}})_j - a| < \varepsilon\,).$$
Let $K := 1$. Then $K \in \mathbb{N}$.
Want: $\forall j \in \mathbb{N}, \quad (\, j \geqslant K\,) \Rightarrow (\, |(C_a^{\mathbb{N}})_j - a| < \varepsilon\,)$.
Given $j \in \mathbb{N}$. Assume $j \geqslant K$. **Want:** $|(C_a^{\mathbb{N}})_j - a| < \varepsilon$.
We have $|(C_a^{\mathbb{N}})_j - a| = |a - a| = |0| = 0 < \varepsilon$.

□

THEOREM 2.1.7. $(1, 1/2, 1/3, \ldots) \to 0$.

Proof. Let $s := (1, 1/2, 1/3, \ldots)$. $\forall j \in \mathbb{N}$, $s_j = 1/j$.
Want: $s \to 0$.
Want: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\ j \geqslant K\ ) \Rightarrow (\ |s_j - 0| < \varepsilon\ ).$$
Given $\varepsilon > 0$. **Want:** $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\ j \geqslant K\ ) \Rightarrow (\ |s_j - 0| < \varepsilon\ ).$$
By the AP, choose $K \in \mathbb{N}$ s.t. $K > 1/\varepsilon$. Then $K \in \mathbb{N}$.
Want: $\forall j \in \mathbb{N}$, $(\ j \geqslant K\ ) \Rightarrow (\ |s_j - 0| < \varepsilon\ )$.
Given $j \in \mathbb{N}$. Assume $j \geqslant K$. **Want:**$|s_j - 0| < \varepsilon$.
Since $\varepsilon > 0$, we get: $1/\varepsilon > 0$ and $1/(1/\varepsilon) = \varepsilon$.
Since $j \geqslant K > 1/\varepsilon$, we get $j > 1/\varepsilon$.
Since $j > 1/\varepsilon > 0$, we get $1/j < 1/(1/\varepsilon)$.
Since $j \in \mathbb{N} > 0$, we get $j > 0$, so $1/j > 0$, so $|1/j| = 1/j$.
Then $|s_j - 0| = |s_j| = |1/j| = 1/j < 1/(1/\varepsilon) = \varepsilon$. □

DEFINITION 2.1.8. *Let $s \in \mathbb{R}^{\mathbb{N}}$. Then $\boxed{s \to \infty}$ means:*
$\forall M \in \mathbb{R}$, $\exists K \in \mathbb{N}$ *s.t.*, $\forall j \in \mathbb{N}$,
$$(\ j \geqslant K\ ) \Rightarrow (\ s_j > M\ ).$$

DEFINITION 2.1.9. *Let $s \in \mathbb{R}^{\mathbb{N}}$. Then $\boxed{s \to -\infty}$ means:*
$\forall N \in \mathbb{R}$, $\exists K \in \mathbb{N}$ *s.t.*, $\forall j \in \mathbb{N}$,
$$(\ j \geqslant K\ ) \Rightarrow (\ s_j < N\ ).$$

THEOREM 2.1.10. *Let* $s \in \mathbb{R}^{\mathbb{N}}$*,* $a, c \in \mathbb{R}$*.*
Assume $s \to a$*. Then* $c \cdot s \to c \cdot a$*.*

Proof. **Want:** $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\, j \geqslant K \,) \;\Rightarrow\; (\, |(c \cdot s)_j - c \cdot a| < \varepsilon \,).$$
Given $\varepsilon > 0$. **Want:** $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\, j \geqslant K \,) \;\Rightarrow\; (\, |(c \cdot s)_j - c \cdot a| < \varepsilon \,).$$
Let $\rho := \varepsilon/(|c| + 1)$. Then $\rho > 0$.
Since $s_j \to a$, choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\, j \geqslant K \,) \;\Rightarrow\; (\, |s_j - a| < \rho \,).$$
Then $K \in \mathbb{N}$. **Want:** $\forall j \in \mathbb{N}$, $(\, j \geqslant K \,) \Rightarrow (\, |(c \cdot s)_j - c \cdot a| < \varepsilon \,)$.
Given $j \in \mathbb{N}$. Assume $j \geqslant K$. **Want:** $|(c \cdot s)_j - c \cdot a| < \varepsilon$.
Since $j \geqslant K$, by the choice of K, we have $|s_j - a| < \rho$.
By definition of ρ, we have $|c| \cdot \rho < \varepsilon$.
Then
$$\begin{aligned} |\,(c \cdot s)_j \,-\, c \cdot a\,| &= |\,c \cdot s_j \,-\, c \cdot a\,| \\ &= |\,c \cdot (s_j \,-\, a)\,| \\ &= |\,c\,| \cdot |\,s_j \,-\, a\,| \\ &\leqslant |\,c\,| \cdot \rho \,<\, \varepsilon. \end{aligned}$$
□

THEOREM 2.1.11. *Let* $s, t \in \mathbb{R}^{\mathbb{N}}$*,* $a, b \in \mathbb{R}$*.*
Assume $s \to a$ *and* $t \to b$*. Then* $s + t \to a + b$*.*

Proof. **Want:** $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\, j \geqslant K \,) \;\Rightarrow\; (\, |(s + t)_j - (a + b)| < \varepsilon \,).$$
Given $\varepsilon > 0$. **Want:** $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\, j \geqslant K \,) \;\Rightarrow\; (\, |(s + t)_j - (a + b)| < \varepsilon \,).$$
Since $s \to a$, choose $L \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\, j \geqslant L \,) \;\Rightarrow\; (\, |s_j - a| < \varepsilon/2 \,).$$
Since $t \to b$, choose $M \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\, j \geqslant M \,) \;\Rightarrow\; (\, |t_j - b| < \varepsilon/2 \,).$$
Let $K := \max\{L, M\}$. Then $K \in \mathbb{N}$.
Want: $\forall j \in \mathbb{N}$, $(\, j \geqslant K \,) \Rightarrow (\, |(s + t)_j - (a + b)| < \varepsilon \,)$.
Given $j \in \mathbb{N}$. Assume $j \geqslant K$. **Want:** $|(s + t)_j - (a + b)| < \varepsilon$.
Since $j \geqslant K \geqslant L$, by the choice of L, we have $|s_j - a| < \varepsilon/2$.
Since $j \geqslant K \geqslant M$, by the choice of M, we have $|t_j - b| < \varepsilon/2$.
Then
$$\begin{aligned} |(s + t)_j - (a + b)| &= |(s_j + t_j) - (a + b)| \\ &= |(s_j - a) + (t_j - b)| \\ &\leqslant |s_j - a| + |t_j - b| \\ &< (\varepsilon/2) + (\varepsilon/2) \;=\; \varepsilon. \end{aligned}$$
□

THEOREM 2.1.12. *Let* $s, t \in \mathbb{R}^{\mathbb{N}}$, $a, b \in \mathbb{R}$.
Assume $s \to a$ *and* $t \to b$. *Then* $s \cdot t \to a \cdot b$.

Proof. **Want:** $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\, j \geqslant K \,) \Rightarrow (\, |(s \cdot t)_j - (a \cdot b)| < \varepsilon \,).$$
Given $\varepsilon > 0$. **Want:** $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\, j \geqslant K \,) \Rightarrow (\, |(s \cdot t)_j - (a \cdot b)| < \varepsilon \,).$$
Let $\rho := \min\{\, 1 \,,\, \varepsilon \,/\, (|b| + |a| + 2) \,\}$. Then $\rho > 0$.
Since $s \to a$, choose $L \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\, j \geqslant L \,) \Rightarrow (\, |s_j - a| < \rho \,).$$
Since $t \to b$, choose $M \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\, j \geqslant M \,) \Rightarrow (\, |t_j - b| < \rho \,).$$
Let $K := \max\{L, M\}$. Then $K \in \mathbb{N}$.
Want: $\forall j \in \mathbb{N}$, $(\, j \geqslant K \,) \Rightarrow (\, |(s \cdot t)_j - (a \cdot b)| < \varepsilon \,)$.
Given $j \in \mathbb{N}$. Assume $j \geqslant K$. **Want:** $|(s \cdot t)_j - (a \cdot b)| < \varepsilon$.
Since $j \geqslant K \geqslant L$, by the choice of L, we have $|s_j - a| < \rho$.
Since $j \geqslant K \geqslant M$, by the choice of M, we have $|t_j - b| < \rho$.
By definition of ρ, we have: $\rho \leqslant 1$ and $\rho \cdot (|b| + |a| + 1) < \varepsilon$.
By the Naive Product Rule,
$$(s_j \cdot t_j) - (a \cdot b) = (s_j - a) \cdot b + a \cdot (t_j - b) + (s_j - a) \cdot (t_j - b).$$
Then
$$\begin{aligned} |(s \cdot t)_j - (a \cdot b)| &= |(s_j \cdot t_j) - (a \cdot b)| \\ &= |(s_j - a) \cdot b + a \cdot (t_j - b) + (s_j - a) \cdot (t_j - b)| \\ &\leqslant |s_j - a| \cdot |b| + |a| \cdot |t_j - b| + |s_j - a| \cdot |t_j - b| \\ &\leqslant \rho \cdot |b| + |a| \cdot \rho + \rho \cdot \rho \\ &= \rho \cdot (|b| + |a| + \rho) \\ &\leqslant \rho \cdot (|b| + |a| + 1) \;<\; \varepsilon. \end{aligned}$$
□

THEOREM 2.1.13. *Let* $s \in (\mathbb{R}_0^\times)^{\mathbb{N}}$, $a \in \mathbb{R}_0^\times$.
Assume $s \to a$. *Then* $1/s \to 1/a$.

Proof. **Want:** $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\, j \geqslant K \,) \Rightarrow (\, |(1/s)_j - (1/a)| < \varepsilon \,).$$
Given $\varepsilon > 0$. **Want:** $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\, j \geqslant K \,) \Rightarrow (\, |(1/s)_j - (1/a)| < \varepsilon \,).$$
Since $a \in \mathbb{R}_0^\times$, we get $|a| > 0$ and $a^2 > 0$.
Let $\rho := \min\{\, |a|/2 \,,\, \varepsilon \cdot a^2/2 \,\}$. Then $\rho > 0$.
Since $s \to a$, choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\, j \geqslant K \,) \Rightarrow (\, |s_j - a| < \rho \,).$$
Let $K := \max\{L, M\}$. Then $K \in \mathbb{N}$.
Want: $\forall j \in \mathbb{N}$, $(\, j \geqslant K \,) \Rightarrow (\, (1/s)_j - (1/a)\,)| < \varepsilon \,)$.

Given $j \in \mathbb{N}$. Assume $j \geqslant K$. **Want:** $|(1/s)_j - (1/a)| < \varepsilon$.
Since $j \geqslant K$, by the choice of K, we have $|s_j - a| < \rho$.
By definition of ρ, we have: $\rho \leqslant |a|/2$ and $2 \cdot \rho \,/\, a^2 \;\leqslant\; \varepsilon$.
We have $|a| - \rho \geqslant |a| - (|a|/2) = |a|/2$, so $|a| - \rho \geqslant |a|/2$.
Since $|\bullet|$ is distance semi-decreasing, we get $|\,|s_j| - |a|\,| \;\leqslant\; |\,s_j - a\,|$.
Since $|\,|s_j| - |a|\,| \;\leqslant\; |\,s_j - a\,| \;<\; \rho$, we get $|a| - \rho \;<\; |s_j| \;<\; |a| + \rho$.
Then $|s_j| > |a| - \rho \geqslant |a|/2$, so $|s_j| > |a|/2$.

$$\begin{aligned} \text{Then}\quad |(1/s)_j - (1/a)| &= \left|\frac{a - s_j}{a \cdot s_j}\right| \\ &= \frac{|a - s_j|}{|a| \cdot |s_j|} \\ &< \frac{|s_j - a|}{|a| \cdot (|a|/2)} \\ &< \frac{\rho}{|a|^2/2} \;=\; \frac{\rho}{a^2/2} \;=\; 2 \cdot \rho \,/\, a^2 \;\leqslant\; \varepsilon. \end{aligned}$$ □

THEOREM 2.1.14. *Let $s \in \mathbb{R}^{\mathbb{N}}$, $t \in (\mathbb{R}_0^{\times})^{\mathbb{N}}$, $a \in \mathbb{R}$, $b \in \mathbb{R}_0^{\times}$.*
Assume $s \to a$ and $t \to b$. Then $s/t \to a/b$.

Proof. By Theorem 2.1.13, $1/t \to 1/b$.
So, since $s \to a$, by Theorem 2.1.12, we get $s \cdot (1/t) \to a \cdot (1/b)$.
By Theorem 1.30.12, $s \cdot (1/t) = s/t$. Also $a \cdot (1/b) = a/b$.
Then $s/t \to a/b$. □

THEOREM 2.1.15. *Let $s \in \mathbb{R}^{\mathbb{N}}$, $a \in \mathbb{R}$, $\varepsilon > 0$.*
Assume $s \to a$. Then $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$, $s_j \in B(a, \varepsilon)$.

Proof. Since $s \to a$, choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$(\, j \geqslant K \,) \;\Rightarrow\; (\, |s_j - a| < \varepsilon \,).$$

Then $K \in \mathbb{N}$. **Want:** $\forall j \in [K..\infty)$, $s_j \in B(a, \varepsilon)$.
Given $j \in [K..\infty)$. **Want:** $s_j \in B(a, \varepsilon)$.
We have $j \in [K..\infty) \subseteq \mathbb{N}$ and $j \in [K..\infty) \geqslant K$,
so, by choice of K, we get $|s_j - a| < \varepsilon$, and so $s_j \in B(a, \varepsilon)$. □

THEOREM 2.1.16. *Let $s := (1, -1, 1, -1, 1, -1, \ldots)$.*
Then, $\forall a \in \mathbb{R}$, $\neg(s_j \to a)$.

Proof. Given $a \in \mathbb{R}$. **Want:** $\neg(s_j \to a)$.
Assume $s_j \to a$. **Want:** Contradiction.

Claim: $\forall j \in \mathbb{N}$, $|s_{j+1} - s_j| = 2$.
Proof of claim: Given $j \in \mathbb{N}$. **Want:** $|s_{j+1} - s_j| = 2$.
We have: (1) $j \in 2\mathbb{N}$ or (2) $j \in 2\mathbb{N} + 1$.

Case (1):
We have $s_j = -1$ and $s_{j+1} = 1$, so $s_{j+1} - s_j = 2$, so $|s_{j+1} - s_j| = 2$.
End of Case (1).

Case (2):
We have $s_j = 1$ and $s_{j+1} = -1$, so $s_{j+1} - s_j = -2$, so $|s_{j+1} - s_j| = 2$.
End of Case (2).
End of proof of claim.

Since $s_j \to a$, choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$

$$(\, j \geqslant K \,) \Rightarrow (\, |s_j - a| < 1 \,).$$

Let $j := K$. By the claim $|s_{j+1} - s_j| = 2$.
Since $j \geqslant K$, by the choice of K, we get $|s_j - a| < 1$. Then $|a - s_j| < 1$.
Since $j + 1 \geqslant K$, by the choice of K, we get $|s_{j+1} - a| < 1$.
Then $2 = |s_{j+1} - s_j| \leqslant |s_{j+1} - a| + |a - s_j| < 1 + 1 = 2$.
Then $2 < 2$. Contradiction $\square$

The preceding theorem shows some sequences that have no limit.

DEFINITION 2.1.17. *Let $s \in \mathbb{R}^{\mathbb{N}}$. Then* $\boxed{\mathrm{LIMS}_s} := \{a \in \mathbb{R} \mid s \to a\}$.

Alternate notations: $\mathrm{LIMS}\, s$ and $\mathrm{LIMS}(s)$.
The preceding theorem asserts that $\mathrm{LIMS}(1, -1, 1, -1, 1, -1, \ldots) = \varnothing$.
The next theorem asserts that, $\forall s \in \mathbb{R}^{\mathbb{N}}$, $\#\mathrm{LIMS}_s \leqslant 1$.

THEOREM 2.1.18. *Let $s \in \mathbb{R}^{\mathbb{N}}$, $a, b \in \mathbb{R}$.*
Assume $s \to a$ and $s \to b$. Then $a = b$.

Proof. Assume $a \neq b$. **Want:** Contradiction.
By the Hausdorff property of $\mathbb{R}$, choose $\varepsilon > 0$ s.t.

$$(B(a, \varepsilon)) \cap (B(b, \varepsilon)) = \varnothing.$$

By Theorem 2.1.15, choose $K \in \mathbb{N}$ s.t., $\forall j \in [K..\infty)$, $s_j \in B(a, \varepsilon)$.
By Theorem 2.1.15, choose $L \in \mathbb{N}$ s.t., $\forall j \in [L..\infty)$, $s_j \in B(b, \varepsilon)$.
Let $j := \max\{K, L\}$. Then $j \in [K..\infty)$ and $j \in [L..\infty)$.
Since $j \in [K..\infty)$, by choice of K, we get $s_j \in B(a, \varepsilon)$.
Since $j \in [L..\infty)$, by choice of L, we get $s_j \in B(b, \varepsilon)$.

Then $s_j \in (B(a,\varepsilon)) \cap (B(b,\varepsilon))$.
Then $(B(a,\varepsilon)) \cap (B(b,\varepsilon)) \neq \varnothing$. Contradiction. □

DEFINITION 2.1.19. *Let* $s \in \mathbb{R}^{\mathbb{N}}$. *Then* $\boxed{\lim\ s} := \mathrm{UE}(\mathrm{LIMS}_s)$.

Alternate notation: $\lim(s)$.

THEOREM 2.1.20. *Let* $s \in \mathbb{R}^{\mathbb{N}}$, $a \in \mathbb{R}$.
Then: $(\, s \to a\,) \Leftrightarrow (\,\lim s = a\,)$.

Proof.
Proof of $\Rightarrow$*:*
Assume $s \to a$. **Want:** $\lim\ s = a$.
Since $s \to a$, we get $a \in \mathrm{LIMS}\, s$.
Then, by Theorem 2.1.18, we have: $\forall b \in \mathrm{LIMS}\, s,\quad a = b$.
Then $\mathrm{LIMS}\, s = \{a\}$. Then $\lim\ s = \mathrm{UE}\{a\} = a$.
End of proof of $\Rightarrow$.

Proof of $\Leftarrow$*:*
Assume $\lim\ s = a$. **Want:** $s \to a$.
We have $a = \lim\ s = \mathrm{UE}(\mathrm{LIMS}\, s)\ {}^*\!\!\in\ \mathrm{LIMS}\, s$, so $a \in \mathrm{LIMS}\, s$.
Since $a \in \mathrm{LIMS}\, s = \{x \in \mathbb{R} \,|\, s \to x\}$, we get $s \to a$.
End of proof of $\Leftarrow$. □

2.2. Compact subsets of $\mathbb{R}$.

DEFINITION 2.2.1. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$. *Then:*
f *is* **strictly-increasing** *means:* $\forall w, x \in \mathbb{D}_f,\ (w < x) \Rightarrow (f_w < f_x)$
and
f *is* **strictly-decreasing** *means:* $\forall w, x \in \mathbb{D}_f,\ (w < x) \Rightarrow (f_w > f_x)$.

DEFINITION 2.2.2. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$.
Then f *is* **strictly-monotone** *means:*
f *is strictly-increasing* *or* f *is strictly-decreasing.*

THEOREM 2.2.3. $(1, 4, 9, 16, 25, 36, 49\ldots)$ *is strictly-increasing.*

If we reverse 1 and 4 in the sequence above, we get a new sequence,
$(4, 1, 9, 16, 25, 36, 49, \ldots)$,
which is NOT strictly-increasing.

THEOREM 2.2.4. *Let* $s \in \mathbb{R}^{\mathbb{N}}$. *Then:*

$$[\,(\,s \textit{ is strictly-increasing}\,) \Leftrightarrow (\,\forall j \in \mathbb{N},\, s_j < s_{j+1}\,)\,]$$

and

$$[\,(\,s \textit{ is strictly-decreasing}\,) \Leftrightarrow (\,\forall j \in \mathbb{N},\, s_j > s_{j+1}\,)\,].$$

DEFINITION 2.2.5. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$. *Then:*

f *is* **semi-increasing** *means:* $\forall w, x \in \mathbb{D}_f,\ (w \leqslant x) \Rightarrow (f_w \leqslant f_x)$,

and

f *is* **semi-decreasing** *means:* $\forall w, x \in \mathbb{D}_f,\ (w \leqslant x) \Rightarrow (f_w \geqslant f_x)$.

DEFINITION 2.2.6. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$.

Then f *is* **semi-monotone** *means:*

f *is semi-increasing* *or* f *is semi-decreasing.*

THEOREM 2.2.7.

$(1, 1, 2, 2, 3, 3, 4, 4, 5, 5, \ldots)$ *is semi-increasing,*

but NOT strictly-increasing.

THEOREM 2.2.8. *Let* $s \in \mathbb{R}^{\mathbb{N}}$. *Then:*

$$[\,(\,s \textit{ is semi-increasing}\,) \Leftrightarrow (\,\forall j \in \mathbb{N},\, s_j \leqslant s_{j+1}\,)\,]$$

and

$$[\,(\,s \textit{ is semi-decreasing}\,) \Leftrightarrow (\,\forall j \in \mathbb{N},\, s_j \geqslant s_{j+1}\,)\,].$$

THEOREM 2.2.9. *Let* $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$. *Then:*

$$[\,(\,f \textit{ is strictly-increasing}\,) \,\&\, (\,g \textit{ is strictly-increasing}\,)\,] \Rightarrow [\,g \circ f \textit{ is strictly-increasing}\,].$$

THEOREM 2.2.10. *Let* $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$. *Then:*

$$[\,(\,f \textit{ is strictly-decreasing}\,) \,\&\, (\,g \textit{ is strictly-increasing}\,)\,] \Rightarrow [\,g \circ f \textit{ is strictly-decreasing}\,].$$

THEOREM 2.2.11. *Let* $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$. *Then:*

$$[\,(\,f \textit{ is strictly-increasing}\,) \,\&\, (\,g \textit{ is strictly-decreasing}\,)\,] \Rightarrow [\,g \circ f \textit{ is strictly-decreasing}\,].$$

THEOREM 2.2.12. *Let* $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$. *Then:*

$$[\,(\,f \textit{ is strictly-decreasing}\,) \,\&\, (\,g \textit{ is strictly-decreasing}\,)\,] \Rightarrow [\,g \circ f \textit{ is strictly-increasing}\,].$$

Recall that $\sup \varnothing = -\infty$. Also:

THEOREM 2.2.13. *Let* $X \subseteq \mathbb{R}$.
Assume $X \neq \varnothing$. *Then* $\sup X \neq -\infty$.

Proof. We have $X \leqslant \sup X$.
Since $X \neq \varnothing$, choose z s.t. $z \in X$
Since $z \in Z \subseteq \mathbb{R} > -\infty$, we get $z > -\infty$.
Then $-\infty < z \in X \leqslant \sup X$, so $-\infty < \sup X$. Then $\sup X \neq -\infty$. ☐

THEOREM 2.2.14. *Let* $X \subseteq \mathbb{R}$.
Assume X *is bounded above.* *Then* $\sup X \neq \infty$.

Proof. Since X is bounded above, choose $z \in \mathbb{R}$ s.t. $X \leqslant z$.
Then $z \in \mathrm{UB}_X \geqslant \min \mathrm{UB}_X = \sup X$.
Then $\sup X \leqslant z \in \mathbb{R} < \infty$. Then $\sup X < \infty$, so $\sup X \neq \infty$. ☐

THEOREM 2.2.15. *Let* $X \subseteq \mathbb{R}$.
Assume $X \neq \varnothing$ *and* X *is bounded above.* *Then* $\sup X \in \mathbb{R}$.

Proof. We have $\sup X \in \mathbb{R}^*$. **Want:** $\sup X \neq -\infty$ and $\sup X \neq \infty$.
By Theorem 2.2.13, $\sup X \neq -\infty$. **Want:** $\sup X \neq \infty$.
By Theorem 2.2.14, $\sup X \neq \infty$. ☐

THEOREM 2.2.16. *Let* $s \in \mathbb{R}^{\mathbb{N}}$.
Assume: s *is semi-increasing* *and* $\mathbb{I}_s$ *is bounded above.*
Then: $s \to \sup \mathbb{I}_S$.

Proof. Since $\mathbb{D}_s = \mathbb{N} \neq \varnothing$, it follows that $\mathbb{I}_s \neq \varnothing$.
So, since $\mathbb{I}_s$ is bounded above, by Theorem 2.2.15, we get: $\sup \mathbb{I}_s \in \mathbb{R}$.
Let $a := \sup \mathbb{I}_s$. Then $a \in \mathbb{R}$. Want $s \to a$.
Want: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall j \in \mathbb{N}$, $(\, j \geqslant K\,) \Rightarrow (\, |s_J - a| < \varepsilon\,)$.
Given $\varepsilon > 0$. **Want:** $\exists \delta > 0$ s.t., $\forall j \in \mathbb{N}$, $(\, j \geqslant K\,) \Rightarrow (\, |s_J - a| < \varepsilon\,)$.
Since $a - \varepsilon < a = \sup \mathbb{I}_s$, we see that $\neg(a - \varepsilon \geqslant \sup \mathbb{I}_s)$.
Then $\neg(a - \varepsilon \geqslant \mathbb{I}_s)$, so choose $x \in \mathbb{I}_s$ s.t. $a - \varepsilon < x$.
Since $x \in \mathbb{I}_s$, choose $K \in \mathbb{D}_s$ s.t. $x = s_K$. Then $K \in \mathbb{D}_s = \mathbb{N}$.
Want: $\forall j \in \mathbb{N}$, $(\, j \geqslant K\,) \Rightarrow (\, |s_J - a| < \varepsilon\,)$.
Given $j \in \mathbb{N}$. Assume $j \geqslant K$. **Want:** $|s_J - a| < \varepsilon$.
Want: $a - \varepsilon < s_j < a + \varepsilon$.
Since $s_j \in \mathbb{I}_s \leqslant \sup \mathbb{I}_s = a < a + \varepsilon$, we get $s_j < a + \varepsilon$.
Want: $a - \varepsilon < s_j$.
Since s is semi-increasing, since $j, K \in \mathbb{N} = \mathbb{D}_s$ and since $j \geqslant K$,
it follows that $s_j \geqslant s_K$.
Then $s_j \geqslant s_K = x$, so $x \leqslant s_j$. Then $a - \varepsilon < x \leqslant s_j$. ☐

THEOREM 2.2.17. *Let* $s \in \mathbb{R}^{\mathbb{N}}$.
Assume: s *is semi-decreasing* *and* $\mathbb{I}_s$ *is bounded below.*
Then: $s \to \inf \mathbb{I}_S$.

Proof. Unassigned HW. □

THEOREM 2.2.18. *Let* $s \in \mathbb{R}^{\mathbb{N}}$.
Assume: s *is semi-increasing* *and* $\mathbb{I}_s$ *is not bounded above.*
Then: $s \to \infty$.

Proof. **Want:** $\forall M \in \mathbb{R}$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\, j \geqslant K \,) \Rightarrow (\, s_j > M \,).$$
Given $M \in \mathbb{R}$. **Want:** $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\, j \geqslant K \,) \Rightarrow (\, s_j > M \,).$$
Since $\mathbb{I}_s$ is not bounded above, we get: $\neg(\mathbb{I}_s \leqslant M)$.
Choose $a \in \mathbb{I}_s$ s.t. $a > M$.
Since $a \in \mathbb{I}_s$, choose $K \in \mathbb{D}_s$ s.t. $a = s_K$.
Since $s \in \mathbb{R}^{\mathbb{N}}$, we get $\mathbb{D}_s = \mathbb{N}$. Then $K \in \mathbb{D}_s = \mathbb{N}$.
Want: $\forall j \in \mathbb{N}$, $(\, j \geqslant K \,) \Rightarrow (\, s_j > M \,)$.
Given $j \in \mathbb{N}$. Assume $j \geqslant K$. **Want:** $s_j > M$.
By hypothesis, s is semi-increasing, so, since $j \geqslant K$, we get: $s_j \geqslant s_K$.
By choice of K, we get: $s_K = a$. Then $s_j \geqslant s_K = a$. □

THEOREM 2.2.19. *Let* $s \in \mathbb{R}^{\mathbb{N}}$.
Assume: s *is semi-decreasing* *and* $\mathbb{I}_s$ *is not bounded below.*
Then: $s \to -\infty$.

Proof. Unassigned HW. □

THEOREM 2.2.20. *Let* $s \in \mathbb{R}^{\mathbb{N}}$.
Assume s *is semi-monotone and* $\mathbb{I}_s$ *is bounded. Then* s *is convergent.*

Proof. At least one of the following must be true:
(1) s is semi-increasing or (2) s is semi-decreasing.

Case (1): By Theorem 2.2.16, s is convergent. *End of Case (1).*

Case (2): By Theorem 2.2.17, s is convergent. *End of Case (2).* □

THEOREM 2.2.21. *Let* s *be a sequence,* $k \in \mathbb{N}^{\mathbb{N}}$.
$$\textit{Then } s \circ k \;=\; (s_{k_1}, s_{k_2}, s_{k_3}, \ldots).$$

THEOREM 2.2.22.
$$(1, 1/2, 1/3, \ldots) \circ (1, 4, 9, 16, \ldots) \;=\; (1, 1/4, 1/9, 1/16, \ldots).$$

THEOREM 2.2.23. *$\forall$sequences s,*

$$s \circ (1, 4, 9, 16, \ldots) \;=\; (s_1, s_4, s_9, s_{16}, \ldots).$$

In the next theorem, we consider the sequence $k : \mathbb{N} \to \mathbb{R}$ defined by:

$$k_1 = 4 \quad \text{and} \quad k_2 = 1 \quad \text{and} \quad \forall j \in [3..\infty), \;\; k_j = j^2.$$

Note that $k = (4, 1, 9, 16, \ldots)$.

THEOREM 2.2.24. *$\forall$sequences s,*

$$s \circ (4, 1, 9, 16, \ldots) \;=\; (s_4, s_1, s_9, s_{16}, \ldots).$$

THEOREM 2.2.25.

$$(1, 1/2, 1/3, \ldots) \circ (4, 1, 9, 16, \ldots) \;=\; (1/4, 1, 1/9, 1/16, \ldots).$$

THEOREM 2.2.26.

$$(1, -1, 1, -1, -1, 1, -1, \ldots) \circ (1, 3, 5, 7, \ldots) \;=\; (1, 1, 1, 1, \ldots).$$

DEFINITION 2.2.27. *Let s and t be sequences.*

By t is a **subsequence** *of s, we mean:*

$\exists$strictly-increasing $k \in \mathbb{N}^{\mathbb{N}}$ s.t. $t = s \circ k$.

THEOREM 2.2.28.

$(1, 1/4, 1/9, 1/16, \ldots)$ *is a subsequence of* $(1, 1/2, 1/3, \ldots)$.

$(1/4, 1, 1/9, 1/16, \ldots)$ *is NOT a subsequence of* $(1, 1/2, 1/3, \ldots)$.

$(1, 1, 1, \ldots)$ *is a subsequence of* $(1, -1, 1, -1, 1, -1, 1, -1, \ldots)$.

THEOREM 2.2.29. *Let s be a sequence.*

Then s is a subsequence of s.

Proof. Since $\mathrm{id}^{\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ and $\mathrm{id}^{\mathbb{N}}$ is strictly-increasing and $s \circ \mathrm{id}^{\mathbb{N}} = s$,
we conclude that: s is a subsequence of s. $\square$

THEOREM 2.2.30. *Let s be a sequence and let t be a subsequence of s.*

Then $\mathbb{I}_t \subseteq \mathbb{I}_s$.

Proof. Choose a strictly-increasing $\ell \in \mathbb{N}^{\mathbb{N}}$ s.t. $t = s \circ \ell$.
Then $\mathbb{I}_t = \mathbb{I}_{s \circ \ell} \subseteq \mathbb{I}_s$. $\square$

THEOREM 2.2.31. *Let A be a set, $s \in \mathbb{A}^{\mathbb{N}}$.*
Let t be a subsequence of s. Then $t \in \mathbb{A}^{\mathbb{N}}$.

Proof. Choose a strictly-increasing $\ell \in \mathbb{N}^{\mathbb{N}}$ s.t. $t = s \circ \ell$.
Since $\ell : \mathbb{N} \to \mathbb{N}$ and $s : \mathbb{N} \to A$, we get $s \circ \ell : \mathbb{N} \to A$.
So, since $t = s \circ \ell$, we get $t : \mathbb{N} \to A$. Then $t \in A^{\mathbb{N}}$. $\square$

DEFINITION 2.2.32. *Let* $s \in \mathbb{R}^{\mathbb{N}}$.

By s *is* **convergent**, *we mean:* $\exists a \in \mathbb{R}$ *s.t.* $s \to a$.

THEOREM 2.2.33.

$(1, 1/2, 1/3, \ldots)$ *is convergent.*

$(1, -1, 1, -1, 1, -1, \ldots)$ *is not convergent.*

The following is left as unassigned HW:

THEOREM 2.2.34. $\forall s \in \mathbb{R}^{\mathbb{N}}$,

$(\ s \to \infty\ ) \ \Rightarrow\ (\ s$ *is not convergent* $)$.

THEOREM 2.2.35.

$(2, 4, 6, 8, 10, 12, \ldots)$ *is NOT convergent.*

While the set of extended reals $\mathbb{R}^*$ does not have a standard "distance", as $\mathbb{R}$ does, it does have a standard topology, if you happen to know what that means. We have:

$(2, 4, 6, 8, 10, 12, \ldots)$ is convergent in $\mathbb{R}^*$, but NOT in $\mathbb{R}$ and

$(1, -1, 1, -1, 1, -1, \ldots)$ is NEITHER convergent in $\mathbb{R}^*$ NOR in $\mathbb{R}$.

For any $s \in \mathbb{R}^{\mathbb{N}}$, if we say s is convergent, and if we want to be completely clear, we should say "in $\mathbb{R}$" or "in $\mathbb{R}^*$"; however, in this course we will always mean "in $\mathbb{R}$". We do not even assume the reader knows what a topological space is, so "convergent in $\mathbb{R}^*$" is not defined.

THEOREM 2.2.36. *Let* s, t, u *be sequences.*

Assume: u *is a subsequence of* t *and* t *is a subsequence of* s.

Then: u *is a subsequence of* s.

Proof. Since u is a subsequence of t,

choose a strictly-increasing $\ell \in \mathbb{N}^{\mathbb{N}}$ s.t. $u = t \circ \ell$.

Since t is a subsequence of s,

choose a strictly-increasing $k \in \mathbb{N}^{\mathbb{N}}$ s.t. $t = s \circ k$.

Since $k \circ \ell \in \mathbb{N}^{\mathbb{N}}$ and $k \circ \ell$ is strictly-increasing and $u = s \circ k \circ \ell$,

we conclude that: u is a subsequence of s. □

THEOREM 2.2.37. *Let* $k \in \mathbb{N}^{\mathbb{N}}$. *Assume* k *is strictly increasing.*

Then: $\forall j \in \mathbb{N}, \quad k_j \geqslant j$.

An informal proof is as follows:

Since k is increasing, we have: $k_1 < k_2 < k_3 < \ldots$.

Want: $k_1 \geqslant 1$, $k_2 \geqslant 2$, $k_3 \geqslant 3$, *etc.*

We have: $k_1 \in \mathbb{N} \geqslant 1$, so $k_1 \geqslant 1$.

Then $k_2 > k_1 \geqslant 1$ and $k_2 \in \mathbb{N}$, so $k_2 \in (1..\infty) \geqslant 2$.
Then $k_3 > k_2 \geqslant 2$ and $k_3 \in \mathbb{N}$, so $k_3 \in (2..\infty) \geqslant 3$.
Then $k_4 > k_3 \geqslant 3$ and $k_4 \in \mathbb{N}$, so $k_4 \in (3..\infty) \geqslant 4$.
Etc.

A formal proof, by math induction, is left as unassigned HW.

THEOREM 2.2.38. *Let* $s, t \in \mathbb{R}^{\mathbb{N}}$, $a \in \mathbb{R}$.
Assume: $s \to a$ *and* t *is a subsequence of* s.
Then: $t \to a$.

Proof. **Want:** $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$(\, j \geqslant K \,) \Rightarrow (\, |t_j - a| < \varepsilon \,)$.
Given $\varepsilon > 0$. **Want:** $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$(\, j \geqslant K \,) \Rightarrow (\, |t_j - a| < \varepsilon \,)$.
Since $s \to a$, choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$(\, j \geqslant K \,) \Rightarrow (\, |s_j - a| < \varepsilon \,)$.
Then $K \in \mathbb{N}$. **Want:** $\forall j \in \mathbb{N}$, $(\, j \geqslant K \,) \Rightarrow (\, |t_j - a| < \varepsilon \,)$.
Given $j \in \mathbb{N}$. Assume $j \geqslant K$. **Want:** $|t_j - a| < \varepsilon$.
Since t is a subsequence of s,
choose a strictly-increasing $\ell \in \mathbb{N}^{\mathbb{N}}$ s.t. $t = s \circ \ell$.
By Theorem 2.2.37, $\ell_j \geqslant j$.
Since $\ell_j \geqslant j \geqslant K$, by the choice of K, we have $|s_{\ell_j} - a| < \varepsilon$.
Then $|t_j - a| = |(s \circ \ell)_j - a| = |s_{\ell_j} - a| < \varepsilon$. □

The proof of the following two theorems are both
similar to that of the preceding theorem.
They are left as unassigned HWs.

THEOREM 2.2.39. *Let* $s, t \in \mathbb{R}^{\mathbb{N}}$.
Assume: $s \to \infty$ *and* t *is a subsequence of* s.
Then: $t \to \infty$.

THEOREM 2.2.40. *Let* $s, t \in \mathbb{R}^{\mathbb{N}}$.
Assume: $s \to -\infty$ *and* t *is a subsequence of* s.
Then: $t \to -\infty$.

Because of the following three theorems, we know:
Let $s, t \in \mathbb{R}^{\mathbb{N}}$, $a \in \mathbb{R}^*$.
Assume: $s \to a$ and t is a subsequence of s.
Then: $t \to a$.
It is natural to wonder if there might be some way to prove this,

without breaking the proof up into three cases:

$$a = -\infty \quad \text{or} \quad a \in \mathbb{R} \quad \text{or} \quad a = \infty.$$

There IS such a proof, but it requires the reader to understand the basics of topology.

DEFINITION 2.2.41. *Let $s \in \mathbb{R}^{\mathbb{N}}$.*
By s is **subconvergent***, we mean:*
$\exists$ *subsequence t of s s.t. t is convergent.*

In this course, for any $s \in \mathbb{R}^{\mathbb{N}}$, saying
s is subconvergent
will always mean
s is subconvergent in $\mathbb{R}$.
By Theorem 2.2.36, convergent implies subconvergent.

THEOREM 2.2.42. $\forall$ *convergent $s \in \mathbb{R}^{\mathbb{N}}$, s is subconvergent.*

THEOREM 2.2.43. $(1, 1/2, 1/3, \ldots)$ *is subconvergent.*

THEOREM 2.2.44.
$(1, -1, 1, -1, 1, -1, 1, -1, \ldots)$ *is subconvergent, but NOT convergent.*

THEOREM 2.2.45. $(2, 4, 6, 8, \ldots)$ *is NOT subconvergent.*

THEOREM 2.2.46. $(1, 2, 1, 4, 1, 6, 1, 8, \ldots)$ *is subconvergent.*

DEFINITION 2.2.47. *Let $X \subseteq \mathbb{R}$, $s \in X^{\mathbb{N}}$.*
By s is **convergent in** *X, we mean:*
$\exists z \in X$ *s.t.* $s \to z$.

THEOREM 2.2.48. $(1, 1/2, 1/3, \ldots)$ *is convergent in* $[0; 1]$.

THEOREM 2.2.49. $(1, 1/2, 1/3, \ldots)$ *is NOT convergent in* $(0; 1]$.

DEFINITION 2.2.50. *Let $X \subseteq \mathbb{R}$, $s \in X^{\mathbb{N}}$.*
By s is **subconvergent in** *X, we mean:*
$\exists$*subsequence t of s s.t. t is convergent in X.*

By Theorem 2.2.36, convergent in X implies subconvergent in X.

THEOREM 2.2.51.
$(1, -1, 1, -1, 1, -1, 1, -1, \ldots)$ *is subconvergent in* $[-1; 1]$.

THEOREM 2.2.52. $(1, 1/2, 1/3, \ldots)$ *is subconvergent in* $[0; 1]$.

THEOREM 2.2.53. $(1, 1/2, 1/3, \ldots)$ *is NOT subconvergent in* $(0; 1]$.

DEFINITION 2.2.54. *Let $K \subseteq \mathbb{R}$.*

By K is **compact***, we mean:*

$\forall s \in K^{\mathbb{N}}$, *$s$ is subconvergent in K.*

DEFINITION 2.2.55. *Let $a \in \mathbb{R}$, $r \geqslant 0$.*

Then $\boxed{\overline{B}(a,r)} := \{ x \in \mathbb{R} \text{ s.t. } |x-a| \leqslant r \}$.

In the preceding definition, $\overline{B}(a,r)$ is called

the **closed ball** about a of radius r.

Note: $\forall a \in \mathbb{R}, \forall r \geqslant 0, \quad \overline{B}(a,r) = [a-r; a+r]$.

Then: $\forall r \geqslant 0, \quad \overline{B}(0,r) = [-r; r]$.

THEOREM 2.2.56. *Let $X \subseteq \mathbb{R}$.*

Assume X is compact. Then X is bounded.

Proof. Assume X is not bounded. **Want:** Contradiction.

Claim 1: $\forall j \in \mathbb{N}, \quad X \backslash (\overline{B}(0,j)) \neq \varnothing$.

Proof of Claim 1: Given $j \in \mathbb{N}$. **Want:** $X \backslash (\overline{B}(0,j)) \neq \varnothing$.

Since $\overline{B}(0,j) \subseteq B(0,j+1)$, we see that $\overline{B}(0,j)$ is bounded.

Since $\overline{B}(0,j)$ is bounded and X is not bounded, $X \nsubseteq (\overline{B}(0,j))$.

Then $\exists x$ s.t. both $x \in X$ and $x \notin \overline{B}(0,j)$.

Then $\exists x$ s.t. $x \in X \backslash (\overline{B}(0,j))$. Then $X \backslash (\overline{B}(0,j)) \neq \varnothing$.

End of proof of Claim 1.

By Claim 1, $\forall j \in \mathbb{N}, \quad X \backslash (\overline{B}(0,j)) \neq \varnothing$.

Define $s \in X^{\mathbb{N}}$ by: $\forall j \in \mathbb{N}, \quad s_j = \mathrm{CH}_{X \backslash (\overline{B}(0,j))}$.

Since $s \in X^{\mathbb{N}}$ and X is compact, s is subconvergent in X.

Choose a subsequence t of s s.t. t is convergent in X.

Then t is convergent, and so $\mathbb{I}_t$ is bounded.

By Theorem 1.32.17, choose $r > 0$ s.t. $\mathbb{I}_t \subseteq B(0,r)$.

By the AP, choose $j \in \mathbb{N}$ s.t. $j > r$.

Since t is a subsequence of s,

choose a strictly-increasing $\ell \in \mathbb{N}^{\mathbb{N}}$ s.t. $t = s \circ \ell$.

Then $\ell_j \geqslant j$. Let $m := \ell_j$. Then $m \geqslant j$.

Claim 2: $|s_m| > m$.

Proof of Claim 2: By Claim 1, $X \backslash \overline{B}(0,m) \neq \varnothing$.

By definition of s, we have $s_m = \mathrm{CH}_{X \backslash (\overline{B}(0,m))}$.

Then $s_m \in X \backslash (\overline{B}(0,m))$.
Then $\neg(s_m \in \overline{B}(0,m))$. Then $\neg(|s_m| \leqslant m)$. Then $|s_m| > m$.
End of proof of Claim 2.

We have $t_j \in \mathbb{I}_t \subseteq B(0,r)$, so $|t_j| < r$.
We have $s_m = s_{\ell_j} = (s \circ \ell)_j = t_j$, and so $|t_j| = |s_m|$.
By Claim 2, $|s_m| > m$. Then $|t_j| = |s_m| > m \geqslant j > r$, so $r < |t_j|$.
Then $r < |t_j| < r$, so $r < r$. Contradiction. □

2.3. **Maximizing compact subsets of $\mathbb{R}$.**

THEOREM 2.3.1. *Let $L \subseteq \mathbb{R}$.*
Assume L is compact. Then L is bounded above.

Proof. This follows from Theorem 2.2.56. □

By HW#7-3 (The Squeeze Theorem), we have:

THEOREM 2.3.2. *Let $t \in \mathbb{R}^{\mathbb{N}}$, $a \in \mathbb{R}$.*
Assume: $\forall j \in \mathbb{N}$, $a - (1/j) \leqslant t_j \leqslant a$.
Then: $t \to a$.

We restate the theorem with t replaced by s:

THEOREM 2.3.3. *Let $s \in \mathbb{R}^{\mathbb{N}}$, $a \in \mathbb{R}$.*
Assume: $\forall j \in \mathbb{N}$, $a - (1/j) \leqslant s_j \leqslant a$.
Then: $s \to a$.

THEOREM 2.3.4. *Let $L \subseteq \mathbb{R}$.*
Assume L is compact and nonempty. Then $\max L \neq$ ☹.

Proof. Since L is compact, L is bounded.
Since L is bounded and nonempty, $\sup L \in \mathbb{R}$.
Let $a := \sup L$. Then $a \in \mathbb{R}$. **Want:** $\max L = a$.
Want: $a \in L$ and $a \geqslant L$.
We have $a = \sup L \geqslant L$. **Want:** $a \in L$.
For all $j \in \mathbb{N}$, let $X_j := L \cap (\, a - (1/j)\, ;\, \infty\,)$.

Claim 1: $\forall j \in \mathbb{N}$, $X_j \neq \varnothing$.
Proof of Claim 1: Given $j \in \mathbb{N}$. **Want:** $X_j \neq \varnothing$.
Because $a - (1/j) < a = \sup L$,
we get $\neg(a - (1/j) \geqslant \sup L)$,
and so $\neg(a - (1/j) \geqslant L)$,

and so $\exists x \in L$ s.t. $a - (1/j) < x$,
and so $\exists x \in L$ s.t. $x \in (a - (1/j); \infty)$,
and so $L \cap (a - (1/j); \infty) \neq \varnothing$,
and so $X_j \neq \varnothing$.
End of proof of Claim 1.

Define $s \in L^{\mathbb{N}}$ by: $\forall j \in \mathbb{N}$, $s_j = \mathrm{CH}_{X_j}$.

Claim 2: $\forall j \in \mathbb{N}$, $a - (1/j) \leqslant s_j \leqslant a$.
Proof of Claim 2: Given $j \in \mathbb{N}$. **Want:** $a - (1/j) \leqslant s_j \leqslant a$.
We have $s_j \in X_j = L \cap (a-(1/j); \infty) \subseteq (a-(1/j); \infty) > a-(1/j)$.
Want: $s_j \leqslant a$. $s_j \in L \leqslant \sup L = a$.
End of proof of Claim 2.

By Claim 2 and the Squeeze Theorem, we know that $s \to a$.
Since $s \in L^{\mathbb{N}}$ and since L is compact,
it follows that s is subconvergent in L.
Choose a subsequence t of s such that t is convergent in L.
Choose $b \in L$ s.t. $t \to b$.
Since $s \to a$ and t is a subsequence of s,
it follows that $t \to b$.
Since $t \to a$ and $\to b$, it follows that $a = b$.
Then $a = b \in L$. □

2.4. **Sums of sequences.**

THEOREM 2.4.1. *Let* $a \in [0; \infty)^{\mathbb{N}}$.
Define $s \in \mathbb{R}^{\mathbb{N}}$ *by:* $\forall j \in \mathbb{N}$, $s_j = a_1 + \cdots + a_j$.
Then: *(i) if* $\mathbb{I}_s$ *is bounded, then* s *is convergent* *and*
(ii) if $\mathbb{I}_s$ *is unbounded, then* $s \to \infty$.

Proof. Part (i) follows from Theorem 2.2.16.
Part (ii) follows from Theorem 2.2.18. □

2.5. **Cauchy sequences and convergence.**

DEFINITION 2.5.1. *Let* $s \in \mathbb{R}^{\mathbb{N}}$. *By* s *is* **Cauchy**, *we mean:*

$$\forall \varepsilon > 0,\ \exists K \in \mathbb{N} \text{ s.t., } \forall i, j \in \mathbb{N},$$
$$(i, j \geqslant K) \Rightarrow (|s_i - s_j| < \varepsilon).$$

We sometimes say:

"s is Cauchy iff,
for every $\varepsilon > 0$, s has an strictly-ε-bounded tail."

More accurately:

"s is Cauchy iff,
for every $\varepsilon > 0$, s has a tail whose image is strictly-ε-bounded."

THEOREM 2.5.2. *Let $s \in \mathbb{R}^{\mathbb{N}}$. Assume s is Cauchy. Then $\mathbb{I}_s$ is bounded.*

Proof. Choose $K \in \mathbb{N}$ s.t., $\forall i, j \in \mathbb{N}$, $(\, i, j \geqslant K\,) \Rightarrow (\, |s_i - s_j| < 1\,)$.
Then: $\forall i \in \mathbb{N}$, $(\, i \geqslant K\,) \Rightarrow (\, |s_i - s_K| < 1\,)$.
Then: $\forall i \in \{K, K+1, K+2, \ldots\}$, $|s_i - s_K| < 1$.
Then: $\forall i \in \{K, K+1, K+2, \ldots\}$, $s_i \in B(s_K, 1)$.
Then $\{s_K, s_{K+1}, s_{K+2}, \ldots\} \subseteq B(s_K, 1)$.
Then $\{s_K, s_{K+1}, s_{K+2}, \ldots\}$ is bounded.
Also, $\{s_1, \ldots, s_K\}$ is finite, and therefore bounded.
Then $\{s_1, \ldots, s_K\} \cup \{s_K, s_{K+1}, s_{K+2}, \ldots\}$ is bounded.
So, since $\mathbb{I}_s = \{s_1, s_2, s_3, \ldots\} = \{s_1, \ldots, s_K\} \cup \{s_K, s_{K+1}, s_{K+2}, \ldots\}$,
we conclude that $\mathbb{I}_s$ is bounded. □

THEOREM 2.5.3. *Let $s \in \mathbb{R}^{\mathbb{N}}$. Then:*

$$(\ s \text{ is Cauchy}\ ) \Leftrightarrow (\ s \text{ is convergent}\ ).$$

Proof. This is HW#8-3 and HW#8-5. □

3. Continuity and limits of functions $\mathbb{R} \to \mathbb{R}$

3.1. Continuity of functions $\mathbb{R} \dashrightarrow \mathbb{R}$.

We next define **continuity of function $\mathbb{R} \dashrightarrow \mathbb{R}$ at a point**:

DEFINITION 3.1.1. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{D}_f$.*
By f is **continuous** *at a, we mean:*

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ s.t.}, \forall x \in \mathbb{D}_f,\quad (\, |x - a| < \delta\,) \Rightarrow (\, |f_x - f_a| < \varepsilon\,).$$

It is our convention that, for any function f, for any object a,
if $a \notin \mathbb{D}_f$, then f is NOT continuous at a.

THEOREM 3.1.2. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{D}_f$, $\varepsilon > 0$, $\delta > 0$.*
Then:

$$(\ \forall x \in \mathbb{D}_f,\ (\, |x - a| < \delta\,) \Rightarrow (\, |f_x - f_a| < \varepsilon\,)\ ) \Leftrightarrow (\ f_*(B(a, \delta)) \subseteq B(f_a, \varepsilon)\ ).$$

THEOREM 3.1.3. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{D}_f$.
Then: (f *is continuous at* a) $\Leftrightarrow$
($\forall \varepsilon > 0$, $\exists \delta > 0$ *s.t.* $f_*(B(a,\delta)) \subseteq B(f_a, \varepsilon)$).

THEOREM 3.1.4. *Define* $f : \mathbb{R} \to \mathbb{R}$ *by:* $\forall x \in \mathbb{R}$, $f_x = x^2$.
Then $\forall a \in \mathbb{R}$, f *is continuous at* a.

Proof. Given $a \in \mathbb{R}$. **Want:** f is continuous at a.
Want: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall x \in \mathbb{D}_f$,
$$(|x-a| < \delta) \Rightarrow (|f_x - f_a| < \varepsilon).$$
Given $\varepsilon > 0$. **Want:** $\exists \delta > 0$ s.t., $\forall x \in \mathbb{D}_f$,
$$(|x-a| < \delta) \Rightarrow (|f_x - f_a| < \varepsilon).$$
Let $\delta := \min\{ 1 , \varepsilon/(2 + 2 \cdot |a|) \}$. Then $\delta > 0$.
Want: $\forall x \in \mathbb{D}_f$, $(|x-a| < \delta) \Rightarrow (|f_x - f_a| < \varepsilon)$.
Given $x \in \mathbb{D}_f$. Assume $|x-a| < \delta$. **Want:** $|f_x - f_a| < \varepsilon$).
We have $\delta \leqslant 1$ and $\delta \cdot (1 + 2 \cdot |a|) < \varepsilon$.
We have $|f_x - f_a| = |x^2 - a^2| = |(x-a)(x+a)|$
$= |x-a| \cdot |x+a| \leqslant \delta \cdot (|x| + |a|)$.
Also, $|x| = |x - a + a| \leqslant |x-a| + |a| < \delta + |a|$,
so $|x| + |a| < \delta + 2 \cdot |a|$. So, since $\delta \leqslant 1$, we get $|x| + |a| < 1 + 2 \cdot |a|$.
Then $|x^2 - a^2| \leqslant \delta \cdot (|x| + |a|) \leqslant \delta(1 + 2 \cdot |a|) < \varepsilon$. □

THEOREM 3.1.5. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{D}_f$, $c \in \mathbb{R}$.
Assume f *is continuous at* a. *Then* $c \cdot f$ *is continuous at* a.

Proof. **Want:** $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall x \in \mathbb{D}_{c \cdot f}$,
$$(|x-a| < \delta) \Rightarrow (|(c \cdot f)_x - (c \cdot f)_a| < \varepsilon).$$
Given $\varepsilon > 0$. **Want:** $\exists \delta > 0$ s.t., $\forall x \in \mathbb{D}_{c \cdot f}$,
$$(|x-a| < \delta) \Rightarrow (|(c \cdot f)_x - (c \cdot f)_a| < \varepsilon).$$
Let $\rho := \varepsilon/(1 + |c|)$. Then $\rho > 0$ and $|c| \cdot \rho < \varepsilon$.
Since f is continuous at a, choose $\delta > 0$ s.t., $\forall x \in \mathbb{D}_f$,
$$(|x-a| < \delta) \Rightarrow (|f_x - f_a| < \rho).$$
Then $\delta > 0$. **Want:** $\forall x \in \mathbb{D}_{c \cdot f}$, $(|x-a| < \delta) \Rightarrow (|(c \cdot f)_x - (c \cdot f)_a| < \varepsilon)$.
Given $x \in \mathbb{D}_{c \cdot f}$. Assume $|x-a| < \delta$. **Want:** $|(c \cdot f)_x - (c \cdot f)_a| < \varepsilon$.
We have $x \in \mathbb{D}_{c \cdot f} = \mathbb{D}_f$ and $|x-a| < \delta$,
so, by the choice of δ, we get: $|f_x - f_a| < \rho$.
Then
$$\begin{aligned} |(c \cdot f)_x - (c \cdot f)_a| &= |c \cdot f_x - c \cdot f_a| \\ &= |c \cdot (f_x - f_a)| \\ &= |c| \cdot |f_x - f_a| \leqslant |c| \cdot \rho < \varepsilon. \end{aligned}$$
□

The following is HW#5-1:

THEOREM 3.1.6. *Let* $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$.
Assume f *and* g *are both continuous at* a.
Then $f + g$ *is continuous at* a.

THEOREM 3.1.7. *Let* $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$.
Assume f *and* g *are both continuous at* a.
Then $f \cdot g$ *is continuous at* a.

Proof. **Want:** $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall x \in \mathbb{D}_{f\cdot g}$,
$$(\,|x-a| < \delta\,) \Rightarrow (\,|(f\cdot g)_x - (f\cdot g)_a| < \varepsilon\,).$$
Given $\varepsilon > 0$. **Want:** $\exists \delta > 0$ s.t., $\forall x \in \mathbb{D}_{f\cdot g}$,
$$(\,|x-a| < \delta\,) \Rightarrow (\,|(f\cdot g)_x - (f\cdot g)_a| < \varepsilon\,).$$
Let $\rho := \min\{\,1\,,\, \varepsilon/(\,|g_a| + |f_a| + 2\,)\,\}$.
Then $\rho > 0$ and $\rho \leqslant 1$ and $\rho \cdot (\,|g_a| + |f_a| + 1\,) < \varepsilon$.
Since f is continuous at a, choose $\lambda > 0$ s.t., $\forall x \in \mathbb{D}_f$,
$$(\,|x-a| < \lambda\,) \Rightarrow (\,|f_x - f_a| < \rho\,).$$
Since g is continuous at a, choose $\mu > 0$ s.t., $\forall x \in \mathbb{D}_f$,
$$(\,|x-a| < \mu\,) \Rightarrow (\,|g_x - g_a| < \rho\,).$$
Let $\delta := \min\{\lambda, \mu\}$. Then $\delta > 0$.
Want: $\forall x \in \mathbb{D}_{f\cdot g}$, $(\,|x-a| < \delta\,) \Rightarrow (\,|(f\cdot g)_x - (f\cdot g)_a| < \varepsilon\,)$.
Given $x \in \mathbb{D}_{f\cdot g}$. Assume $|x-a| < \delta$. **Want:** $|(f\cdot g)_x - (f\cdot g)_a| < \varepsilon$.
We have $\delta \leqslant \lambda$ and $\delta \leqslant \mu$.
We have $x \in \mathbb{D}_{f\cdot g} = \mathbb{D}_f \bigcap \mathbb{D}_g \subseteq \mathbb{D}_f$ and $|x-a| < \delta \leqslant \lambda$,
so, by the choice of λ, we get: $|f_x - f_a| < \rho$.
We have $x \in \mathbb{D}_{f\cdot g} = \mathbb{D}_f \bigcap \mathbb{D}_g \subseteq \mathbb{D}_g$ and $|x-a| < \delta \leqslant \mu$,
so, by the choice of μ, we get: $|g_x - g_a| < \rho$.
By the Naive Product Rule,
$$f_x \cdot g_x - f_a \cdot g_a = (f_x - f_a)\cdot g_a + f_a \cdot (g_x - g_a) + (f_x - f_a)\cdot(g_x - g_a).$$
Recall: $\rho \cdot (|g_a| + |f_a| + 1) < \varepsilon$.
Since $\rho \leqslant 1$, we get $\rho \cdot (\,|g_a| + |f_a| + \rho\,) \leqslant \rho \cdot (\,|g_a| + |f_a| + 1\,)$.
Then
$$\begin{aligned} |\,(f\cdot g)_x - (f\cdot g)_a\,| &= |\,f_x \cdot g_x - f_a \cdot g_a\,| \\ &= |(f_x - f_a)\cdot g_a + f_a \cdot (g_x - g_a) + (f_x - f_a)\cdot(g_x - g_a)| \\ &\leqslant |f_x - f_a|\cdot|g_a| + |f_a|\cdot|g_x - g_a| + |f_x - f_a|\cdot|g_x - g_a| \\ &\leqslant \rho\cdot|g_a| + |f_a|\cdot\rho + \rho\cdot\rho \\ &= \rho\cdot(\,|g_a| + |f_a| + \rho\,) \leqslant \rho\cdot(\,|g_a| + |f_a| + 1\,) < \varepsilon. \end{aligned}$$ □

Unassigned HW:

THEOREM 3.1.8. *Let* $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$.

Then: $(\ a \in \mathbb{D}_{g \circ f}\ ) \Leftrightarrow (\ f_a \in \mathbb{D}_g\ )$.

The following is HW#5-2:

THEOREM 3.1.9. *Let* $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$.

Assume: f *is continuous at* a *and* g *is continuous at* f_a.

Then $g \circ f$ *is continuous at* a.

We next define **continuity of function** $\mathbb{R} \dashrightarrow \mathbb{R}$ **on a subset of** $\mathbb{R}$:

DEFINITION 3.1.10. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $S \subseteq \mathbb{R}$.

By f *is* **continuous** *on* S, *we mean:*

$\forall a \in S$, f *is continuous at* a.

We next define **continuity of function** $\mathbb{R} \dashrightarrow \mathbb{R}$:

DEFINITION 3.1.11. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$.

By f *is* **continuous**, *we mean:*

f *is continuous on* $\mathbb{D}_f$.

THEOREM 3.1.12. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $S \subseteq \mathbb{D}_f$.

Assume f *is continuous on* S. *Then* $f|S$ *is continuous.*

Proof. **Want:** $\forall a \in \mathbb{D}_{f|S}$, $f|S$ is continuous at a.
Given $a \in \mathbb{D}_{f|S}$. **Want:** $f|S$ is continuous at a.
Want: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall x \in \mathbb{D}_{f|S}$,

$$(\ |x - a| < \delta\ ) \Rightarrow (\ |(f|S)_x - (f|S)_a| < \varepsilon\ ).$$

Given $\varepsilon > 0$. **Want:** $\exists \delta > 0$ s.t., $\forall x \in \mathbb{D}_{f|S}$,

$$(\ |x - a| < \delta\ ) \Rightarrow (\ |(f|S)_x - (f|S)_a| < \varepsilon\ ).$$

Since $a \in \mathbb{D}_{f|S} = S$ and since f is continuous on S,
it follows that f is continuous at a.
Then choose $\delta > 0$ s.t., $\forall x \in \mathbb{D}_f$, $(\ |x - a| < \delta\ ) \Rightarrow (\ |f_x - f_a| < \varepsilon\ )$.
Then $\delta > 0$.
Want: $\forall x \in \mathbb{D}_{f|S}$, $(\ |x - a| < \delta\ ) \Rightarrow (\ |(f|S)_x - (f|S)_a| < \varepsilon\ )$.
Given $x \in \mathbb{D}_{f|S}$. Assume $|x - a| < \delta$. **Want:** $|(f|S)_x - (f|S)_a| < \varepsilon$.
We have $\mathbb{D}_{f|S} = S$. By assumption, $S \subseteq \mathbb{D}_f$. Then $x \in \mathbb{D}_{f|S} = S \subseteq \mathbb{D}_f$.
So, as $|x - a| < \delta$, by choice of δ, we get: $|f_x - f_a| < \varepsilon$.
Since $x, a \in \mathbb{D}_{f|S} = S$, we get $(f|S)_x = f_x$ and $(f|S)_a = f_a$.
Then $|(f|S)_x - (f|S)_a| = |f_x - f_a| < \varepsilon$. □

The following is HW#5-3:

THEOREM 3.1.13. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $s \in (D_f)^{\mathbb{N}}$, $a \in \mathbb{R}$.

Assume f *is continuous at* a *and* $s \to a$. *Then* $f \circ s \to f_a$.

3.2. **Lipschitz and uniformaly continuous functions $\mathbb{R} \dashrightarrow \mathbb{R}$.**

The next two definitions explain **Lipschitz, for functions $\mathbb{R} \dashrightarrow \mathbb{R}$.**

DEFINITION 3.2.1. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $L \geqslant 0$.*
*By f is **L-Lipschitz**, we mean:*

$$\forall w, x, \in \mathbb{D}_f, \quad |f_x - f_w| \leqslant L|x - w|.$$

DEFINITION 3.2.2. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$.*
*By f is **Lipschitz**, we mean:*

$$\exists L \geqslant 0 \text{ s.t. } f \text{ is } L\text{-Lipschitz.}$$

The next two theorems are left as unassigned exercises.
The first asserts that we need not check w and x when $w = x$.
The second asserts that it's okay to lable the smaller of the two as w.

THEOREM 3.2.3. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $L \geqslant 0$.*
Then: f *is* **L-Lipschitz** *if and only if*

$$\forall w, x, \in \mathbb{D}_f, \quad (\, w \neq x \,) \Rightarrow (\, |f_x - f_w| \leqslant L|x - w| \,).$$

THEOREM 3.2.4. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $L \geqslant 0$.*
Then: f *is* **L-Lipschitz** *if and only if*

$$\forall w, x, \in \mathbb{D}_f, \quad (\, w < x \,) \Rightarrow (\, |f_x - f_w| \leqslant L|x - w| \,).$$

We record a quadruply quantified equivalence to continuity:

THEOREM 3.2.5. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$.*
By f is continuous *if and only if*

$$\forall \varepsilon > 0, \forall w \in \mathbb{D}_f, \exists \delta > 0 \text{ s.t.}, \forall x \in \mathbb{D}_f,$$
$$(\, |x - w| < \delta \,) \Rightarrow (\, |f_x - f_w| < \varepsilon \,).$$

The preceding theorem is left as a unassigned HW.
The next definition covers

uniformly continuous, for functions $\mathbb{R} \dashrightarrow \mathbb{R}$.

DEFINITION 3.2.6. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$.*
By f is **uniformly continuous** *we mean:*

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t.}, \forall w \in \mathbb{D}_f, \forall x \in \mathbb{D}_f,$$
$$(\, |x - w| < \delta \,) \Rightarrow (\, |f_x - f_w| < \varepsilon \,).$$

Note that
the quantified equivalence for continuity
is similiar to
the definition of uniform continuity;

only the order of the quantified causes has been changed.

In the preceding definition, we sometimes replace

"$\forall w \in \mathbb{D}_f$, $\forall x \in \mathbb{D}_f$" by "$\forall w, x \in \mathbb{D}_f$",

for brevity. The reader must remember that

the single $\forall$ in "$\forall w, x \in \mathbb{D}_f$" counts twice.

We sometimes abbreviate uniformly continuous by **u.c.**

The following is HW#5-4:

THEOREM 3.2.7. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$.*
Assume f is Lipschitz. Then f is uniformly continuous.

The following is HW#5-5:

THEOREM 3.2.8. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$.*
Assume f is uniformly continuous. Then f is continuous.

Thus, Lipschitz $\Rightarrow$ u.c. $\Rightarrow$ continuous.
We eventually show that neither of these implications can be reversed:

DEFINITION 3.2.9. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $w, x \in \mathbb{D}_f$. Assume $w \neq x$.*

$$\text{Then } \boxed{\mathrm{DQ}_f(w,x)} := \frac{f_x - f_w}{x - w}.$$

Note that $\mathrm{DQ}_f(w,x)$ is equal to

the slope of the secant line between $(w, f(w))$ and $(x, f(x))$.

DEFINITION 3.2.10. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$.*

$$\text{Then } \boxed{\mathrm{DQ}_f} := \{\, \mathrm{DQ}_f(w,x) \,|\, (\, w, x \in \mathbb{D}_f \,) \,\&\, (\, w \neq x \,) \,\}.$$

Thus DQ_f collects all of the slopes of secant lines for the graph of f.

THEOREM 3.2.11. *Let $f : \mathbb{R} \to \mathbb{R}$, $L \geqslant 0$.*
Then: f is L-Lipschitz if and only if

$$\forall w, x, \in \mathbb{D}_f, \quad (\, w \neq x \,) \Rightarrow (\, |\mathrm{DQ}_f(w,x)| \leqslant L \,).$$

THEOREM 3.2.12. *Let $f : \mathbb{R} \to \mathbb{R}$, $L \geqslant 0$.*
Then: f is L-Lipschitz if and only if

$$\forall w, x, \in \mathbb{D}_f, \quad (\, w \neq x \,) \Rightarrow (\, -L \leqslant \mathrm{DQ}_f(w,x) \leqslant L \,).$$

THEOREM 3.2.13. *Let $f : \mathbb{R} \to \mathbb{R}$, $L \geqslant 0$.*
Then: f is L-Lipschitz if and only if $-L \leqslant \mathrm{DQ}_f \leqslant L$.

That is, a function $\mathbb{R} \dashrightarrow \mathbb{R}$ is L-Lipschitz iff
the slopes of its secant lines are bounded above and below.

THEOREM 3.2.14. *Define* $f : \mathbb{R} \to \mathbb{R}$ *by* $\forall x \in \mathbb{R},\ f_x = \dfrac{x}{\sqrt{1+x^2}}$.
Then f *is* 1*-Lipschitz.*

An examination of the graph of the function f above indicates that
all secant line slopes are strictly between 0 and 1,
or, in other words,

$$0 < \mathrm{DQ}_f < 1.$$

Proving this formally is left as an exercise for the reader.

By contrast, for the squaring function, the graph is a parabola, so
its secant line slopes are neither bounded above nor below.
This indicates that the squaring function is not Lipschitz;
in fact it is not even uniformly continuous:

THEOREM 3.2.15. *Define* $f : \mathbb{R} \to \mathbb{R}$ *by* $\forall x \in \mathbb{R},\ f_x = x^2$.
Then f *is NOT uniformly continuous.*

Proof. Assume that f is uniformly continuous. **Want:** Contradiction.
Since f is uniformly continuous, choose $\delta > 0$ s.t., $\forall w, x \in \mathbb{D}_f$,

$$(\,|x - w| < \delta\,) \Rightarrow (\,|f_x - f_w| < 1\,).$$

Let $w := 1/\delta$ and let $x := w + (\delta/2)$. Then $w > 0$.
Then $w, x \in \mathbb{R} = \mathbb{D}_f$ and $|x - w| = |\delta/2| = \delta/2 < \delta$,
so, by choice of δ, we get $|f_x - f_w| < 1$.
Since $w > 0$ and $\delta > 0$, we get

$$w \cdot \delta + (\delta/2)^2 \quad > \quad 0$$

and so $\quad |w \cdot \delta + (\delta/2)^2| \quad = \quad w \cdot \delta + (\delta/2)^2$.

Then

$$\begin{aligned} 1 > |f_x - f_w| &= |x^2 - w^2| = |(w + (\delta/2))^2 - w^2| \\ &= |w^2 + 2 \cdot w \cdot (\delta/2) + (\delta/2)^2 - w^2| \\ &= |w \cdot \delta + (\delta/2)^2| = w \cdot \delta + (\delta/2)^2 \\ &> w \cdot \delta = (1/\delta) \cdot \delta = 1, \end{aligned}$$

so $1 > 1$. Contradiction. □

Define $f : \mathbb{R} \to \mathbb{R}$ by $\forall x \in \mathbb{R},\ f_x = x^2$.
By Theorem 3.1.4, f is continuous.
However, according to the preceding theorem,
f is not uniformly continuous.

We therefore see that continuous does not imply uniformly continuous.

Now define $f : \mathbb{R} \to \mathbb{R}$ by $\forall x \in \mathbb{R}$, $f_x = \sqrt[3]{x}$.
We will argue that f is uniformly continuous, but not Lipschitz.
The formal proof that f is not Lipschitz will be left to the reader,
but an examination of the graph of f will show that
if w is close to zero, then $\mathrm{DQ}_f(-w, w)$ is very large.
In fact the slopes of secant lines are not bounded above.
We will supply a formal proof that f is uniformly continuous:

THEOREM 3.2.16. *Define $f : \mathbb{R} \to \mathbb{R}$ by $\forall x \in \mathbb{R}$, $f_x = \sqrt[3]{x}$. Then f is uniformly continuous.*

Proof. **Want:** $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall w, x \in \mathbb{D}_f$,
$$(\,|x - w| < \delta\,) \Rightarrow (\,|f_x - f_w| < \varepsilon\,).$$
Given $\varepsilon > 0$. **Want:** $\exists \delta > 0$ s.t., $\forall w, x \in \mathbb{D}_f$,
$$(\,|x - w| < \delta\,) \Rightarrow (\,|f_x - f_w| < \varepsilon\,).$$
Let $\delta := \varepsilon^2/8$. Then $\delta > 0$.
Want: $\forall w, x \in \mathbb{D}_f$, $(\,|x - w| < \delta\,) \Rightarrow (\,|f_x - f_w| < \varepsilon\,)$.
Given $w, x \in \mathbb{D}_f$. Assume $|x - w| < \delta$. **Want:** $|f_x - f_w| < \varepsilon$.
We have $0 \leqslant |x - w| < \varepsilon^2/8$.
Let $s := \min\{w, x\}$ and $t := \max\{w, x\}$.
Since $s \leqslant t$, we get $\sqrt[3]{s} \leqslant \sqrt[3]{t}$, and so $f_s \leqslant f_t$.
Then $|w - x| = t - s$ and $|f_x - f_w| = f_t - f_s$.
Then $0 \leqslant t - s < \varepsilon^2/8$. **Want:** $f_t - f_s < \varepsilon$.
Let $\sigma := f_s$ and $\tau := f_t$. **Want:** $\tau - \sigma < \varepsilon$.
Assume $\tau - \sigma > \varepsilon$. **Want:** Contradiction.
Since $\tau \geqslant \sigma + \varepsilon$, we get $\tau^3 \geqslant (\sigma + \varepsilon)^3$.
Then $\tau^3 \geqslant \sigma^3 + 3\sigma^2\varepsilon + 3\sigma\varepsilon^2 + \varepsilon^3$.
Then $\tau^3 - \sigma^3 \geqslant 3\sigma^2\varepsilon + 3\sigma\varepsilon^2 + \varepsilon^3$.
Then $3\sigma^2\varepsilon + 3\sigma\varepsilon^2 + \varepsilon^3 \leqslant \tau^3 - \sigma^3 = t - s < \varepsilon^3/8$.
Then $3\sigma^2\varepsilon + 3\sigma\varepsilon^2 + \varepsilon^3 - (\varepsilon^3/8) < 0$.
Then $3\sigma^2\varepsilon + 3\sigma\varepsilon^2 + (7/8)\varepsilon^3 < 0$.
As $3\varepsilon^3 > 0$, dividing by $3\varepsilon^3$, we get: $\dfrac{3\sigma^2\varepsilon}{3\varepsilon^3} + \dfrac{3\sigma\varepsilon^2}{3\varepsilon^3} + \dfrac{7}{8} \cdot \dfrac{\varepsilon^3}{3\varepsilon^3} < 0$.

Then $\dfrac{\sigma^2}{\varepsilon^2} + \dfrac{\sigma}{\varepsilon} + \dfrac{7}{24} < 0$. Then $\left(\dfrac{\sigma^2}{\varepsilon^2} + 2 \cdot \dfrac{\sigma}{\varepsilon} \cdot \dfrac{1}{2} + \dfrac{1}{4}\right) - \dfrac{1}{4} + \dfrac{7}{24} < 0$.

We have $0 \leqslant \left(\dfrac{\sigma}{\varepsilon} + \dfrac{1}{2}\right)^2$, so $-\dfrac{1}{4} + \dfrac{7}{24} \leqslant \left(\dfrac{\sigma}{\varepsilon} + \dfrac{1}{2}\right)^2 - \dfrac{1}{4} + \dfrac{7}{24}$.

Then $\displaystyle\frac{1}{24} = -\frac{6}{24} + \frac{7}{24} = -\frac{1}{4} + \frac{7}{24} \leqslant \left(\frac{\sigma}{\varepsilon} + \frac{1}{2}\right)^2 - \frac{1}{4} + \frac{7}{24}$

$$= \left(\frac{\sigma^2}{\varepsilon^2} + 2 \cdot \frac{\sigma}{\varepsilon} \cdot \frac{1}{2} + \frac{1}{4}\right) - \frac{1}{4} + \frac{7}{24} < 0.$$

Then $\displaystyle\frac{1}{24} < 0$. Contradiction. □

3.3. **Continuity and topological preimages.**

THEOREM 3.3.1. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$.
Assume: $\forall$*closed* $C \subseteq \mathbb{R}$, *we have:* f^*C *is closed.*
Then: f *is continuous.*

Proof. **Want:** $\forall a \in \mathbb{D}_f$, f is continuous at a.
Given $a \in \mathbb{D}_f$. **Want:** f is continuous at a.
Want: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall x \in \mathbb{D}_f$,
$(\,|x - a| < \delta\,) \Rightarrow (\,|f_x - f_a| < \varepsilon\,)$.
Given $\varepsilon > 0$. **Want:** $\exists \delta > 0$ s.t., $\forall x \in \mathbb{D}_f$,
$(\,|x - a| < \delta\,) \Rightarrow (\,|f_x - f_a| < \varepsilon\,)$.
By the Subset Recentering Theorem $B(f_a, \varepsilon)$ is open.
Let $C := \mathbb{R}\backslash(B(f_a, \varepsilon))$. Then C is closed.
So, by assumption, f^*C is closed. Then $\mathbb{R}\backslash(f^*C)$ is open.
We have $|f_a - f_a| = 0 < \varepsilon$, so $f_a \in B(f_a, \varepsilon)$, so $f_a \notin C$, so $a \notin f^*C$.
We have $a \in \mathbb{D}_f \subseteq \mathbb{R}$ and $a \notin f^*C$, so $a \in \mathbb{R}\backslash(f^*C)$.
So, since $\mathbb{R}\backslash(f^*C)$ is open, by a class theorem,
choose $\delta > 0$ s.t. $B(a, \delta) \subseteq \mathbb{R}\backslash(f^*C)$. Then $\delta > 0$.
Want: $\forall x \in \mathbb{D}_f$, $(\,|x - a| < \delta\,) \Rightarrow (\,|f_x - f_a| < \varepsilon\,)$.
Given $x \in \mathbb{D}_f$. Assume: $|x - a| < \delta$. **Want:** $|f_x - f_a| < \varepsilon$.
Since $|x - a| < \delta$, we get: $x \in B(a, \delta)$.
Then $x \in B(a, \delta) \subseteq \mathbb{R}\backslash(f^*C)$. Then $x \notin f^*C$, and so $f_x \notin C$.
We have $x \in \mathbb{D}_f$, so $f_x \in \mathbb{I}_f$. Since $f : \mathbb{R} \dashrightarrow \mathbb{R}$, we get: $\mathbb{I}_f \subseteq \mathbb{R}$.
Since $B(f_a, \varepsilon) \subseteq \mathbb{R}$, we get: $\mathbb{R}\backslash(\mathbb{R}\backslash(B(f_a, \varepsilon))) = B(f_a, \varepsilon)$.
Since $f_x \in \mathbb{I}_f \subseteq \mathbb{R}$ and since $f_x \notin C = \mathbb{R}\backslash(B(f_a, \varepsilon))$,
we conclude: $f_x \in \mathbb{R}\backslash(\mathbb{R}\backslash(B(f_a, \varepsilon)))$.
Then $f_x \in \mathbb{R}\backslash(\mathbb{R}\backslash(B(f_a, \varepsilon))) = B(f_a, \varepsilon)$. Then $|f_x - f_a| < \varepsilon$. □

THEOREM 3.3.2. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$.
Assume: $\forall$*closed* $C \subseteq \mathbb{R}$, *we have:* f^*C *is closed.*
Then: f *is continuous.*

Proof. This is HW#11-5. □

3.4. **The Intermediate Value Theorem.**

Recall: $\forall S \subseteq \mathbb{R}^*$, $(\, S \leqslant \sup S \leqslant \mathrm{UB}_S \,) \,\&\, (\, \mathrm{LB}_S \leqslant \inf S \leqslant S \,)$.

THEOREM 3.4.1. *Let* $a \in \mathbb{R}$, $b \geqslant a$, $S \subseteq [a;b]$.
Assume $S \neq \varnothing$. *Then* $\sup S \in [a;b]$.

Proof. **Want:** $b \geqslant \sup S \geqslant a$.
We have $b \geqslant [a;b] \supseteq S$, so $b \in \mathrm{UB}_S$. Then $b \in \mathrm{UB}_S \geqslant \sup S$.
Want: $\sup S \geqslant a$.
Since $S \subseteq [a;b] \geqslant a$, we get $S \geqslant a$.
Then $\sup S \geqslant S \geqslant a$, so, since $S \neq \varnothing$, we get $\sup S \geqslant a$. □

THEOREM 3.4.2. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a, b, c, v \in \mathbb{R}$.
Assume $a < b$ *and* $[a;b] \subseteq \mathbb{D}_f$. *Assume* $c \in [a;b)$ *and* $f_c < v$.
Assume f *is continuous at* c.
Then $\exists \delta > 0$ *s.t.,* $\forall x \in [c; c+\delta)$, $(\; x \in [a;b] \;) \;\&\; (\; f_x < v \;)$.

Proof. Let $\varepsilon := v - f_c$. Since $f_c < v$, we get: $\varepsilon > 0$.
Then, since f is continuous at c, choose $\alpha > 0$ s.t., $\forall x \in \mathbb{D}_f$,
$$(\, |x - c| < \alpha \,) \Rightarrow (\, |f_x - f_c| < \varepsilon \,).$$
Since $c \in [a;b) < b$, we get $b - c > 0$.
Let $\delta := \min\{\alpha, b - c\}$. Then $\delta > 0$.
Want: $\forall x \in [c; c+\delta)$, $(\; x \in [a;b] \;) \;\&\; (\; f_x < v \;)$.
Given $x \in [c; c+\delta)$. **Want:** $(\; x \in [a;b] \;) \;\&\; (\; f_x < v \;)$.
We have $x \in [c; c+\delta) \geqslant c \in [a;b) \geqslant a$, so $x \geqslant a$, so $a \leqslant x$.
We have $\delta = \min\{\alpha, b-c\} \leqslant b - c$, so $\delta \leqslant b - c$, so $c + \delta \leqslant b$.
Then $x \in [c; c+\delta) < c + \delta \leqslant b$, so $x < b$.
Since $a \leqslant x$ and $x < b$, we see that $x \in [a;b)$. Then $x \in [a;b) \subseteq [a;b]$.

It remains to show: $f_x < v$.

Since $x \in [c; c+\delta)$, we get both $c \leqslant x$ and $x < c + \delta$.
Since $\delta > 0$, we get $c - \delta < c$.
Since $c - \delta < c \leqslant x$, we get $c - \delta < x$.
Then $c - \delta < x < c + \delta$, and so $|x - c| < \delta$.

By definition of δ, we have $\delta \leqslant \alpha$ and $\delta \leqslant b - c$.
By hypothesis, $[a;b] \subseteq \mathbb{D}_f$.

Since $x \in [a;b] \subseteq \mathbb{D}_f$, and $|x - c| < \delta \leqslant \alpha$, by choice of α,
we conclude that $|f_x - f_c| < \varepsilon$, and so $f_c - \varepsilon < f_x < f_c + \varepsilon$.
Then $f_x < f_c + \varepsilon = f_c + (v - f_c) = v$. □

The proof of the following is similar to the proof of the preceding. It is left as unassigned HW.

THEOREM 3.4.3. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a, b, c, v \in \mathbb{R}$.
Assume $a < b$ *and* $[a;b] \subseteq \mathbb{D}_f$. *Assume* $c \in (a;b]$ *and* $v < f_c$.
Assume f *is continuous at* c.
Then $\exists \delta > 0$ *s.t.*, $\forall x \in (c - \delta; c]$, $(\, x \in [a;b] \,) \; \& \; (\, v < f_x \,)$.

THEOREM 3.4.4. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a, b, v \in \mathbb{R}$.
Assume $a < b$. *Assume* f *is continuous on* $[a;b]$. *Assume* $f_a \leqslant v \leqslant f_b$.
Then $\exists c \in [a;b]$ *s.t.* $f_c = v$.

Proof. Exactly one of the following holds:
(A) $v = f_a$ or (B) $v = f_b$ or (C) $f_a < v < f_b$.

Case (A): Let $c := a$. Then $c \in [a;b]$. **Want:** $f_c = v$.
We have $f_c = f_a = v$.
End of Case (A).

Case (B): Let $c := b$. Then $c \in [a;b]$. **Want:** $f_c = v$.
We have $f_c = f_b = v$.
End of Case (B).

Case (C): Let $S := \{x \in [a;b] \mid f_x < v\}$. Then $c = \sup S$.
By hypothesis $f_a < v$. Since $a \in [a;b]$ and $f_a < v$, we get $a \in S$.
Then $S \neq \varnothing$. Then, by Theorem 3.4.1, $\sup S \in [a;b]$.
Then $c = \sup S \in [a;b]$. **Want:** $f_c = v$.

We wish to show: (1) $f_c \geqslant v$ and (2) $f_c \leqslant v$.

Proof of (1): Assume $f_c < v$. **Want:** Contradiction.
We have $f_c < v$. By hypothesis, $v < f_b$.
Then $f_c < v < f_b$, so $f_c \neq f_b$, so $c \neq b$.
So, since $c \in [a;b]$, we get $c \in [a;b)$.
Then, by Theorem 3.4.2, choose $\delta > 0$ s.t., $\forall x \in [c; c + \delta)$,

($x \in [a;b]$) & ($f_x < v$).
Then $[c;c+\delta) \subseteq \{x \in [a;b] \,|\, f_x < v\}$.
Then $c + (\delta/2) \in [c;c+\delta) \subseteq \{x \in [a;b] \,|\, f_x < v\} = S \leqslant \sup S = c$.
Then $c + (\delta/2) \leqslant c$, so $\delta/2 \leqslant 0$, so $\delta \leqslant 0$.
Then $0 < \delta \leqslant 0$, so $0 < 0$. Contradiction.
End of proof of (1).

Proof of (2): Assume $f_c > v$. **Want:** Contradiction.
We have $v < f_c$. By hypothesis, $f_a < v$.
Then $f_a < v < f_c$, so $f_a \neq f_c$, so $a \neq c$.
So, since $c \in [a;b]$, we get $c \in (a;b]$.
Then, by Theorem 3.4.3, choose $\delta > 0$ s.t., $\forall x \in (c-\delta;c]$,
($x \in [a;b]$) & ($v < f_x$).
Since $\delta > 0$, we get $c - \delta < c$.
Since $c - \delta < c = \sup S = \min \mathrm{UB}_S$, we get $c - \delta < \min \mathrm{UB}_S$.
Then $c - \delta \notin \mathrm{UB}_S$, so $\neg(S \leqslant c - \delta)$, so $\neg(\forall x \in S,\, x \leqslant c - \delta)$.
Then choose $x \in S$ s.t. $x > c - \delta$.
Since $x \in S$, by definition of S, we get $f_x < v$.
Since $x \in S \leqslant \sup S = c$, we get $x \leqslant c$.
Since $x > c - \delta$, we get $c - \delta < x$.
Since $c - \delta < x \leqslant c$, we get $x \in (c-\delta;c]$.
Then, by choice of δ, we get ($x \in [a;b]$) & ($v < f_x$).
Then $v < f_x < v$, so $v < v$. Contradiction.
End of proof of (2).

End of Case (C). □

THEOREM 3.4.5. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a, b, v \in \mathbb{R}$.
Assume $a < b$. *Assume* f *is continuous on* $[a;b]$. *Assume* $f_a \geqslant v \geqslant f_b$.
Then $\exists c \in [a;b]$ *s.t.* $f_c = v$.

Proof. Let $g := -f$. Then $g_a \leqslant -v \leqslant g_b$.
By Theorem 3.4.4, choose $c \in [a;b]$ s.t. $g_c = -v$.
Then $c \in [a;b]$. **Want:** $f_c = v$.
Since $g = -f$, it follows that $g_c = (-f)_c$.
Then $f_c = -(-f_c) = -((-f)_c) = -g_c = -(-v) = v$. □

DEFINITION 3.4.6. *$\forall a, b \in \mathbb{R}^*$, we define:*

$$\boxed{[a|b]} \;:=\; [\,\min\{a,b\}\,;\,\max\{a,b\}\,]$$

and

$$\boxed{(a|b)} \;:=\; (\,\min\{a,b\}\,;\,\max\{a,b\}\,).$$

THEOREM 3.4.7. $[7|1] = [1;7] = [1|7]$ *and* $(7|1) = (1;7) = (1|7)$.

THEOREM 3.4.8. *$\forall a, b \in \mathbb{R}^*$,* $(\,[a|b] = [b|a]\,)$ & $(\,(a|b) = (b|a)\,)$.

THEOREM 3.4.9. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $\alpha, \beta, v \in \mathbb{R}$.
Assume $\alpha < \beta$. Assume f is continuous on $[\alpha;\beta]$. Assume $v \in [f_\alpha|f_\beta]$.
Then $\exists c \in [\alpha;\beta]$ s.t. $f_c = v$.

Proof. At least one of the following is true:
(1) $f_\alpha \leqslant f_\beta$ or (2) $f_\beta \leqslant f_\alpha$.

Case (1): Since $f_\alpha \leqslant f_\beta$, we get $[f_\alpha|f_\beta] = [f_\alpha;f_\beta]$.
Then $v \in [f_\alpha|f_\beta] = [f_\alpha;f_\beta]$, so $f_\alpha \leqslant v \leqslant f_\beta$.
Then by Theorem 3.4.4, $\exists c \in [\alpha;\beta]$ s.t. $f_c = v$.
End of Case (1).

Case (2): Since $f_\beta \leqslant f_\alpha$, we get $[f_\alpha|f_\beta] = [f_\beta;f_\alpha]$.
Then $v \in [f_\alpha|f_\beta] = [f_\beta;f_\alpha]$, so $f_\beta \leqslant v \leqslant f_\alpha$.
Then by Theorem 3.4.5, $\exists c \in [\alpha;\beta]$ s.t. $f_c = v$.
End of Case (2). □

The following is the **Intermediate Value Theorem** or **IVT**:

THEOREM 3.4.10. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a, b \in \mathbb{R}$.
Assume f is continuous on $[a|b]$. Then $[f_a|f_b] \subseteq f_*([a|b])$.

Proof. **Want:** $\forall v \in [f_a|f_b]$, $v \in f_*([a|b])$.
Given $v \in [f_a|f_b]$. **Want:** $v \in f_*([a|b])$.
Want: $\exists c \in [a|b]$ s.t. $f_c = v$.
Exactly one the following is true:
(1) $a = b$ or (2) $a < b$ or (3) $b < a$.

Case (1): $v \in [f_a|f_b] = [f_a|f_a] = \{f_a\}$, so $v = f_a$.
Let $c := a$. Then $c \in [a|b]$. **Want:** $f_c = v$.
We have $f_c = f_a = v$.
End of Case (1).

Case (2): Since $a < b$, we get $[a|b] = [a;b]$.
Let $\alpha := a$, $\beta := b$.
Then $\alpha < \beta$, Also, $[\alpha;\beta] = [a;b] = [a|b]$, so $[\alpha;\beta] = [a|b]$.
Want: $\exists c \in [\alpha;\beta]$ s.t. $f_c = v$.
Then f is continuous on $[\alpha;\beta]$ and $v \in [f_a|f_b] = [f_\alpha|f_\beta]$.
Then, by Theorem 3.4.9, $\exists c \in [\alpha;\beta]$ s.t. $f_c = v$.
End of Case (2).

Case (3): Since $b < a$, we get $[a|b] = [b;a]$.
Let $\alpha := b$, $\beta := a$.
Then $\alpha < \beta$. Also, $[\alpha;\beta] = [b;a] = [a|b]$, so $[\alpha;\beta] = [a|b]$.
Want: $\exists c \in [\alpha;\beta]$ s.t. $f_c = v$.
Then f is continuous on $[a|b]$ and $[\alpha;\beta] = [a|b]$, so f is continuous on $[\alpha;\beta]$. So, since $v \in [f_a|f_b] = [f_b|f_a] = [f_\alpha|f_\beta]$,
by Theorem 3.4.9, $\exists c \in [\alpha;\beta]$ s.t. $f_c = v$.
End of Case (3). □

A **power function on** $\mathbb{R}$ is a power of $\mathrm{id}^{\mathbb{R}}$,
e.g., $x \mapsto x^4 : \mathbb{R} \to \mathbb{R}$.
A **monomial on** $\mathbb{R}$ is a scalar multiple of a power function on $\mathbb{R}$,
e.g., $x \mapsto 7x^4 : \mathbb{R} \to \mathbb{R}$.
The function f defined in the next proof is an example of
a **polynomial on** $\mathbb{R}$,
i.e., a finite sum of monomials on $\mathbb{R}$,
e.g., $x \mapsto 7x^4 + 2x^3 - 5x + 8 : \mathbb{R} \to \mathbb{R}$.
We leave it as an exercise to show that any polynomial is continuous.

THEOREM 3.4.11. *$\forall a \in \mathbb{R}$, $\exists x \in \mathbb{R}$ s.t. $x^5 - x^3 + x = a$.*

Proof. Given $a \in \mathbb{R}$. **Want:** $\exists x \in \mathbb{R}$ s.t. $x^5 - x^3 + x = a$.
We have: $\forall t \in \mathbb{R}$, $-|t| \leqslant t \leqslant |t|$. Then $-|a| \leqslant a \leqslant |a|$.
Define $f : \mathbb{R} \to \mathbb{R}$ by $\forall x \in \mathbb{R}$, $f_x = x^5 - x^3 + x$.
Then f is a polynomial on $\mathbb{R}$, and so f is continuous.
Want: $\exists x \in \mathbb{R}$ s.t. $f_x = a$. **Want:** $a \in \mathbb{I}_f$.
Let $b := \max\{1, |a|\}$. Then $b \geqslant 1$.
Also, $b \geqslant |a|$. Negating this, we get $-b \leqslant -|a|$.
Since $b \geqslant 1$, we get $b^5 \geqslant b^3$, and so $b^5 - b^3 \geqslant 0$.
Then $b^5 - b^3 + b \geqslant b$. Negating this, we get $-b^5 + b^3 - b \leqslant -b$.
Then $f_b = b^5 - b^3 + b \geqslant b \geqslant |a| \geqslant a$,

so $a \leqslant f_b$.
Also, $f_{-b} = (-b)^5 - (-b)^3 + (-b) = -b^5 + b^3 - b \leqslant -b \leqslant -|a| \leqslant a$,
so $f_{-b} \leqslant a$.
Then $f_{-b} \leqslant a \leqslant f_b$, so $a \in [f_{-b}|f_b]$.
By the IVT, $[f_{-b}|f_b] \subseteq f_*[-b|b]$.
For any function g, for any set S, $g_*S \subseteq \mathbb{I}_g$. Then $f_*[-b|b] \subseteq \mathbb{I}_f$.
Then $a \in [f_{-b}|f_b] \subseteq f_*[-b|b] \subseteq \mathbb{I}_f$. □

3.5. **Limits at extended real numbers of functions $\mathbb{R} \dashrightarrow \mathbb{R}$.**

DEFINITION 3.5.1. *Let $a \in \mathbb{R}$, $\delta > 0$. Then* $\boxed{B^\times(a,\delta)} := (B(a,\delta))_a^\times$.

The set $B^\times(a,\delta)$ is called
the **punctured open ball** about a of radius δ.

THEOREM 3.5.2. *Let $a \in \mathbb{R}$, $\delta > 0$. Then:*

$$(\ x \in B^\times(a,\delta)\ ) \ \Leftrightarrow\ (\ 0 < |x-a| < \delta\ ).$$

DEFINITION 3.5.3. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a, z \in \mathbb{R}$.*
By **as** $x \to a$, $f_x \to z$, *we mean:*

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ s.t., } \forall x \in \mathbb{D}_f,$$
$$(\,0 < |x-a| < \delta\,) \Rightarrow (\,|f_x - z| < \varepsilon\,).$$

Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a, z \in \mathbb{R}$, $\delta, \varepsilon > 0$. Then the quantified statement

$$\forall x \in \mathbb{D}_f, \qquad (\,0 < |x-a| < \delta\,) \Rightarrow (\,|f_x - z| < \varepsilon\,)$$

is equivalent to

$$f_*(B^\times(a,\delta)) \ \subseteq\ B(z,\varepsilon).$$

DEFINITION 3.5.4. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$.*
By **as** $x \to a$, $f_x \to \infty$, *we mean:*

$$\forall M \in \mathbb{R},\ \exists \delta > 0 \text{ s.t., } \forall x \in \mathbb{D}_f,$$
$$(\,0 < |x-a| < \delta\,) \Rightarrow (\,f_x > M\,).$$

DEFINITION 3.5.5. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$.*
By **as** $x \to a$, $f_x \to -\infty$, *we mean:*

$$\forall N \in \mathbb{R},\ \exists \delta > 0 \text{ s.t., } \forall x \in \mathbb{D}_f,$$
$$(\,0 < |x-a| < \delta\,) \Rightarrow (\,f_x < N\,).$$

DEFINITION 3.5.6. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $z \in \mathbb{R}$.*
By **as** $x \to -\infty$, $f_x \to z$, *we mean:*

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{R} \text{ s.t., } \forall x \in \mathbb{D}_f,$$
$$(\,x < N\,) \Rightarrow (\,|f_x - z| < \varepsilon\,).$$

DEFINITION 3.5.7. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$.
By **as** $x \to \infty$**,** $f_x \to \infty$, *we mean:*
$\forall M \in \mathbb{R}$, $\exists L \in \mathbb{R}$ *s.t.*, $\forall x \in \mathbb{D}_f$,
$(\, x > L\,) \Rightarrow (\, f_x > M\,)$.

DEFINITION 3.5.8. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$.
By **as** $x \to \infty$**,** $f_x \to -\infty$, *we mean:*
$\forall N \in \mathbb{R}$, $\exists L \in \mathbb{R}$ *s.t.*, $\forall x \in \mathbb{D}_f$,
$(\, x > L\,) \Rightarrow (\, f_x < N\,)$.

THEOREM 3.5.9. *Let* $f : \mathbb{R} \to \mathbb{R}$, $y \in \mathbb{R}_0^\times$.
Assume: *as* $x \to -\infty$, $f_x \to y$.
Then: *as* $x \to -\infty$, $(1/f)_x \to (1/y)$.

Proof. This is HW#12-3. □

3.6. **Forward image of a compact set.**

THEOREM 3.6.1. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $K \subseteq \mathbb{D}_f$.
Assume f *is continuous on* K *and* K *is compact.*
Then f_*K *is compact.*

Proof. **Want:** $\forall s \in (f_*K)^{\mathbb{N}}$, s is subconvergent in f_*K.
Given $s \in (f_*K)^{\mathbb{N}}$. **Want:** s is subconvergent in f_*K.
Want: $\exists$subsequence t of s s.t. t is convergent in f_*K.
For all $j \in \mathbb{N}$, let $A_j := (f^*\{s_j\}) \cap K$.

Claim 1: $\forall j \in \mathbb{N}$, $A_j \neq \varnothing$.
Proof of Claim 1: Given $j \in \mathbb{N}$. Want $A_j \neq \varnothing$.
Since $s \in (f_*K)^{\mathbb{N}}$, we get $s_j \in f_*K$,
so choose $x \in K$ s.t. $s_j = f_x$.
Since $f_x = s_j \in \{s_j\}$, we get $x \in f^*\{s_j\}$.
So, since $x \in K$, we get $x \in (f^*\{s_j\}) \cap K$.
So, since $A_j := (f^*\{s_j\}) \cap K$, we get $x \in A_j$. Then $A_j \neq \varnothing$.
End of proof of Claim 1.

By definition of A_j, we have: $\forall j \in \mathbb{N}$, $A_j \subseteq K$.
By the Claim, we have: $\forall j \in \mathbb{N}$, $A_j \neq \varnothing$.
Define $\sigma \in K^{\mathbb{N}}$ by $\forall j \in \mathbb{N}$, $\sigma_j = \mathrm{CH}_{A_j}$.
By hypothesis K is compact, so σ is subconvergent in K.

Choose a subsequence τ of σ s.t. τ is convergent in K.
Choose a strictly-increasing $\ell \in \mathbb{N}^{\mathbb{N}}$ s.t. $\tau = \sigma \circ \ell$.
Let $t := s \circ \ell$. Then t is a subsequence of s.
Want: t is convergent in $f_* K$.

Claim 2: $f \circ \sigma = s$.
Proof of Claim 2: **Want:** $\forall j \in \mathbb{N}$, $(f \circ \sigma)_j = s_j$.
Given $j \in \mathbb{N}$. **Want:** $(f \circ \sigma)_j = s_j$.
We have $\sigma_j \in A_j = (f^*\{s_j\}) \cap K \subseteq f^*\{s_j\}$, so $\sigma_j \in f^*\{s_j\}$, so $f_{\sigma_j} \in \{s_j\}$.
Then $f_{\sigma_j} = s_j$. Then $(f \circ \sigma)_j = f_{\sigma_j} = s_j$.
End of proof of Claim 2.

By Claim 2, $f \circ \sigma = s$. Then $f \circ \sigma \circ \ell = s \circ \ell$.
So since $\tau = \sigma \circ \ell$ and $t = s \circ \ell$, we get $f \circ \tau = t$.
Since τ is convergent in K, choose $\xi \in K$ s.t. $\tau \to \xi$.
By hypothesis f is continuous on K. Then f is continuous at ξ.
Then, by HW#6-2, $f \circ \tau \to f_\xi$.
So, since $f \circ \tau = t$, we get $t \to f_\xi$.
Since $\xi \in K$ and $K \subseteq \mathbb{D}_f$, we get $f_\xi \in f_* K$.
So, since $t \to f_\xi$, we see that t is convergent in $f_* K$. □

3.7. **Semi-monotone subsequences of real-valued sequences.**

Note that, in Case (1) of the proof of the following theorem,
ℓ is the strict-forward-orbit of $\min P$ under f
and that, in Case (2) of the proof of the following theorem,
ℓ is the strict-forward-orbit of $(\max P) + 1$ under f.

THEOREM 3.7.1. *Let $s \in \mathbb{R}^{\mathbb{N}}$.*
Then $\exists$subsequence t of s s.t. t is semi-monotone.

Proof. Let $P := \{j \in \mathbb{N} \,|\, \forall k \in (j..\infty),\ s_k < s_j\}$.
Then $P \subseteq \mathbb{N}$, so $P^{\mathbb{N}} \subseteq \mathbb{N}^{\mathbb{N}}$.
Exactly one of the following is true:
(1) P is infinite or (2) P is finite.

Case (1): Since P is infinite, we get: $\forall j \in \mathbb{N}$, $\varnothing \neq P\backslash[1..j] \subseteq \mathbb{N}$.
So, by the Well-Ordering Axiom, we get: $\forall j \in \mathbb{N}$, $\min(P\backslash[1..j]) \neq$ ☹.
Define $f : P \to P$ by: $\forall j \in \mathbb{N}$, $f(j) = \min(P\backslash[1..j])$.

Then, $\forall j \in \mathbb{N}$, $f(j) \in P\backslash[1..j]$.
Define $\ell \in P^{\mathbb{N}}$ by: $\forall j \in \mathbb{N}$, $\ell_j = f_\circ^j(\min P)$.
Then, $\forall j \in \mathbb{N}$, we have $f(\ell_j) = \ell_{j+1}$.
Then, $\forall j \in \mathbb{N}$, we have $\ell_{j+1} = f(\ell_j) \in P\backslash[1..\ell_j]$.

Claim A: ℓ is strictly increasing.
Proof of Claim A: **Want:** $\forall j \in \mathbb{N}$, $\ell_{j+1} > \ell_j$.
Given $j \in \mathbb{N}$. **Want:** $\ell_{j+1} > \ell_j$.
We have $\ell_{j+1} \in P\backslash[1..\ell_j] \subseteq \mathbb{N}\backslash[1..\ell_j] = (\ell_j..\infty) > \ell_j$.
End of proof of Claim A.

We have $\ell \in P^{\mathbb{N}} \subseteq \mathbb{N}^{\mathbb{N}}$, so, by Claim A,
$s \circ \ell$ is a subsequence of s.
Let $t := s \circ \ell$. Then t is a subsequence of s.
Want: t is semi-monotone.
It suffices to show: t is semi-decreasing.
It suffices to show: t is strictly-decreasing.
Want: $\forall j \in \mathbb{N}$, $t_{j+1} < t_j$.
Given $j \in \mathbb{N}$. **Want:** $t_{j+1} < t_j$.
By Claim A, $\ell_{j+1} > \ell_j$. Since $\ell \in P^{\mathbb{N}}$, we get $\ell_{j+1} \in P$.
Since $\ell_{j+1} > \ell_j$ and $\ell_{j+1} \in P \subseteq \mathbb{N}$, we get: $\ell_{j+1} \in (\ell_j..\infty)$.
Since $\ell \in P^{\mathbb{N}}$, we see that $\ell_j \in P$,
so, by definition of P, we get: $\forall k \in (\ell_j..\infty)$, $s_k < s_{\ell_j}$.
So, since $\ell_{j+1} \in (\ell_j..\infty)$, we get: $s_{\ell_{j+1}} < s_{\ell_j}$.
Then $t_{j+1} = (s \circ \ell)_{j+1} = s_{\ell_{j+1}} < s_{\ell_j} = (s \circ \ell)_j = t_j$.
End of Case (1).

Case (2): Since P is finite, we get: $\max P \neq ☹$.
Then $\max P \in P$. Let $m := \max P$. Then $m \in P$.
Since $m \in P \subseteq \mathbb{N}$, we get $m \in \mathbb{N}$. Then $m + 1 \in (m..\infty)$.
For all $j \in \mathbb{N}$, let $X_j := \{k \in (j..\infty) \,|\, s_k \geqslant s_j\}$.
Then, $\forall j \in \mathbb{N}$, $X_j \subseteq (j..\infty)$.

Claim B: $\forall j \in (m..\infty)$, $X_j \neq \varnothing$.
Proof of Claim B: Given $j \in (m..\infty)$. **Want:** $X_j \neq \varnothing$.
Since $j > m = \max P$, we see that $j \notin P$.
Then, by definition of P, choose $k \in (j..\infty)$ s.t. $s_k \geqslant s_j$.
Then, by definition of X_j, we get: $k \in X_j$. Then $X_j \neq \varnothing$.

End of proof of Claim B.

We have: $\forall j \in \mathbb{N}, \quad X_j \subseteq (j..\infty) \subseteq \mathbb{N}, \quad$ so $X_j \subseteq \mathbb{N}$.
So, by Claim B, we have: $\forall j \in \mathbb{N}$, $\emptyset \neq X_j \subseteq \mathbb{N}$.
Then, by the Well-Ordering Axiom, we get: $\forall j \in \mathbb{N}$, $\min X_j \neq ☹$.
Then, $\forall j \in \mathbb{N}$, $\min X_j \in X_j$.
We have: $\forall j \in (m..\infty), \quad j > m, \quad$ so $(j..\infty) \subseteq (m..\infty)$.
We have: $\forall j \in (m..\infty), \quad \min X_j \in X_j \subseteq (j..\infty) \subseteq (m..\infty)$.
Then: $\forall j \in (m..\infty), \quad \min X_j \in (m..\infty)$.
Define $f : (m..\infty) \to (m..\infty)$ by: $\forall j \in (m..\infty)$, $f(j) = \min X_j$.
Then, $\forall j \in \mathbb{N}$, $f(j) \in X_j$.
We have: $\forall j \in \mathbb{N}, \quad f(j) \in X_j \subseteq (j..\infty), \quad$ so $f(j) \in (j..\infty)$.
Define $\ell \in (m..\infty)^{\mathbb{N}}$ by: $\forall j \in \mathbb{N}$, $\ell_j = f_\circ^j(m+1)$.
Then, $\forall j \in \mathbb{N}$, we have $f(\ell_j) = \ell_{j+1}$.
Then, $\forall j \in \mathbb{N}$, we have $\ell_{j+1} = f(\ell_j) \in X_{\ell_j}$, so $\ell_{j+1} \in X_{\ell_j}$.
Also, $\forall j \in \mathbb{N}$, we have $\ell_{j+1} = f(\ell_j) \in (\ell_j..\infty) > \ell_j$.
Then ℓ is strictly-increasing.
So, as $\ell \in P^{\mathbb{N}} \subseteq \mathbb{N}^{\mathbb{N}}$, we get: $s \circ \ell$ is a subsequence of s.
Let $t := s \circ \ell$. Then t is a subsequence of s.
Want: t is semi-monotone.
It suffices to show: t is semi-increasing.
Want: $\forall j \in \mathbb{N}$, $t_{j+1} \geqslant t_j$.
Given $j \in \mathbb{N}$. **Want:** $t_{j+1} \geqslant t_j$.
We have $\ell_{j+1} \in X_{\ell_j} = \{k \in (\ell_j..\infty) \,|\, s_k \geqslant s_{\ell_j}\}$. Then $s_{\ell_{j+1}} \geqslant s_{\ell_j}$.
Then $t_{j+1} = (s \circ \ell)_{j+1} = s_{\ell_{j+1}} \geqslant s_{\ell_j} = (s \circ \ell)_j = t_j$.
End of Case (2). □

3.8. **Sequentially-closed subsets of $\mathbb{R}$.**

DEFINITION 3.8.1. *Let $A \subseteq \mathbb{R}$.*
By A is **sequentially-closed**, *we mean:*
$$\forall s \in A^{\mathbb{N}}, \quad (\, s \textit{ is convergent}\,) \Rightarrow (\, s \textit{ is convergent in } A\,).$$

THEOREM 3.8.2. *Let $A \subseteq \mathbb{R}$.*
Then A is sequentially-closed if and only if:
$$\forall s \in A^{\mathbb{N}}, \forall z \in R, \quad (\, s \to z\,) \Rightarrow (\, z \in A\,).$$

THEOREM 3.8.3. *Let $a, z \in \mathbb{R}$, $t \in \mathbb{R}^{\mathbb{N}}$.*
Assume $\forall j \in \mathbb{N}$, $t_j \leqslant a$.
Assume $t \to z$. Then $z \leqslant a$.

Proof. Assume $z > a$. **Want:** Contradiction.
Let $\varepsilon := z - a$. Then $\varepsilon > 0$.
Since $t \to z$, choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$, $(\, j \geqslant K \,) \Rightarrow (\, |t_j - z| < \varepsilon \,)$.
By assumption, $t_K \leqslant a$. By the choice of K, $|t_K - z| < \varepsilon$.
Since $|t_K - z| < \varepsilon$, we get $z - \varepsilon < t_K < z + \varepsilon$.
Then $t_K \leqslant a = z-(z-a) = z-\varepsilon < t_K$, so $t_K < t_K$. Contradiction. □

THEOREM 3.8.4. *Let $a, z \in \mathbb{R}$, $t \in \mathbb{R}^{\mathbb{N}}$.*
Assume $\forall j \in \mathbb{N}$, $a \leqslant t_j$.
Assume $t \to z$. Then $a \leqslant z$.

Proof. Assume $a > z$. **Want:** Contradiction.
Let $\varepsilon := a - z$. Then $\varepsilon > 0$.
Since $t \to z$, choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$, $(\, j \geqslant K \,) \Rightarrow (\, |t_j - z| < \varepsilon \,)$.
By assumption, $a \leqslant t_K$. By the choice of K, $|t_K - z| < \varepsilon$.
Since $|t_K - z| < \varepsilon$, we get $z - \varepsilon < t_K < z + \varepsilon$.
Then $t_K < z+\varepsilon = z+(a-z) = a \leqslant t_K$, so $t_K < t_K$. Contradiction. □

THEOREM 3.8.5. *Let $a, b \in \mathbb{R}$. Assume $a \leqslant b$.*
Then $[a; b]$ is sequentially-closed.

Proof. **Want:** $\forall s \in [a; b]^{\mathbb{N}}$, $\forall z \in \mathbb{R}$, $(\, s \to z \,) \Rightarrow (\, z \in [a; b] \,)$.
Given $s \in [a; b]^{\mathbb{N}}$, $z \in \mathbb{R}$. Assume $s \to z$. **Want:** $z \in [a; b]$.
Since $s \in [a; b]^{\mathbb{N}}$, we get: $\forall j \in \mathbb{N}$, $s_j \in [a; b]$.
Then $\forall j \in \mathbb{N}$, we have $a \leqslant s_j \leqslant b$.
By Theorem 3.8.3, $z \leqslant b$. By Theorem 3.8.4, $a \leqslant z$.
Then $a \leqslant z \leqslant b$. Then $z \in [a; b]$. □

THEOREM 3.8.6. *Let $X \subseteq \mathbb{R}$.*
Then: $(\, X$ is closed$\,) \Leftrightarrow (X$ is sequentially-closed$)$.

Proof. Proof of $\Rightarrow$:
Assume X is closed. **Want:** X is sequentially-closed.
Want: $\forall s \in X^{\mathbb{N}}$, $\forall q \in \mathbb{R}$, $(\, s \to q \,) \Rightarrow (\, q \in X \,)$.
Given $s \in X^{\mathbb{N}}$, $q \in \mathbb{R}$. Assume $s \to q$. **Want:** $q \in X$.
Assume $q \notin X$. **Want:** Contradiction.
Since $q \in \mathbb{R}$ and $q \notin X$, we get: $q \in \mathbb{R} \backslash X$.
Since S is closed, we get: $\partial X \subseteq X$.
Let $t := (q, q, q, q, \ldots)$. Then $t \in (\mathbb{R} \backslash)^{\mathbb{N}}$ and $t \to q$.
So, since $s \in X^{\mathbb{N}}$ and $s \to q$, we conclude: $q \in \partial X$.
Then $q \in \partial X \subseteq X$, so $q \in X$.

Then $q \in X$ and $q \notin X$. Contradiction.
End of proof of $\Rightarrow$.

Proof of $\Leftarrow$:
Assume X is sequentially-closed. **Want:** X is closed.
Want: $\partial X \subseteq X$.
Want: $\forall q \in \partial X$, $q \in X$.
Given $q \in \partial X$. **Want:** $q \in X$.
Since $q \in \partial X$, we know: $\exists s \in X^{\mathbb{N}}$ s.t. $s \to q$.
So, since X is sequentially-closed, $q \in X$.
End of proof of $\Leftarrow$. □

THEOREM 3.8.7. *Let* $X \subseteq \mathbb{R}$. *Then:*

$$(\text{ } X \text{ is compact }) \quad \Leftrightarrow \quad (\text{ } X \text{ is closed and bounded }).$$

Proof. Proof of $\Rightarrow$:
Assume: X is compact.
Want: X is closed and bounded.
By Theorem 2.2.56, X is bounded. **Want:** X is closed.
Want: X is sequentially-closed.
Want: $\forall s \in X^{\mathbb{N}}$, $\forall q \in \mathbb{R}$, $(s \to q) \Rightarrow (q \in X)$.
Given $s \in X^{\mathbb{N}}$, $q \in \mathbb{R}$. Assume $s \to q$. **Want:** $q \in X$.
Since X is compact and $s \in X^{\mathbb{N}}$, we know: s is subconvergent in X.
Choose a subsequence t of s s.t. t is convergent in X.
Choose $z \in X$ s.t. $t \to z$.
Since $s \to q$ and since t is a subsequence of s, we get: $t \to q$.
Since $t \to q$ and $t \to z$, we get: $q = z$.
Since $q = z \in X$, we get: $q \in X$.
End of proof of $\Rightarrow$.

Proof of $\Leftarrow$:
Assume: X is closed and bounded.
Want: X is compact.
Want: $\forall s \in X^{\mathbb{N}}$, s is subconvergent in X.
Given $s \in X^{\mathbb{N}}$. **Want:** s is subconvergent in X.
Want: $\exists$subsequence t of s s.t. t is convergent in X.
Since $s \in X^{\mathbb{N}} \subseteq \mathbb{R}^{\mathbb{N}}$, by Theorem 3.7.1,
choose a subsequence t of s s.t. t is semi-monotone.
Then t is a subsequence of s. **Want:** t is convergent in X.

Since X is closed, we know: X is sequentially-closed.
Want: t is convergent.
Since $\mathbb{I}_t \subseteq \mathbb{I}_s \subseteq X$ and since X is bounded,
we conclude: $\mathbb{I}_t$ is bounded.
So, since t is semi-monotone, t is convergent.
End of proof of $\Leftarrow$. □

THEOREM 3.8.8. *Let* $C, K \subseteq \mathbb{R}$.
Assume that C *is closed and that* K *is compact.*
Then $C \cap K$ *is compact.*

Proof. Since K is compact, we get: K is closed and bounded.
Since C and K are both closed, we get: $C \cap K$ is closed.
Since K is bounded and since $C \cap K \subseteq K$, we get: $C \cap K$ is bounded.
Since $C \cap K$ is closed and bounded, $C \cap K$ is compact. □

3.9. Extreme values of continuous functionals on $[0; 1]$.

Our goal, in this section, is to prove:
$\forall$continuous $f : [0; 1] \to \mathbb{R}$, $\max \mathbb{I}_f \neq$ ☹.
We indicated, in class, why
this is NOT true when $[0; 1]$ is replaced by $[0; 1)$.

By Theorem 2.2.53, we see that $(0; 1]$ is NOT compact.

THEOREM 3.9.1. *Let* $a, b \in \mathbb{R}$. *Assume* $a \leqslant b$.
Then $[a; b]$ *is compact.*

Proof. **Want:** $\forall s \in [a; b]^{\mathbb{N}}$, s is subconvergent in $[a; b]$.
Given $s \in [a; b]^{\mathbb{N}}$. **Want:** s is subconvergent in $[a; b]$.
Want: $\exists$subsequence t of s s.t. t is convergent in $[a; b]$.
By Theorem 3.7.1, choose a subsequence t of s s.t. t is semi-monotone.
Then t is a subsequence of s. **Want:** t is convergent in $[a; b]$.
Since $\mathbb{I}_t \subseteq \mathbb{I}_s \subseteq [a; b] \subseteq B((a + b)/2; (b - a + 2)/2)$, we see that $\mathbb{I}_t$ is bounded.
Since t is semi-monotone and $\mathbb{I}_t$ is bounded, t is convergent.
By Theorem 3.8.5, $[a; b]$ is sequentially-closed.
So, since $t \in [a; b]^{\mathbb{N}}$ and t is convergent,
it follows that t is convergent in $[a; b]$. □

THEOREM 3.9.2. *Let* $K \subseteq \mathbb{R}$, $f : \mathbb{R} \dashrightarrow \mathbb{R}$.
Assume: f *is continuous on* K *and* K *is compact and nonempty.*
Then: $\max f_* K \neq ☹$.

Proof. Theorem 3.6.1, we get: $f_* K$ is compact.
Since K is nonempty, $f_* K$ is nonempty.
Let $L := f_* K$. Then L is compact and nonempty.
By Theorem 2.3.4, $\max L \neq ☹$.
So, since $L = f_* K$, we see that $\max f_* K \neq ☹$. □

Recall that our goal for this section was to prove:
∀continuous $f : [0; 1] \to \mathbb{R}$, $\max \mathbb{I}_f \neq ☹$.
With the preceding three theorems, we are now ready to prove more:

THEOREM 3.9.3. *Let* $a \in \mathbb{R}$, $b > a$. *Let* $f : [a; b] \to \mathbb{R}$.
Assume f *is continuous. Then* $\max \mathbb{I}_f \neq ☹$.

Proof. Let $K := [a; b]$. Then $\mathbb{I}_f = f_* \mathbb{D}_f = f_*[a; b] = f_* K$.
Also, K is nonempty and f is continuous on K.
By Theorem 3.9.1, K is compact.
Then, by Theorem 3.9.2, $\max f_* K \neq ☹$.
So, since $\mathbb{I}_f = f_* K$, we conclude that $\max \mathbb{I}_f \neq ☹$. □

3.10. Uniform convergence of sequences of functions $\mathbb{R} \dashrightarrow \mathbb{R}$.

DEFINITION 3.10.1. *Let* D *and* Y *be sets,* $s \in (Y^D)^{\mathbb{N}}$, $x \in D$.
Then $\boxed{s_\bullet(x)} \in Y^{\mathbb{N}}$ *is defined by:* $\forall j \in \mathbb{N}$, $((s_\bullet(x))_j = s_j(x)$.

We define **pointwise convergence**.

DEFINITION 3.10.2. *Let* $D \subseteq \mathbb{R}$, $s \in (\mathbb{R}^D)^{\mathbb{N}}$, $f \in \mathbb{R}^D$.
By $s \to f$ **pointwise**, *we mean:* $\forall x \in D$, $s_\bullet(x) \to f(x)$.

Let $D \subseteq \mathbb{R}$, $s \in (\mathbb{R}^D)^{\mathbb{N}}$, $f \in \mathbb{R}^D$.
Then $s \to f$ pointwise iff
$\forall x \in D$, $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\ j \geqslant K\ ) \ \Rightarrow\ (\ |[\,s_j(x)\,] - [\,f(x)\,]| < \varepsilon\ ).$$
We define **uniform convergence**:

DEFINITION 3.10.3. *Let* $D \subseteq \mathbb{R}$, $s \in (\mathbb{R}^D)^{\mathbb{N}}$, $f \in \mathbb{R}^D$.
By $s \to f$ **uniformly**, *we mean:* $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ *s.t.,* $\forall j \in \mathbb{N}$, $\forall x \in D$,
$$(\ j \geqslant K\ ) \ \Rightarrow\ (\ |[\,s_j(x)\,] - [\,f(x)\,]| < \varepsilon\ ).$$

THEOREM 3.10.4. *Define $\phi : \mathbb{R} \to \mathbb{R}$ by: $\forall x \in \mathbb{R}$, $\phi(x) = 1/(1+x^2)$. Define $s \in (\mathbb{R}^{\mathbb{R}})^{\mathbb{N}}$ by: $\forall j \in \mathbb{N}$, $\forall x \in \mathbb{R}$, $s_j(x) = \phi(j \cdot x)$. Let $f : \chi_{\{0\}}^{\mathbb{R}}$. Then $s \to f$ pointwise.*

From the preceding theorem, we see that:
a pointwise limit of continuous functions can be discontinuous.
By contrast, uniform limits of continuous functions are continuous:

THEOREM 3.10.5. *Let $D \subseteq \mathbb{R}$, $s \in (\mathbb{R}^D)^{\mathbb{N}}$, $f \in \mathbb{R}^D$.*
Assume: $s \to f$ uniformly and $\forall j \in \mathbb{N}$, s_j is continuous.
Then: f is continuous.

Proof. This is HW#9-2. □

3.11. Open Mapping Theorem.

Let $A := [1; 2]$, $B := (3; 4]$, $C := [5; 7]$.
Define $f : A \cup B \to C$ by: $\forall x \in A$, $f_x = x + 4$
and $\forall x \in B$, $f_x = x + 3$.
Then $f : A \cup B \hookrightarrow\!\!> C$ and f is continuous.
However f^{-1} is not continuous at 6.
So the inverse of a continuous function is not always continuous.
Basically, f glues two intervals, A and B, together,
whereas, whereas f^{-1} tears C apart;
gluing is continuous, while tearing apart is discontinouus. Our goal in this section is to show, that
if a continuous injection has compact domain,
then its inverse is continuous:

THEOREM 3.11.1. *Let $K \subseteq \mathbb{R}$, $f : K \hookrightarrow \mathbb{R}$.*
Assume: K is compact and f is continuous.
Then: f^{-1} is continuous.

Proof. Since $f : K \hookrightarrow \mathbb{R}$, we get $\mathbb{D}_f = K$.
By Theorem 3.3.1, want: $\forall$closed $C \subseteq \mathbb{R}$, $(f^{-1})^*C$ is closed.
Given a closed $C \subseteq \mathbb{R}$. **Want:** $(f^{-1})^*C$ is closed.
By Theorem 3.8.8, $C \cap K$ is compact.
Then, by Theorem 3.6.1, $f_*(C \cap K)$ is compact.
Then $f_*(C \cap K)$ is closed.
So, since $(f^{-1})^*C = f_*C = f_*(C \cap \mathbb{D}_f) = f_*(C \cap K)$,
we conclude: $(f^{-1})^*C$ is closed. □

3.12. Continuity and uniform continuity on a compact set.

THEOREM 3.12.1. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$.*
Assume f is continuous and $\mathbb{D}_f$ is compact. Then f is uniformly continuous.

Proof. Assume f is not uniformly continuous. **Want:** Contradiction.
Choose $\varepsilon > 0$ s.t., $\forall \delta > 0$, $\exists w, x \in \mathbb{D}_f$ s.t.
$$(\,|w - x| < \delta\,) \,\&\, (\,|f_w - f_x| \geqslant \varepsilon\,).$$
Let $K := \mathbb{D}_f$. Then K is compact.
Also, $\forall \delta > 0$, $\exists w, x \in K$ s.t.
$$(\,|w - x| < \delta\,) \,\&\, (\,|f_w - f_x| \geqslant \varepsilon\,).$$
Then: $\forall j \in \mathbb{N}$, $\exists w, x \in K$ s.t.
$$(\,|w - x| < 1/j\,) \,\&\, (\,|f_w - f_x| \geqslant \varepsilon\,).$$
By the Axiom of Choice, choose $w, x \in K^{\mathbb{N}}$ s.t., $\forall j \in \mathbb{N}$,
$$(\,|w_j - x_j| < 1/j\,) \,\&\, (\,|f_{w_j} - f_{x_j}| \geqslant \varepsilon\,).$$
Since K is compact, w is subconvergent in K.
Choose a subsequence v of w s.t. v is convergent in K.
Choose $q \in K$ s.t. $v \to q$.
Choose a strictly-increasing $\ell \in \mathbb{N}^{\mathbb{N}}$ s.t. $v = w \circ \ell$.
Then $w \circ \ell \to q$, so, by HW#10-1, $x \circ \ell \to q$.
Since $w \circ \ell \to q$ and since f is continuous at q, we get: $f \circ w \circ \ell \to f_q$.
Choose $A \in \mathbb{N}$ s.t., $\forall i \in \mathbb{N}$, $(\,i \geqslant A\,) \Rightarrow (\,|(f \circ w \circ \ell)_i - f_q| < \varepsilon/2\,)$.
Since $x \circ \ell \to q$ and since f is continuous at q, we get: $f \circ x \circ \ell \to f_q$.
Choose $B \in \mathbb{N}$ s.t., $\forall i \in \mathbb{N}$, $(\,i \geqslant B\,) \Rightarrow (\,|(f \circ x \circ \ell)_i - f_q| < \varepsilon/2\,)$.
Let $i := \max\{A, B\}$. Then $i \in \mathbb{N}$ and $i \geqslant A$ and $i \geqslant B$.
Since $i \geqslant A$, we get: $|(f \circ w \circ \ell)_i - f_q| < \varepsilon/2$.
Since $i \geqslant B$, we get: $|(f \circ x \circ \ell)_i - f_q| < \varepsilon/2$.
Let $j := \ell_i$. Then $|(f \circ w)_j - f_q| < \varepsilon/2$ and $|(f \circ x)_j - f_q| < \varepsilon/2$.
Then $|f_{w_j} - f_q| < \varepsilon/2$ and $|f_{x_j} - f_q| < \varepsilon/2$.
Since $\ell \in \mathbb{N}^{\mathbb{N}}$ and $j = \ell_i$, we get: $j \in \mathbb{N}$.
Then, by the choice of w and x, we get: $|f_{w_j} - f_{x_j}| \geqslant \varepsilon$.
Then $\varepsilon \leqslant |f_{w_j} - f_{x_j}| \leqslant |f_{w_j} - f_q| + |f_q - f_{x_j}|$.
Then $\varepsilon \leqslant |f_{w_j} - f_{x_j}| \leqslant |f_{w_j} - f_q| + |f_{x_j} - f_q| < (\varepsilon/2) + (\varepsilon/2) = \varepsilon$.
Then $\varepsilon < \varepsilon$. Contradiction. □

4. Differentiability of functions $\mathbb{R} \dashrightarrow \mathbb{R}$

4.1. The double translate.

The function $f_a^{\mathbb{T}}$ in the next definition is called the **Double Translate** of f based at a.

DEFINITION 4.1.1. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$*,* $a \in \mathbb{R}$*.*
Then $\boxed{f_a^{\mathbb{T}}} : \mathbb{R} \dashrightarrow \mathbb{R}$ *is defined by:* $\quad \forall h \in \mathbb{R}, \quad (f_a^{\mathbb{T}})_h = f_{a+h} - f_a$.

Note that: $\forall f : \mathbb{R} \dashrightarrow \mathbb{R}$, $\forall a \in \mathbb{R}\backslash\mathbb{D}_f$, we have: $f_a^{\mathbb{T}} = \varnothing$.

By HW#6-4, we have:

THEOREM 4.1.2. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$*,* $a \in \mathbb{D}_f$*.*
Assume that $f_a^{\mathbb{T}}$ *is continuous at* 0*. Then* f *is continuous at* a*.*

The following is HW#6-5. It is the **Precalculus Chain Rule**.

THEOREM 4.1.3. *Let* $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$*,* $a \in \mathbb{D}_{g \circ f}$*.*
Then: $\quad (g \circ f)_a^{\mathbb{T}} \quad = \quad g_{f_a}^{\mathbb{T}} \ \circ \ f_a^{\mathbb{T}}$.

The following is HW#7-5. It is the **Precalculus Product Rule**.

THEOREM 4.1.4. *Let* $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$*,* $a \in \mathbb{D}_{f \cdot g}$*.*
Then: $\quad (f \cdot g)_a^{\mathbb{T}} \quad = \quad f_q^{\mathbb{T}} \cdot g_q \ + \ f_q \cdot g_q^{\mathbb{T}} \ + \ f_q^{\mathbb{T}} \cdot g_q^{\mathbb{T}}$.

4.2. $\widehat{\mathcal{O}}$ and $\wp$.

DEFINITION 4.2.1.
$\boxed{(\bullet)} : \mathbb{R} \to \mathbb{R}$ *is defined by:* $\quad \forall x \in \mathbb{R}, \quad (\bullet)_x = x$.
$\boxed{|\bullet|} : \mathbb{R} \to \mathbb{R}$ *is defined by:* $\quad \forall x \in \mathbb{R}, \quad |\bullet|_x = |x|$.
$\boxed{\sqrt{\bullet}} : \mathbb{R} \dashrightarrow \mathbb{R}$ *is defined by:* $\quad \forall x \in \mathbb{R}, \quad \sqrt{\bullet}_x = \sqrt{x}$.

DEFINITION 4.2.2. *Let* $k \in \mathbb{N}_0$*. Then*
$\boxed{(\bullet)^k} : \mathbb{R} \to \mathbb{R}$ *is defined by:* $\quad \forall x \in \mathbb{R}, \quad (\bullet)_x^k = x^k$ $\quad$ *and*
$\boxed{|\bullet|^k} : \mathbb{R} \to \mathbb{R}$ *is defined by:* $\quad \forall x \in \mathbb{R}, \quad |\bullet|_x^k = |x|^k$.

THEOREM 4.2.3.
$(\bullet)^0 = |\bullet|^0 = C_1^{\mathbb{R}}$ $\quad$ *and* $\quad (\bullet)^1 = (\bullet) = \mathrm{id}^{\mathbb{R}}$ *and* $|\bullet|^1 = |\bullet|$.

THEOREM 4.2.4. $\forall k \in \mathbb{N}_0$, $|\bullet|^{2k} = (\bullet)^{2k}$.

THEOREM 4.2.5. $(\bullet)^2|[0;\infty) : [0;\infty) \hookrightarrow\!\!> [0;\infty)$.

THEOREM 4.2.6. $\sqrt{\bullet} = ((\bullet)^2|[0;\infty))^{-1}$.

THEOREM 4.2.7. $\sqrt{\bullet} : [0;\infty) \hookrightarrow\!\!> [0;\infty)$.

DEFINITION 4.2.8. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$.
Then f *is* **defined at** a *means:* $a \in D_f$.
Also, f *is* **defined near** a *means:*
$\exists B \in \mathcal{B}(a)$ *s.t.* $B \subseteq \mathbb{D}_f$.

Because $\mathbb{D}_{\sqrt{\bullet}} = [0; \infty)$, we get:

THEOREM 4.2.9.
$\sqrt{\bullet}$ *is defined near* 0.01 *and*
$\sqrt{\bullet}$ *is defined at* 0 *and*
$\sqrt{\bullet}$ *is NOT defined near* 0.

Convention: Each of $a \leqslant b$ or $a < b$ or $a \geqslant b$ or $a > b$,
implies that $a \neq ☹ \neq b$.

THEOREM 4.2.10. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$. *Then:*
(f *is defined near* a *and continuous at* a) $\Leftrightarrow$
($\forall \varepsilon > 0, \exists \delta > 0$ *s.t.,* $\forall x \in \mathbb{R}$,
($|x - a| < \delta$) $\Rightarrow$ ($|f_x - f_a| \leqslant \varepsilon$)).

DEFINITION 4.2.11. *Let* $k \in \mathbb{N}_0$. *Then:*
$$\boxed{o(k)} := \{ f : \mathbb{R} \dashrightarrow \mathbb{R} \,|\, \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in \mathbb{R}, \; (|x| < \delta) \Rightarrow (|f_x| \leqslant \varepsilon \cdot |x|^k) \}.$$

Let $k \in \mathbb{N}_0$ and $f : \mathbb{R} \dashrightarrow \mathbb{R}$. Then: $f \in o(k)$ iff
$\forall \varepsilon > 0$, near 0 we have $-\varepsilon \cdot |\bullet|^k \leqslant f \leqslant \varepsilon \cdot |\bullet|^k$.

Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$. Then: $f \in o(7)$ iff
near 0 we have $-|\bullet|^7 \leqslant f \leqslant |\bullet|^7$ and
near 0 we have $-|\bullet|^7/2 \leqslant f \leqslant |\bullet|^7/2$ and
near 0 we have $-|\bullet|^7/3 \leqslant f \leqslant |\bullet|^7/3$ *etc.*

Note that $(\bullet)^7 \notin o(7)$ and that $|\bullet|^7 \notin o(7)$.

THEOREM 4.2.12. *Let* $k \in \mathbb{N}_0$, $f \in o(k)$.
Then: $f_0 = 0$ *and* f *is defined near* 0.

THEOREM 4.2.13. *Let* $f \in o(k)$. *Then:* $f \in o(0)$ *iff*
(f *is defined near* 0) & (f *is continuous at* 0) & ($f_0 = 0$).

DEFINITION 4.2.14. $\forall x \in \mathbb{R}$, $\forall k \in \mathbb{N}_0$, $\boxed{x^{k+(1/101)}} := x^k \cdot \sqrt[101]{x}$.

THEOREM 4.2.15.

$\forall a \in \mathbb{R},\ a^{1/101} = \sqrt[101]{a}$ *and*

$\forall a \in \mathbb{R},\ (a^{101})^{1/101} = a = (a^{1/101})^{101}$ *and*

$\forall a \in \mathbb{R},\ |a^{3+(1/101)}| = |a|^{3+(1/101)} = |a|^3 \cdot |a|^{1/101}$ *and*

$\forall a, b \in \mathbb{R},\ (\, a < b \,) \Rightarrow (\, a^{3+(1/101)} < b^{3+(1/101)} \,)$.

THEOREM 4.2.16.

(a) $(\bullet)^4 \in o(3)$ *and*

(b) $(\bullet)^{3+(1/101)} \in o(3)$.

Proof. Unassigned HW. (Hint: Use Theorem 4.2.15.) □

DEFINITION 4.2.17. *Let* $k \in \mathbb{N}_0$.

Then $\boxed{\widehat{\mathcal{O}}(k)} := \{\, f : \mathbb{R} \dashrightarrow \mathbb{R} \,|\, \exists C \geqslant 0, \exists \delta > 0$ *s.t.*, $\forall x \in \mathbb{R},$

$(\, |x| < \delta \,) \Rightarrow (\, |f_x| \leqslant C \cdot |x|^k \,) \,\}$.

NOTE: For any $k \in \mathbb{N}_0$, for any $f : \mathbb{R} \dashrightarrow \mathbb{R}$, we have:

$[\ f \in \widehat{\mathcal{O}}(k)\ ]$ iff $[$ near 0, $-C \cdot |\bullet|^k \leqslant f \leqslant C \cdot |\bullet|^k\ ]$.

Here, "near 0" means "on some ball in $\mathbb{R}$ centered at 0".

THEOREM 4.2.18. $\forall k \in \mathbb{N},\ \forall f \in \widehat{\mathcal{O}}(k),\quad f_0 = 0.$

Note that the preceding theorem is not true when $k = 0$: $C_1^{\mathbb{R}} \in \widehat{\mathcal{O}}(0)$.

On the other hand the next result holds for all $k \in \mathbb{N}_0$:

THEOREM 4.2.19. $\forall k \in \mathbb{N}_0,\ \forall f \in \widehat{\mathcal{O}}(k),$ *f is defined near* 0.

THEOREM 4.2.20. $\forall k \in \mathbb{N}_0,\quad (\bullet)^k, |\bullet|^k \in \widehat{\mathcal{O}}(k)$.

The next result is called the **chain of $\widehat{\mathcal{O}}$, o spaces**:

THEOREM 4.2.21.

$\widehat{\mathcal{O}}(1) \supseteq o(1) \supseteq \widehat{\mathcal{O}}(2) \supseteq o(2) \supseteq \widehat{\mathcal{O}}(3) \supseteq o(3) \supseteq \widehat{\mathcal{O}}(4) \supseteq o(4) \supseteq$
$\widehat{\mathcal{O}}(5) \supseteq o(5) \supseteq \widehat{\mathcal{O}}(6) \supseteq o(6) \supseteq \widehat{\mathcal{O}}(7) \supseteq o(7) \supseteq \widehat{\mathcal{O}}(8) \supseteq o(8) \supseteq \cdots$.

THEOREM 4.2.22. *Let* $k \in \mathbb{N}_0$. *Then:*

$\forall f, g \in o(k),\quad f + g \in o(k)$ *and*

$\forall c \in \mathbb{R},\ \forall f \in o(k),\quad c \cdot f \in o(k)$.

The preceding and following theorem are both unassigned HW.
NOTE: The "**linear operations**" are: addition, scalar multiplication.

The preceding theorem says that $\mathcal{o}(k)$ is closed under linear operations. The phrase "$\mathcal{o}(k)$ is **linearly closed**" expresses that.
The set $\widehat{\mathcal{O}}(k)$ is also linearly closed:

THEOREM 4.2.23. *Let $k \in \mathbb{N}_0$. Then:*
$\forall f, g \in \widehat{\mathcal{O}}(k), \quad f + g \in \widehat{\mathcal{O}}(k)$ *and*
$\forall c \in \mathbb{R}, \forall f \in \widehat{\mathcal{O}}(k), \quad c \cdot f \in \widehat{\mathcal{O}}(k).$

THEOREM 4.2.24. *Let $k, \ell \in \mathbb{N}_0$, $f \in \mathcal{o}(k)$, $g \in \mathcal{o}(\ell)$. Then $g \circ f \in \mathcal{o}(\ell \cdot k)$.*

Proof. **Want:** $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall x \in \mathbb{R}$,
$$(\,|x| < \delta\,) \Rightarrow (\,|(g \circ f)_x| \leqslant \varepsilon \cdot |x|^{\ell \cdot k}\,).$$
Given $\varepsilon > 0$. **Want:** $\exists \delta > 0$ s.t., $\forall x \in \mathbb{R}$,
$$(\,|x| < \delta\,) \Rightarrow (\,|(g \circ f)_x| \leqslant \varepsilon \cdot |x|^{\ell \cdot k}\,).$$
Since $\varepsilon > 0$ and $g \in \mathcal{o}(\ell)$, choose $\mu > 0$ s.t., $\forall y \in \mathbb{R}$,
$$(\,|y| < \mu\,) \Rightarrow (\,|g_y| \leqslant \varepsilon \cdot |y|^{\ell}\,).$$
Let $\tau := \min\{\mu/2, 1\}$. Then $\tau > 0$ and $\tau \leqslant \mu/2$ and $\tau \leqslant 1$.
Since $\tau > 0$ and $f \in \mathcal{o}(k)$, choose $\lambda > 0$ s.t., $\forall x \in \mathbb{R}$,
$$(\,|x| < \lambda\,) \Rightarrow (\,|f_x| \leqslant \tau \cdot |x|^{k}\,).$$
Let $\delta := \min\{\lambda, 1\}$. Then $\delta > 0$ and $\delta \leqslant \lambda$ and $\delta \leqslant 1$.
Want: $\forall x \in \mathbb{R}$, $(\,|x| < \delta\,) \Rightarrow (\,|(g \circ f)_x| \leqslant \varepsilon \cdot |x|^{\ell \cdot k}\,)$.
Given $x \in \mathbb{R}$. Assume $|x| < \delta$. **Want:** $|(g \circ f)_x| \leqslant \varepsilon \cdot |x|^{\ell \cdot k}$.
Since $|x| < \delta \leqslant \lambda$, by choice of λ, we get: $|f_x| \leqslant \tau \cdot |x|^k$.
Let $y := f_x$. Then $|y| = |f_x| \leqslant \tau \cdot |x|^k$, so $|y| \leqslant \tau \cdot |x|^k$.
We have $|x| < \delta \leqslant 1$, so, since $k \in \mathbb{N}_0$, we get: $|x|^k \leqslant 1$.
Then $|y| \leqslant \tau \cdot |x|^k \leqslant \tau \cdot 1 = \tau \leqslant \mu/2 < \mu$, so $|y| < \mu$.
Since $|y| < \mu$, by choice of μ, we get: $|g_y| \leqslant \varepsilon \cdot |y|^{\ell}$.
Since $|y| \leqslant \tau \cdot |x|^k$ and $\tau \leqslant 1$, we get: $|y| \leqslant |x|^k$.
Then $|(g \circ f)_x| = |g_{f_x}| = |g_y| \leqslant \varepsilon \cdot |y|^{\ell} \leqslant \varepsilon \cdot (|x|^k)^{\ell} = \varepsilon \cdot |x|^{\ell \cdot k}$. □

THEOREM 4.2.25. *Let $k, \ell \in \mathbb{N}_0$. Then:*
$\forall f \in \widehat{\mathcal{O}}(k), \forall g \in \widehat{\mathcal{O}}(\ell), \quad f \cdot g \in \widehat{\mathcal{O}}(k + \ell)$ *and*
$\forall f \in \widehat{\mathcal{O}}(k), \forall g \in \mathcal{o}(\ell), \quad f \cdot g \in \mathcal{o}(k + \ell)$ *and*
$\forall f \in \mathcal{o}(k), \forall g \in \widehat{\mathcal{O}}(\ell), \quad f \cdot g \in \mathcal{o}(k + \ell)$ *and*
$\forall f \in \mathcal{o}(k), \forall g \in \mathcal{o}(\ell), \quad f \cdot g \in \mathcal{o}(k + \ell).$

Some of the following theorem fails when $k = 0$ or $\ell = 0$, so note the requirement that $k, \ell \in \mathbb{N}$.

THEOREM 4.2.26. *Let $k, \ell \in \mathbb{N}$. Then:*
$\forall f \in \widehat{\mathcal{O}}(k), \forall g \in \widehat{\mathcal{O}}(\ell), \quad f \circ g \in \widehat{\mathcal{O}}(\ell \cdot k)$ *and*

$$\forall f \in \widehat{\mathcal{O}}(k),\ \forall g \in o(\ell), \quad f \circ g \in o(\ell \cdot k) \qquad \textit{and}$$
$$\forall f \in o(k),\ \forall g \in \widehat{\mathcal{O}}(\ell), \quad f \circ g \in o(\ell \cdot k) \qquad \textit{and}$$
$$\forall f \in o(k),\ \forall g \in o(\ell), \quad f \circ g \in o(\ell \cdot k).$$

We define

agreement, near a point, of two partial functions on $\mathbb{R}$:

DEFINITION 4.2.27. *Let f and g be two functions, $q \in \mathbb{R}$. Assume $\mathbb{D}_f \subseteq \mathbb{R}$ and $\mathbb{D}_g \subseteq \mathbb{R}$. By* **near** q, $f = g$, *we mean: $\exists B \in \mathcal{B}(q)$ s.t. on B, $f = g$.*

4.3. Polynomials $\mathbb{R} \to \mathbb{R}$.

DEFINITION 4.3.1. $\forall k \in \mathbb{N}_0$, $\boxed{\mathcal{H}(k)} := \{c \cdot (\bullet)^k \mid c \in \mathbb{R}\}$.

$\boxed{\mathcal{C}} := \mathcal{H}(0)$, $\boxed{\mathcal{L}} := \mathcal{H}(1)$, $\boxed{\mathcal{Q}} := \mathcal{H}(2)$, $\boxed{\mathcal{K}} := \mathcal{H}(3)$.

Elements of $\mathcal{C}$ are called **constant**.
Elements of $\mathcal{L}$ are called (homogeneous) **linear**.
Elements of $\mathcal{Q}$ are called (homogeneous) **quadratic**.
Elements of $\mathcal{K}$ are called (homogeneous) **cubic**.

For any $k \in \mathbb{N}_0$, elements of $\mathcal{H}(k)$ are called
(homogenous) **polynomials** $\mathbb{R} \to \mathbb{R}$ **of degree** k, or
k-**polynomials** $\mathbb{R} \to \mathbb{R}$.
We may sometimes omit "$\mathbb{R} \to \mathbb{R}$".

THEOREM 4.3.2. *$\forall C \in \mathcal{C}$, C is Lipschitz-0.*

Proof. **Want:** $\forall x, y \in \mathbb{R}$, $|C_x - C_y| \leqslant 0 \cdot |x - y|$.
Given $x, y \in \mathbb{R}$. **Want:** $|C_x - C_y| \leqslant 0 \cdot |x - y|$.
Choose $a \in \mathbb{R}$ s.t. $C = C_a^{\mathbb{R}}$.
Then $C_x = a$ and $C_y = a$.
Then $|C_x - C_y| = |a - a| = |0| = 0 \cdot |x - y|$. □

DEFINITION 4.3.3. $\forall L \in \mathcal{L}$, $\boxed{[L]} := L_1$.

Let $m \in \mathbb{R}$ and let $L := m \cdot (\bullet)$.
Then $[L] = L_1 = m \cdot 1 = m$, so $[L]$ is just the slope of L.
Also, $|[L]|$ is the absolute value of the slope of L,
which we might call the "absolute slope" of L.

We next show: Each linear function is Lipschitz,
with Lipschitz constant equal to the absolute slope:

THEOREM 4.3.4. *$\forall L \in \mathcal{L}$, L is Lipschitz-$|[L]|$.*

Proof. Choose $m \in \mathbb{R}$ s.t. $L = m \cdot (\bullet)$.
Then $[L] = L_1 = m \cdot 1 = m$. Let $a := |m|$.
Then $a = |[L]|$. **Want:** L is Lipschitz-a.
Want: $\forall x, y \in \mathbb{D}_L$, $|L_x - L_y| \leqslant a \cdot |x - y|$.
Given $x, y \in \mathbb{D}_L$. **Want:** $|L_x - L_y| \leqslant a \cdot |x - y|$.
We have $L_x - L_y = m \cdot x - m \cdot y = m \cdot (x - y)$.
Then $|L_x - L_y| = |m| \cdot |x - y| = a \cdot |x - y|$, so $|L_x - L_y| = a \cdot |x - y|$.
Then $|L_x - L_y| \leqslant a \cdot |x - y|$. □

THEOREM 4.3.5. *Let $F \in (\mathcal{H}(0)) \cup (\mathcal{H}(1)) \cup (\mathcal{H}(2)) \cup (\mathcal{H}(3)) \cup \ldots$. Then F is continuous.*

Idea of proof:
If $F \in \mathcal{H}(0)$, then F is constant, hence Lipschitz-0,
hence Lipschitz, hence uniformly continuous, hence continuous.
If $F \in \mathcal{H}(1)$, then F is linear, hence Lipschitz-$[F]$,
hence Lipschitz, hence uniformly continuous, hence continuous.
If $F \in \mathcal{H}(2)$, then F is quadratic,
hence a product of two linear functions,
hence a product of two continuous functions, hence continuous.
If $F \in \mathcal{H}(3)$, then F is cubic,
hence a product of three linear functions,
hence a product of three continuous functions, hence continuous.
Etc.
End of idea of proof.

The next result says that every k-polynomial has order k.
In particular, $\mathcal{C} \subseteq \widehat{\mathcal{O}}(0)$ and $\mathcal{L} \subseteq \widehat{\mathcal{O}}(1)$ and $\mathcal{Q} \subseteq \widehat{\mathcal{O}}(2)$ and $\mathcal{K} \subseteq \widehat{\mathcal{O}}(3)$.

THEOREM 4.3.6. *Let $k \in \mathbb{N}_0$. Then $\mathcal{H}(k) \subseteq \widehat{\mathcal{O}}(k)$.*

Proof. **Want:** $\forall P \in \mathcal{H}(k)$, $P \in \widehat{\mathcal{O}}(k)$.
Given $P \in \mathcal{H}(k)$. **Want:** $P \in \widehat{\mathcal{O}}(k)$.
Want: $\exists C \geqslant 0$, $\exists \delta > 0$ s.t., $\forall x \in \mathbb{R}$,

$$(\,|x| < \delta\,) \Rightarrow (\,|P_x| \leqslant C \cdot |x|^k\,).$$

Since $P \in \mathcal{H}(k)$, choose $a \in \mathbb{R}$ s.t. $P = a \cdot (\bullet)^k$.

Let $C := |a|$, $\delta := 1$. Then $C \geqslant 0$ and $\delta > 0$.
Want: $\forall x \in \mathbb{R}, \quad (\,|x| < \delta\,) \Rightarrow (\,|P_x| \leqslant C \cdot |x|^k\,)$.
Given $x \in \mathbb{R}$. Assume: $|x| < \delta$. **Want:** $|P_x| \leqslant C \cdot |x|^k$.
We have $|P_x| = |a \cdot x^k| = |a| \cdot |x|^k = C \cdot |x|^k$.
Then $|P_x| = C \cdot |x|^k$, so $|P_x| \leqslant C \cdot |x|^k$. □

DEFINITION 4.3.7. $\boxed{\mathbf{0}} := C_0^{\mathbb{R}}$.

THEOREM 4.3.8. *Let $k \in \mathbb{N}_0$. Then: $\mathbf{0} \in \mathcal{o}(k)$ and $\mathbf{0} \in \widehat{\mathcal{O}}(k)$.*

THEOREM 4.3.9. *Let $k \in \mathbb{N}_0$. Then $(\mathcal{H}(k)) \cap (\mathcal{o}(k)) = \{\mathbf{0}\}$.*

Proof. $\mathbf{0} = 0 \cdot (\bullet)^k \in \mathcal{H}(k)$. Also, $\mathbf{0} \in \mathcal{o}(k)$.
Then $\mathbf{0} \in (\mathcal{H}(k)) \cap (\mathcal{o}(k))$, so $\{\mathbf{0}\} \subseteq (\mathcal{H}(k)) \cap (\mathcal{o}(k))$.
Want: $(\mathcal{H}(k)) \cap (\mathcal{o}(k)) \subseteq \{\mathbf{0}\}$.
Want: $\forall f \in (\mathcal{H}(k)) \cap (\mathcal{o}(k)), \quad f \in \{\mathbf{0}\}$.
Given $f \in (\mathcal{H}(k)) \cap (\mathcal{o}(k))$. **Want:** $f \in \{\mathbf{0}\}$.
Since $f \in \mathcal{H}(k)$, choose $c \in \mathbb{R}$ s.t. $f = c \cdot (\bullet)^k$.
Since $0 \cdot (\bullet)^k = \mathbf{0} \in \{\mathbf{0}\}$, **it suffices to show:** $c = 0$.
Assume $c \neq 0$. **Want:** Contradiction.
Since $c \in \mathbb{R}$ and $c \neq 0$, we get: $|c| > 0$.
Let $\varepsilon := |c|/2$. Then $\varepsilon > 0$.
So, since $f \in \mathcal{o}(k)$, choose $\delta > 0$ s.t., $\forall x \in \mathbb{R}$,
$$(\,|x| < \delta\,) \Rightarrow (\,|f_x| \leqslant \varepsilon \cdot |x|^k\,).$$
Since $\delta > 0$, we get: $\delta/2 > 0$, and $\delta/2 < \delta$.
Since $\delta/2 > 0$, we get: $|\delta/2| = \delta/2$.
Let $x := \delta/2$. Then $|x| = |\delta/2| = \delta/2 > 0$, so $|x| > 0$.
Also, $|x| = |\delta/2| = \delta/2 < \delta$, so $|x| < \delta$.
So, by choice of δ, we get: $|f_x| \leqslant \varepsilon \cdot |x|^k$.
Since $f = c \cdot (\bullet)^k$, we get: $f_x = c \cdot x^k$.
Then $|c| \cdot |x|^k = |c \cdot x^k| = |f_x| \leqslant \varepsilon \cdot |x|^k$, so $|c| \cdot |x|^k \leqslant \varepsilon \cdot |x|^k$.
Since $|x| > 0$ and $k \in \mathbb{N}_0$, we get: $|x|^k > 0$.
So, since $|c| \cdot |x|^k \leqslant \varepsilon \cdot |x|^k$, we get $|c| \leqslant \varepsilon$.
Then $2 \cdot \varepsilon = 2 \cdot (|c|/2) = |c| \leqslant \varepsilon$, so $2 \cdot \varepsilon \leqslant \varepsilon$, so $2 \cdot \varepsilon - \varepsilon \leqslant \varepsilon - \varepsilon$, so $\varepsilon \leqslant 0$, so $0 \geqslant \varepsilon$.
Then $0 \geqslant \varepsilon > 0$, so $0 > 0$. Contradiction. □

4.4. **Linearizations and derivatives.**

THEOREM 4.4.1. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $q \in \mathbb{R}$.*
Assume f is defined near q. Then, near q, we have: $f - f = \mathbf{0}$.

THEOREM 4.4.2. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $q \in \mathbb{D}_f$. *Then:*

$$(\text{ } f \text{ is defined near } q \text{ }) \Rightarrow (\text{ } f_q^{\mathbb{T}} \text{ is defined near } 0 \text{ }).$$

DEFINITION 4.4.3. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$.

$$\textit{Then } \boxed{\mathrm{LINS}_a f} := \{ L \in \mathcal{L} \mid f_a^{\mathbb{T}} - L \in o(1) \}.$$

Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{D}_f$.
Elements of $\mathrm{LINS}_a f$ are called **linearizations** of f at a.

We next show that $(\bullet)^2$ has a linearization at 3:

THEOREM 4.4.4. *Define* $f : \mathbb{R} \to \mathbb{R}$ *by:* $\forall x \in \mathbb{R}$, $f_x = x^2$.
Define $L \in \mathcal{L}$ *by:* $\forall h \in \mathbb{R}$, $L_h = 6h$. *Then* $L \in \mathrm{LINS}_3 f$.

Proof. **Want:** $f_3^{\mathbb{T}} - L \in o(1)$.
Since $(\bullet)^2 \in \mathcal{Q} = \mathcal{H}(2) \subseteq \widehat{\mathcal{O}}(2) \subseteq o(1)$,
it suffices to show: $f_3^{\mathbb{T}} - L = (\bullet)^2$.
Want: $\forall h \in \mathbb{R}$, $(f_3^{\mathbb{T}} - L)_h = ((\bullet)^2)_h$.
Given $h \in \mathbb{R}$. **Want:** $(f_3^{\mathbb{T}} - L)_h = ((\bullet)^2)_h$.
We have:

$$\begin{aligned} (f_3^{\mathbb{T}} - L)_h &= (f_3^{\mathbb{T}})_h - L_h \\ &\overset{*}{=} f_{3+h} - f_3 - L_h \\ &\overset{*}{=} (3+h)^2 - 3^2 - 6h \\ &= 9 + 6h + h^2 - 9 - 6h \\ &= h^2 \quad = \quad ((\bullet)^2)_h. \end{aligned}$$

□

We next show that $|\bullet|$ has no linearization at 0:

THEOREM 4.4.5. *Let* $f := |\bullet|$. *Then* $\mathrm{LINS}_0 f = \varnothing$.

Idea of proof: **Want:** $\forall L \in \mathcal{L}$, $L \notin \mathrm{LINS}_0 f$.
Given $L \in L$. **Want:** $L \notin \mathrm{LINS}_0 f$.
Choose $a \in \mathbb{R}$ s.t. $L = a \cdot (\bullet)$.
In general, we would handle $a \geqslant 0$ and $a \leqslant 0$ separately.
We looked only at $a = 1$. **Want:** $f_0^{\mathbb{T}} - (\bullet) \notin o(1)$.
Since $f_0 = |0| = 0$, we know: $f_0^{\mathbb{T}} = f$.
Want: $f - (\bullet) \notin o(1)$.
We graphed $f - (\bullet)$, and saw that, near 0,
that graph is not in the envelope semi-between $-|\bullet|$ and $|\bullet|$.
So, since the graph is not in every linear envelope, we get: $f - (\bullet) \notin o(1)$.
We leave it to the reader to formalize this argument.
We leave it to the reader to generalize the proof to all $a \geqslant 0$.

We leave it to the reader, then to consider the case $a \leqslant 0$.
End of idea of proof.

THEOREM 4.4.6. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$. *Assume* $\mathrm{LINS}_a f \neq \emptyset$.
Then: $\exists L \in \mathcal{L}, \exists R \in \mathcal{o}(1)$ *s.t.* $f_a^{\mathbb{T}} = L + R$.

Proof. Since $\mathrm{LINS}_a f \neq \emptyset$, choose L s.t. $L \in \mathrm{LINS}_a f$.
Then $L \in \mathrm{LINS}_a f \subseteq \mathcal{L}$ and $f_a^{\mathbb{T}} - L \in \mathcal{o}(1)$.
Let $R := f_a^{\mathbb{T}} - L$. Then $R \in \mathcal{o}(1)$. **Want:** $f_a^{\mathbb{T}} = L + R$.
Since $L \in \mathcal{L}$, we get $-L + L = \mathbf{0}$.
Then $f_a^{\mathbb{T}} = f_a^{\mathbb{T}} - L + L = R + L = L + R$. □

THEOREM 4.4.7. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$. *Assume* $\mathrm{LINS}_a f \neq \emptyset$.
Then: $f_a^{\mathbb{T}} \in \widehat{\mathcal{O}}(1)$ *and* f *is defined near* a
and f *is continuous at* a.

Proof. By Theorem 4.4.6, choose $L \in \mathcal{L}$, $R \in \mathcal{o}(1)$ s.t. $f_a^{\mathbb{T}} = L + R$.
Since $L \in \mathcal{L} = \mathcal{H}(1) \subseteq \widehat{\mathcal{O}}(1)$ and $R \in \mathcal{o}(1) \subseteq \widehat{\mathcal{O}}(1)$,
we conclude that $R + L \in \widehat{\mathcal{O}}(1)$. Then $f_a^{\mathbb{T}} = R + L \in \widehat{\mathcal{O}}(1)$.
Want: f is defined near a and f is continuous at a.
Since $f_a^{\mathbb{T}} \in \widehat{\mathcal{O}}(1) \subseteq \mathcal{o}(0)$, we see that
$f_a^{\mathbb{T}}$ is defined near 0 and $f_a^{\mathbb{T}}$ is continuous at 0.
Then: f is defined near a and f is continuous at a. □

We next show that
no function $\mathbb{R} \dashrightarrow \mathbb{R}$ can have two linearizations at one point:

THEOREM 4.4.8. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$.
Let $L, M \in \mathrm{LINS}_a f$. *Then* $L = M$.

Proof. By assumption, $\mathrm{LINS}_a f \neq \emptyset$.
Let $R := f_a^{\mathbb{T}} - L$, $S := f_a^{\mathbb{T}} - M$.
Since $R, S \in \mathcal{o}(1)$, we get: $R - S \in \mathcal{o}(1)$.
Since f We have $R - S = (f_a^{\mathbb{T}} - f_a^{\mathbb{T}}) + (M - L)$
By Theorem 4.4.7, $f_a^{\mathbb{T}}$ is defined near 0.
Then: near 0, $f_a^{\mathbb{T}} - f_a^{\mathbb{T}} = \mathbf{0}$.
Then: near 0, $R - S = M - L$.
So, since $R - S \in \mathcal{o}(1)$, we get: $M - L \in \mathcal{o}(1)$.
Since $L, M \in \mathcal{L} = \mathcal{H}(1)$, we get: $M - L \in \mathcal{H}(1)$.
Then $M - L \in (\mathcal{H}(1)) \cap (\mathcal{o}(1))$.
So, since $(\mathcal{H}(1)) \cap (\mathcal{o}(1)) = \{\mathbf{0}\}$, we get: $M - L \in \{\mathbf{0}\}$.
Then $M - L = \mathbf{0}$, and so $L = M$. □

DEFINITION 4.4.9. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$.

Then $\boxed{D_a f} := \mathrm{UE}(\mathrm{LINS}_a f)$.

In the preceding, $D_a f$ is the **D-derivative** at a of f.

THEOREM 4.4.10. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$, $L \in \mathcal{L}$. *Then:*

$$(\, D_a f = L \,) \;\Leftrightarrow\; (\, L \in \mathrm{LINS}_a f \,) \;\Leftrightarrow\; (\, f_a^{\mathbb{T}} - L \in \wp(1) \,).$$

Idea of proof:
By definition of $\mathrm{LINS}_a f$, we have:
$L \in \mathrm{LINS}_a f$ iff $f_a^{\mathbb{T}} - L \in \wp(1)$.
We therefore need only show: $L \in \mathrm{LINS}_a f$ iff $L = D_a f$.
By Theorem 4.4.8, we have:
$L \in \mathrm{LINS}_a f$ iff $\{L\} = \mathrm{LINS}_a f$.
Then: $L \in \mathrm{LINS}_a f$ iff $L = \mathrm{UE}(\mathrm{LINS}_a f)$.
Then: $L \in \mathrm{LINS}_a f$ iff $L = D_a f$.
End of idea of proof.

By Theorem 4.4.4, we get $D_3((\bullet)^2) = 6 \cdot (\bullet)$,
or, in other words:

THEOREM 4.4.11. *Let* $f = (\bullet)^2$, $L := 6 \cdot (\bullet)$. *Then* $D_3 f = L$.

Proof. By Theorem 4.4.4, we have: $L \in \mathrm{LINS}_3 f$.
Then by Theorem 4.4.10, we get: $D_3 f = L$. □

THEOREM 4.4.12. *Let* $f = |\bullet|$. *Then* $D_0 f = ☹$.

DEFINITION 4.4.13. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$.

Then $\boxed{f'} : \mathbb{D}_f \dashrightarrow \mathbb{R}$ *is defined by:* $\forall a \in \mathbb{D}_f$, $f'_a = [D_a f]$.

Also, $\mathbb{D}'_f := \mathbb{D}_{f'}$.

In the preceding, f' is called the **prime derivative** of f.
Sometimes we simply call f' the **derivative** of f.

Let $f := (\bullet)^2$, $L := 6 \cdot (\bullet)$. By Theorem 4.4.11, $D_3 f = L$.
Then $f'_3 = [D_3 f] = [6 \cdot (\bullet)] = (6 \cdot (\bullet))_1 = 6 \cdot 1 = 6$.
Unassigned HW: Show: $\forall x \in \mathbb{R}$, $f'_x = 2x$.

Let $g := |\bullet|$. Then, since $D_0 g = ☹$, we get: $g'_0 = ☹$.
Unassigned HW: Show: $\forall x < 0$, $g'_x = -1$.
Unassigned HW: Show: $\forall x > 0$, $g'_x = 1$.

THEOREM 4.4.14. *Let* $g := |\bullet|$*. Then* $\mathbb{D}'_g = \mathbb{R}_0^\times \subsetneq \mathbb{R} = \mathbb{D}_g$.

Let $f := (\bullet)^2|[7;9]$. Then f is not defined near 7, so $7 \notin \mathbb{D}'_f$.
In fact, we have:

THEOREM 4.4.15. *Let* $f := (\bullet)^2|[7;9]$*. Then* $\mathbb{D}'_f = (7;9)$.

THEOREM 4.4.16. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$.
Then: $(\, a \in \mathbb{D}'_f \,) \Leftrightarrow (\, f'_a \neq ☹ \,) \Leftrightarrow (\, D_a f \neq ☹ \,) \Leftrightarrow (\, \mathrm{LINS}_a f \neq \varnothing \,)$.

For any $f : \mathbb{R} \dashrightarrow \mathbb{R}$, for any $a \in \mathbb{R}$,
by f is **differentiable at** a, we mean: $a \in \mathbb{D}'_f$.

For any $f : \mathbb{R} \dashrightarrow \mathbb{R}$, for any $S \subseteq \mathbb{R}$,
by f is **differentiable on** S, we mean: $S \subseteq \mathbb{D}'_f$.

For any $f : \mathbb{R} \dashrightarrow \mathbb{R}$,
by f is **differentiable**, we mean: f is differentiable on $\mathbb{D}_f$.
This is equivalent to: $\mathbb{D}_f = \mathbb{D}'_f$.

By the preceding theorem and Theorem 4.4.7, we have:

THEOREM 4.4.17. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{D}'_f$.
Then: $f_a^{\mathbb{T}} \in \widehat{\mathcal{O}}(1)$ *and* f *is defined near* a
and f *is continuous at* a.

DEFINITION 4.4.18. $\forall f : \mathbb{R} \dashrightarrow \mathbb{R}$, *we define:*
$\boxed{f''} = (f')'$ *and* $f\boxed{'''} = ((f')')'$ *and* $\boxed{f''''} = (((f')')')'$ *and*
$\boxed{\mathbb{D}''_f} := \mathbb{D}_{f''}$ *and* $\boxed{\mathbb{D}'''_f} := \mathbb{D}_{f'''}$ *and* $\boxed{\mathbb{D}''''_f} := \mathbb{D}_{f''''}$.

For any $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $\quad \mathbb{D}_f \supseteq \mathbb{D}'_f \supseteq \mathbb{D}''_f \supseteq \mathbb{D}'''_f \supseteq \mathbb{D}''''_f$.

Let $S := \{f \mid f : \mathbb{R} \dashrightarrow \mathbb{R}\}$ be the set of partial functions $\mathbb{R} \dashrightarrow \mathbb{R}$.
Define $\Phi : S \to S$ by: $\forall f \in S$, $\Phi(f) = f'$.
For any $f \in S$, for all $k \in \mathbb{N}_0$, we define $\boxed{f^{(k)}} := \Phi_\circ^k(f)$.
For any $f \in S$, we have $f^{(0)} = f$ and $f^{(1)} = f'$ and $f^{(2)} = f''$
and $f^{(3)} = f'''$ and $f^{(4)} = f''''$.
For any $f \in S$, we have: $\forall k \in \mathbb{N}_0$, $f^{(k+1)} = (f^{(k)})'$.
For any $f \in S$, for all $k \in \mathbb{N}_0$, we define $\mathbb{D}_f^{(k)} := \mathbb{D}_{f^{(k)}}$.
For any $f \in S$, for all $k \in \mathbb{N}_0$, we have $\mathbb{D}_f^{(k)} \supseteq \mathbb{D}_f^{(k+1)}$.
Then: $\quad \forall f \in S$, $\mathbb{D}_f^{(0)} \supseteq \mathbb{D}_f^{(1)} \supseteq \mathbb{D}_f^{(2)} \supseteq \mathbb{D}_f^{(3)} \supseteq \cdots$.

4.5. **Derivatives of Polynomials.**

THEOREM 4.5.1. *Let* $C \in \mathcal{C}$*,* $a \in \mathbb{R}$*. Then* $C_a^{\mathbb{T}} = \mathbf{0}$*.*

Idea of proof: Choose $b \in \mathbb{R}$ s.t. $C = C_b^{\mathbb{R}}$. For any $h \in \mathbb{R}$, we have
$$(C_a^{\mathbb{T}})_h \;=\; C_{a+h} - C_a \;=\; (C_b^{\mathbb{R}})_{a+h} - (C_b^{\mathbb{R}})_a \;=\; b - b \;=\; 0 \;=\; \mathbf{0}_h.$$
End of idea of proof.

THEOREM 4.5.2. *Let* $C \in \mathcal{C}$*,* $a \in \mathbb{R}$*. Then* $D_a C = \mathbf{0}$*.*

Proof. Since $\mathbf{0} \in \mathcal{L}$, **it suffices to show:** $C_a^{\mathbb{T}} - \mathbf{0} \in \mathcal{O}(1)$.
We have $C_a^{\mathbb{T}} - \mathbf{0} = \mathbf{0} - \mathbf{0} = \mathbf{0} \in \mathcal{O}(1)$. □

THEOREM 4.5.3. *Let* $C \in \mathcal{C}$*. Then* $C' = \mathbf{0}$*.*

Idea of proof: For any $x \in \mathbb{R}$, we have
$$C'_x \;=\; [D_x C] \;=\; [\mathbf{0}] \;=\; \mathbf{0}_1 \;=\; 0.$$
End of idea of proof.

THEOREM 4.5.4. *Let* $L \in \mathcal{L}$*,* $a \in \mathbb{R}$*. Then* $L_a^{\mathbb{T}} = L$*.*

Proof. **Want:** $\forall h \in \mathbb{R}$, $(L_a^{\mathbb{T}})_h = L_h$.
Given $h \in \mathbb{R}$. **Want:** $(L_a^{\mathbb{T}})_h = L_h$.
Since L is algebraically linear, we have: $L_{a+h} = L_a + \mathrm{L}_h$.
Then $(L_a^{\mathbb{T}})_h = L_{a+h} - L_a = L_a + L_h - L_a = L_h$. □

Since $(\bullet) = 1 \cdot (\bullet)^1 \in \mathcal{H}_1 = \mathcal{L}$,
the preceding theorem gives: $\forall a \in \mathbb{R}$, $(\bullet)_a^{\mathbb{T}} = (\bullet)$.

THEOREM 4.5.5. *Let* $L \in \mathcal{L}$*,* $a \in \mathbb{R}$*. Then* $D_a L = L$*.*

Proof. We have $L_a^{\mathbb{T}} - L = L - L = \mathbf{0} \in \mathcal{O}(1)$, so $L \in \mathrm{LINS}_a L$.
Then, by uniqueness of linearization, we get: $\mathrm{LINS}_a L = \{L\}$.
Then $D_a L = \mathrm{UE}(\mathrm{LINS}_a L) = \mathrm{UE}\{L\} = L$. □

THEOREM 4.5.6. *Let* $m \in \mathbb{R}$*,* $L := m \cdot (\bullet)$*. Then* $L' = C_m^{\mathbb{R}}$*.*

Proof. **Want:** $\forall a \in \mathbb{R}$, $L'_a = (C_m^{\mathbb{R}})_a$.
Given $a \in \mathbb{R}$. **Want:** $L'_a = (C_m^{\mathbb{R}})_a$.
We have $L'_a \;=\; [D_a L] \;=\; [L] \;=\; L_1 = m \cdot 1 = m = (C_m^{\mathbb{R}})_a$. □

THEOREM 4.5.7. $\forall j \in \mathbb{N}$*,* $((\bullet)^j)_a^{\mathbb{T}} \,-\, j \cdot a^{j-1} \cdot (\bullet) \in \mathcal{O}(1)$*.*

Proof. This is HW#12-1. □

THEOREM 4.5.8. *Let* $j \in \mathbb{N}$*. Then* $((\bullet)^j)' = j \cdot (\bullet)^{j-1}$*.*

Proof. This is HW#12-2. □

4.6. **Sub-k versus order k vanishing.**

Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$. Recall that $\qquad f_0 = 0 \qquad$ iff

f is defined near 0 $\quad$ and $\quad$ f is continuous at 0 $\quad$ and $\quad$ $f_0 = 0$.

Then: $\quad f_0 = 0 \quad \Leftarrow \quad f \in \mathcal{O}(0)$,

but $\quad f_0 = 0 \quad \not\Rightarrow \quad f \in \mathcal{O}(0)$.

So: subconstant implies vanishes at zero, but not conversely.

We will show below that

$$f_0 = f'_0 = 0 \quad \Leftrightarrow \quad f \in \mathcal{O}(1).$$

That is, sublinear iff vanishes to order 1 at zero.

We will also show below (after the MVT) that

$$f_0 = f'_0 = f''_0 = 0 \quad \Rightarrow \quad f \in \mathcal{O}(2),$$

but $\quad f_0 = f'_0 = f''_0 = 0 \quad \not\Leftarrow \quad f \in \mathcal{O}(2)$,

So: vanishes to order 2 at zero implies subquadratic, but not conversely.

Unassigned HW: show, for all $k \in [2..\infty)$, that

vanishes to order k at zero implies sub-k, but not conversely.

THEOREM 4.6.1. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$. *Then:*

$$f \in \mathcal{O}(1) \qquad \Leftrightarrow \qquad f_0 = f'_0 = 0.$$

Proof. Proof of $\Rightarrow$*:*

Assume: $f \in \mathcal{O}(1)$ $\qquad$ **Want:** $f_0 = f'_0 = 0$.

Since $f \in \mathcal{O}(1) \subseteq \mathcal{O}(0)$, it follows that:

f is defined near 0 $\quad$ and $\quad$ f is continuous at 0 $\quad$ and $\quad$ $f_0 = 0$.

Since $f_0 = 0$, **it remains to show:** $f'_0 = 0$.

Since $f_0 = 0$, we get: $f_0^{\mathbb{T}} = f$.

Then $f_0^{\mathbb{T}} - \mathbf{0} = f_0^{\mathbb{T}} = f \in \mathcal{O}(1)$, and so $\mathbf{0} \in \mathrm{LINS}_0 f$.

Then, by uniqueness of linearization, $\mathrm{LINS}_0 f = \{\mathbf{0}\}$.

Then $D_0 f = \mathrm{UE}(\mathrm{LINS}_0 f) = \mathrm{UE}\{\mathbf{0}\} = \mathbf{0}$.

Then $f'_0 = [\mathbb{D}_0 f] = [\mathbf{0}] = \mathbf{0}_1 = 0$.

End of proof of $\Rightarrow$.

Proof of $\Leftarrow$*:*

Assume: $f_0 = f'_0 = 0$ $\qquad$ **Want:** $f \in \mathcal{O}(1)$.

Since $f'_0 \neq \text{☹}$, we get $D_0 f \neq \text{☹}$, so $D_0 f \in \mathrm{LINS}_0 f$. $\quad$ Let $L := D_0 f$.

Then $L \in \mathrm{LINS}_0 f$, $\quad$ so $f_0^{\mathbb{T}} - L \in \mathcal{O}(1)$. $\quad$ **Want:** $f_0^{\mathbb{T}} - L = f$.

Since $f_0 = 0$, we get: $f_0^{\mathbb{T}} = f$. $\quad$ **Want:** $f - L = f$. $\quad$ **Want:** $L = \mathbf{0}$.

Want: $\forall h \in \mathbb{R}$, $L_h = \mathbf{0}_h$. $\qquad$ Given $h \in \mathbb{R}$. $\quad$ **Want:** $L_h = \mathbf{0}_h$.

Since $L \in \mathrm{LINS}_0 f \subseteq \mathcal{L}$, we see that L is algebraically linear.

Then $L_{h \cdot 1} = h \cdot L_1$. $\quad$ We have $L_1 = [L] = [D_0 f] = f'_0 = 0$.

Then $L_h = L_{h\cdot 1} = h \cdot L_1 = h \cdot 0 = 0 = \mathbf{0}_h$.
End of proof of $\Leftarrow$. □

DEFINITION 4.6.2. *Let* $f : \mathbb{R} \to \mathbb{R}$, $a \in \mathbb{R}$.
By f *is* **continuous near** a, *we mean:* $\exists B \in \mathcal{B}(a)$ *s.t.* f *is continuous on* B.
By f *is* **differentiable near** a, *we mean:* $\exists B \in \mathcal{B}(a)$ *s.t.* $B \subseteq \mathbb{D}'_f$.

Recall: $\forall g : \mathbb{R} \dashrightarrow \mathbb{R}$, $0 \in \mathbb{D}'_g$ implies:
$g_0^{\mathbb{T}} \in \widehat{\mathcal{O}}(1)$ and g is defined near 0 and g is differentiable at 0.

Let $A := \{1, 1/2, 1/3, \ldots\}$ and let $f := \chi_A^{\mathbb{R}} \cdot |\bullet|^3$.
Then $f \in \widehat{\mathcal{O}}(3) \subseteq \mathcal{o}(2)$. Also, f is not continuous near 0.
Since $f \in \mathcal{o}(2) \subseteq \mathcal{o}(1)$, by Theorem 4.6.1, we get: $f_0 = f'_0 = 0$.
Let $a := 0$. Then this function f shows:
differentiable at a does not imply continuous near a.
Since f is not continuous near 0,
it follows that f is not differentiable near 0.
Then f' is not defined near 0.
Let $g := f'$. Then g is not defined near 0. Then $0 \notin \mathbb{D}'_g$.
Then $0 \notin \mathbb{D}'_g = \mathbb{D}'_{f'} = \mathbb{D}''_f$.
So this function f shows:
subquadratic does not imply vanishes to order 2 at zero.
Note that the counterexample f fails to vanish to order 2 at zero
because f is not twice differentiable at zero.
This begs the question:

Let $h : \mathbb{R} \dashrightarrow \mathbb{R}$ be twice differentiable at zero.
Then do we have: $h_0 = h'_0 = h''_0 = 0 \quad \Leftrightarrow \quad h \in \mathcal{o}(2)$?
The answer is yes, and, in fact, we'll eventually prove:

THEOREM 4.6.3. *Let* $h : \mathbb{R} \dashrightarrow \mathbb{R}$, $k \in \mathbb{N}_0$. *Assume* $0 \in \mathbb{D}_h^{(k)}$.
Then: $h \in \mathcal{o}(k) \quad \Leftrightarrow \quad h_0 = h'_0 = \cdots = h_0^{(k)} = 0$.

4.7. Algebraic linearity of the D-derivative.

THEOREM 4.7.1. *Let* $f := |\bullet|$, $g := -|\bullet|$.
Then $D_0(f + g) = \mathbf{0}$. *Also,* $(D_0 f) + (D_0 g) = ☹$.

Proof. We have $f + g = \mathbf{0} \in \mathcal{C}$, so $D_0(f + g) = \mathbf{0}$.
Want: $(D_0 f) + (D_0 g) = ☹$.
Since $D_0 f = ☹$, it follows that $(D_0 f) + (D_0 g) = ☹$. □

THEOREM 4.7.2. *Let* $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$.
Then $D_a(f+g) \quad =^* \quad (D_a f) + (D_a g)$.

Proof. **Want:** $(\ (D_a f) + (D_a g) \neq ☹\ ) \Rightarrow (\ D_a(f+g) = (D_a f) + (D_a g)\ )$.
Assume $(D_a f) + (D_a g) \neq ☹$. **Want:** $D_a(f+g) = (D_a f) + (D_a g)$.
Let $L := D_a f$, $M := D_a g$. Then $L + M \neq ☹$, so $L \neq ☹ \neq M$.
Then $L, M \in \mathcal{L}$ and $f_a^{\mathbb{T}} - L, g_a^{\mathbb{T}} - M \in \mathcal{o}(1)$.
Want: $D_a(f+g) = L + M$.
Since $\mathcal{L}$ is linearly closed and $L, M \in \mathcal{L}$, we get: $L + M \in \mathcal{L}$.
By uniqueness of linearization, want: $(f+g)_a^{\mathbb{T}} - (L+M) \in \mathcal{o}(1)$.
Since $f_a^{\mathbb{T}} - L, g_a^{\mathbb{T}} - M \in \mathcal{o}(1)$ and since $\mathcal{o}(1)$ is linearly closed,
we conclude: $(f_a^{\mathbb{T}} - L) + (g_a^{\mathbb{T}} - M) \in \mathcal{o}(1)$.
Then $(f+g)_a^{\mathbb{T}} - (L+M) = f_a^{\mathbb{T}} + g_a^{\mathbb{T}} - L - M$
$= (f_a^{\mathbb{T}} - L) + (g_a^{\mathbb{T}} - M) \in \mathcal{o}(1)$. □

THEOREM 4.7.3. *Let* $f := |\bullet|$.
Then $D_0(0 \cdot f) = \mathbf{0}$. *Also,* $0 \cdot (D_0 f) = ☹$.

Proof. We have $0 \cdot f = \mathbf{0} \in \mathcal{C}$, so $D_0(0 \cdot f) = \mathbf{0}$.
Want: $0 \cdot (D_0 f) = ☹$.
Since $D_0 f = ☹$, it follows that $0 \cdot (D_0 f) = ☹$. □

THEOREM 4.7.4. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a, c \in \mathbb{R}$.
Then $D_a(c \cdot f) \quad =^* \quad c \cdot (D_a f)$.

Proof. This is HW#10-5. □

THEOREM 4.7.5. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$, $c \in \mathbb{R}_0^\times$.
Then $D_a(c \cdot f) \; = \; c \cdot (D_a f)$.

Proof. By Theorem 4.7.4, we have: $D_a(c \cdot f) \quad =^* \quad c \cdot (D_a f)$.
Want: $D_a(c \cdot f) \quad {}^*\!= \quad c \cdot (D_a f)$.
Want: $c \cdot (D_a f) \quad =^* \quad D_a(c \cdot f)$.
Let $\phi := c \cdot f$ and let $\gamma := 1/c$.
By Theorem 4.7.4, $D_a(\gamma \cdot \phi) \quad =^* \quad \gamma \cdot (D_a \phi)$.
Then $c \cdot (D_a(\gamma \cdot \phi)) \quad =^* \quad c \cdot \gamma \cdot (D_a \phi)$.
So, since $\gamma \cdot \phi = (1/c) \cdot c \cdot f = f$ and since $c \cdot \gamma = c \cdot (1/c) = 1$, we get:
$c \cdot (D_a f) \quad =^* \quad 1 \cdot (D_a \phi)$.
Then $c \cdot (D_a f) \quad =^* \quad 1 \cdot (D_a \phi) \; = \; D_a \phi \; = \; D_a(c \cdot f)$. □

4.8. **The D-product and chain rules.**

The following is the D-**product rule**:

THEOREM 4.8.1. *Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$.*
Then $D_a(f \cdot g) \;\; =^* \;\; (D_a f) \cdot g_a \; + \; f_a \cdot (D_a g)$.

Proof. **Want:** $(\ (D_a f) \cdot g_a \; + \; f_a \cdot (D_a g) \neq ☹\ )$
$\Rightarrow (\ D_a(f \cdot g) \; = \; (D_a f) \cdot g_a \; + \; f_a \cdot (D_a g)\ )$.
Assume $(D_a f) \cdot g_a \; + \; f_a \cdot (D_a g) \; \neq \; ☹$.
Want: $D_a(f \cdot g) \; = \; (D_a f) \cdot g_a \; + \; f_a \cdot (D_a g)$.
Let $L := D_a f$, $M := D_a g$, $y := f_a$, $z := g_a$.
Then $L \cdot z + y \cdot M \neq ☹$, so $L \neq ☹$ & $z \neq ☹$ & $y \neq ☹$ & $M \neq ☹$.
Then $y, z \in \mathbb{R}$ and $L, M \in \mathcal{L}$ and $f_a^{\mathbb{T}} - L, g_a^{\mathbb{T}} - M \in \mathcal{o}(1)$.
Want: $D_a(f \cdot g) \; = \; L \cdot z \; + \; y \cdot M$.
Since $\mathcal{L}$ is linearly closed and $L, M \in \mathcal{L}$, we get: $L \cdot z \; + \; y \cdot M \in \mathcal{L}$.
By uniqueness of linearization, want: $(f \cdot g)_a^{\mathbb{T}} - (L \cdot z + y \cdot M) \in \mathcal{o}(1)$.
By the Precalculus Product Rule, $(f \cdot g)_a^{\mathbb{T}} = f_a^{\mathbb{T}} \cdot g_a + f_a \cdot g_a^{\mathbb{T}} + f_a^{\mathbb{T}} \cdot g_a^{\mathbb{T}}$.
Then $(f \cdot g)_a^{\mathbb{T}} - (L \cdot z + y \cdot M) = (f_a^{\mathbb{T}} - L) \cdot g_a + f_a \cdot (g_a^{\mathbb{T}} - M) + f_a^{\mathbb{T}} \cdot g_a^{\mathbb{T}}$.
Want: $(f_a^{\mathbb{T}} - L) \cdot g_a \; + \; f_a \cdot (g_a^{\mathbb{T}} - M) \; + \; f_a^{\mathbb{T}} \cdot g_a^{\mathbb{T}} \; \in \; \mathcal{o}(1)$.
Want: $(f_a^{\mathbb{T}} - L) \cdot g_a, \quad f_a \cdot (g_a^{\mathbb{T}} - M), \quad f_a^{\mathbb{T}} \cdot g_a^{\mathbb{T}} \; \in \; \mathcal{o}(1)$.
Since $f_a^{\mathbb{T}} - L \in \mathcal{o}(1)$ and since $\mathcal{o}(1)$ is linearly closed,
we get: $(f_a^{\mathbb{T}} - L) \cdot g_a \; \in \; \mathcal{o}(1)$.
Since $g_a^{\mathbb{T}} - M \in \mathcal{o}(1)$ and since $\mathcal{o}(1)$ is linearly closed,
we get: $f_a \cdot (g_a^{\mathbb{T}} - M) \; \in \; \mathcal{o}(1)$.
Want: $f_a^{\mathbb{T}} \cdot g_a^{\mathbb{T}} \; \in \; \mathcal{o}(1)$. Since $f_a^{\mathbb{T}}, g_a^{\mathbb{T}} \in \widehat{\mathcal{O}}(1)$, we get: $f_a^{\mathbb{T}} \cdot g_a^{\mathbb{T}} \; \in \; \widehat{\mathcal{O}}(2)$.
Then $f_a^{\mathbb{T}} \cdot g_a^{\mathbb{T}} \; \in \; \widehat{\mathcal{O}}(2) \; \subseteq \; \mathcal{o}(1)$. □

THEOREM 4.8.2. *Let f and g be functionals. Let h be a function.*
Then $(f + g) \circ h = (f \circ h) + (g \circ h)$.

Proof. **Want:** $\forall x, \;\; ((f + g) \circ h)_x \;\; = \;\; ((f \circ h) + (g \circ h))_x$.
Given x. **Want:** $((f + g) \circ h)_x \;\; = \;\; ((f \circ h) + (g \circ h))_x$.
We have $((f + g) \circ h)_x \;\; = \;\; (f + g)_{h_x} \;\; = \;\; f_{h_x} + g_{h_x}$
$\stackrel{*}{=} \;\; (f \circ h)_x + (g \circ h)_x$
$\stackrel{*}{=} \;\; ((f \circ h) + (g \circ h))_x$. □

THEOREM 4.8.3. *Let $L \in \mathcal{L}$. Then:*
$[\ \forall c, x \in \mathbb{R}, \;\; L_{c \cdot x} = c \cdot L_x\ ]$
$\&\ \ [\ \forall w, x \in \mathbb{R}, \;\; L_{w+x} = L_w + L_x\ ]$.

The proof of the preceding theorem is left as an unassigned HW.
The preceding theorem can be used to prove the next two:

THEOREM 4.8.4. *Let $L \in \mathcal{L}$ and let $c \in \mathbb{R}$ and let f be a function. Then $L \circ (c \cdot f) = c \cdot (L \circ f)$.*

THEOREM 4.8.5. *Let $L \in \mathcal{L}$ and let f, g be functions. Then $L \circ (f + g) = (L \circ f) + (L \circ g)$.*

Theorem 4.8.2 is sometimes expressed by saying
$\circ$ is linear on the left.
Theorem 4.8.4 and Theorem 4.8.5 are sometimes expressed by saying
$\circ$ is linear on the right, PROVIDED the left function is linear.
However, if the left function is, say, a quadratic Q,
then we get different formulas for $Q \circ (f + g)$ and $Q(c \cdot f)$:

THEOREM 4.8.6. *Let $Q \in \mathcal{Q}$. Let f and g be functions. Then $Q \circ (f + g) = (Q \circ f) + 2 \cdot f \cdot g + (Q \circ g)$.*

THEOREM 4.8.7. *Let $Q \in \mathcal{Q}$. Let $c \in \mathbb{R}$. Let f be a function. Then $Q \circ (c \cdot f) = c^2 \cdot (Q \circ f)$.*

The next theorem expresses that $\mathcal{L}$ is closed under composition.
It also says that the slope of the composite is the product of the slopes.

THEOREM 4.8.8. *Let $L, M \in \mathcal{L}$. Then:*
$$M \circ L \in \mathcal{L} \qquad \text{and} \qquad [M \circ L] = [M] \cdot [L].$$

Proof. Since $L, M \in \mathcal{L} = \mathcal{H}(1)$, we get $M \circ L \in \mathcal{H}(1 \cdot 1) = \mathcal{H}(1)$.
Want: $[M \circ L] = [M] \cdot [L]$.
Let $a := [L]$, $b := [M]$. **Want:** $[M \circ L] = b \cdot a$.
We have $a = [L] = L_1$, so $a = L_1$.
We have $b = [M] = M_1$, so $b = M_1$.
By algebraic linearity of M, we have $M_{a \cdot 1} = a \cdot M_1$.
Then $[M \circ L] = (M \circ L)_1 = M_{L_1} = M_a = M_{a \cdot 1} = a \cdot M_1 = a \cdot b = b \cdot a$. □

We will be using two properties of $\widehat{\mathcal{O}}$ and o:
$$\forall \alpha \in \widehat{\mathcal{O}}(1), \forall \beta \in o(1), \quad \beta \circ \alpha \in o(1 \cdot 1) = o(1) \qquad \text{and}$$
$$\forall \alpha \in o(1), \forall \beta \in \widehat{\mathcal{O}}(1), \quad \beta \circ \alpha \in o(1 \cdot 1) = o(1).$$
The following is the D-**chain rule**:

THEOREM 4.8.9. *Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$. Then:*
$$D_a(g \circ f) \quad =^* \quad (D_{f_a} g) \circ (D_a f).$$

Proof. **Want:** $(\ (D_{f_a}g)\circ(D_af)\neq ☹\ )$
$\Rightarrow (\ D_a(g\circ f)=(D_{f_a}g)\circ(D_af)\ )$.
Assume $(D_{f_a}g)\circ(D_af)\neq ☹$.
Want: $D_a(g\circ f)=(D_{f_a}g)\circ(D_af)$.
Let $L:=D_af$, $M:=D_{f_a}g$.
Then $M\circ L\neq ☹$, so $L\neq ☹\neq M$.
Then $L,M\in\mathcal{L}$ and $f_a^{\mathbb{T}}-L, g_{f_a}^{\mathbb{T}}-M\in\mathcal{o}(1)$.
Let $R:=f_a^{\mathbb{T}}-L$, $S:=g_{f_a}^{\mathbb{T}}-M$. Then $R,S\in\mathcal{o}(1)$.
Also $L+R=f_a^{\mathbb{T}}$ and $M+S=g_{f_a}^{\mathbb{T}}$.
Want: $D_a(g\circ f)=M\circ L$.
Since $\mathcal{L}$ is closed under composition, and $L,M\in\mathcal{L}$, we get: $M\circ L\in\mathcal{L}$.
By uniqueness of linearization, want: $(g\circ f)_a^{\mathbb{T}}-(M\circ L)\in\mathcal{o}(1)$.
By the Precalculus Chain Rule, $(g\circ f)_a^{\mathbb{T}}=g_{f_a}^{\mathbb{T}}\circ f_a^{\mathbb{T}}$.
Want: $g_{f_a}^{\mathbb{T}}\circ f_a^{\mathbb{T}}-(M\circ L)\in\mathcal{o}(1)$.
We have
$$\begin{aligned} g_{f_a}^{\mathbb{T}}\circ f_a^{\mathbb{T}} &= (M+S)\circ f_a^{\mathbb{T}}\\ &= M\circ f_a^{\mathbb{T}}\ +\ S\circ f_a^{\mathbb{T}}\\ &= M\circ(L+R)\ +\ S\circ f_a^{\mathbb{T}}\\ &= M\circ L\ +\ M\circ R\ +\ S\circ f_a^{\mathbb{T}}.\end{aligned}$$
Then $g_{f_a}^{\mathbb{T}}\circ f_a^{\mathbb{T}}-(M\circ L)=M\circ R\ +\ S\circ f_a^{\mathbb{T}}$.
Want: $M\circ R\ +\ S\circ f_a^{\mathbb{T}}\in\mathcal{o}(1)$.
Want: $M\circ R$, $S\circ f_a^{\mathbb{T}}\in\mathcal{o}(1)$.
Since $M\in\mathcal{L}=\mathcal{H}(1)\subseteq\widehat{\mathcal{O}}(1)$ and $R\in\mathcal{o}(1)$, we get: $M\circ R\in\mathcal{o}(1)$.
Want: $S\circ f_a^{\mathbb{T}}\in\mathcal{o}(1)$.
Since $S\in\mathcal{o}(1)$ and $f_a^{\mathbb{T}}\in\widehat{\mathcal{O}}(1)$, we get: $S\circ f_a^{\mathbb{T}}\in\mathcal{o}(1)$. □

4.9. **Properties of the prime derivative.**

Unassigned HW:
$\forall c\in\mathbb{R}, \forall L\in\mathcal{L},\quad [c\cdot L]=c\cdot[L]$ and
$\forall L,M\in\mathcal{L},\quad [L+M]=[L]+[M]$.

THEOREM 4.9.1. *Let $f,g:R \dashrightarrow \mathbb{R}$, $a\in\mathbb{R}$.*
Then: $(f+g)'_a \ =^*\ f'_a+g'_a$.

Proof. We have $(f+g)'_a\ =\ [D_a(f+g)]\ =^*\ [D_af+D_ag]$
$\stackrel{*}{=}\ [D_af]+[D_ag]\ =\ f'_a+g'_a$. □

THEOREM 4.9.2. *Let $f:R \dashrightarrow \mathbb{R}$, $a,c\in\mathbb{R}$.*
Then: $(c\cdot f)'_a\ =^*\ c\cdot f'_a$.

Proof. We have $(c\cdot f)'_a\ =\ [D_a(c\cdot f)]\ =^*\ [c\cdot D_af]$
$\stackrel{*}{=}\ c\cdot[D_af]\ =\ c\cdot f'_a$. □

The preceding two theorems can be summarized as:
    the prime derivative is algebraically linear.
The following is the **prime product rule**:

THEOREM 4.9.3. *Let* $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$.
Then: $(f \cdot g)'_a \;=^*\; f'_a \cdot g_a \,+\, f_a \cdot g'_a$.

Proof.
$$\begin{aligned}(f \cdot g)'_a &= [D_a(f \cdot g)] \\ &=^* [D_a f \cdot g_a \,+\, f_a \cdot D_a g] \\ &= [D_a f] \cdot g_a \,+\, f_a \cdot [D_a g] \\ &= f'_a \cdot g_a \,+\, f_a \cdot g'_a.\end{aligned}$$
□

The following is the **prime chain rule**:

THEOREM 4.9.4. *Let* $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$.
Then: $(g \circ f)'_a \;=^*\; g'_{f_a} \circ f'_a$.

Proof.
$$\begin{aligned}(g \circ f)'_a &= [D_a(g \circ f)] \\ &=^* [D_{f_a} g \circ D_a f] \\ &= [D_{f_a} g] \cdot [D_a f] \\ &= g'_{f_a} \circ f'_a.\end{aligned}$$
□

The following is the **prime quotient rule**:

THEOREM 4.9.5. *Let* $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{D}'_f \cap \mathbb{D}'_g$.
Assume $g_a \neq 0$. *Then:* $\left(\dfrac{f}{g}\right)'_a = -\dfrac{g_a \cdot f'_a - f_a \cdot g'_a}{g_a^2}$.

Proof. This is HW#11-3. □

4.10. The Mean Value Theorem.

DEFINITION 4.10.1. *We define* $\boxed{\mathrm{sgn}} : \mathbb{R} \to \{-1, 0, 1\}$ *by:*

$$\forall x \in \mathbb{R}, \qquad \mathrm{sgn}_x \;=\; \begin{cases} -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0. \end{cases}$$

The function sgn is read "sign",
not to be confused with the trignonometric function "sine",
which is not defined in this course.

THEOREM 4.10.2. $(\,\mathrm{sgn}_3 = 1\,) \,\&\, (\,\mathrm{sgn}_{-\pi} = -1\,) \,\&\, (\,\mathrm{sgn}_0 = 0\,)$.

The function sgn is multiplicative:

THEOREM 4.10.3. $\forall w, x \in \mathbb{R}, \quad \text{sgn}_{w \cdot x} \;=\; \text{sgn}_w \cdot \text{sgn}_x.$

The function sgn is "robust", in the sense that
small perturbations to an input don't affect the output:

THEOREM 4.10.4. *Let* $a, b \in \mathbb{R}$.
Assume $|a| \leqslant |b|/2$. *Then* $\text{sgn}_{a+b} = \text{sgn}_b$.

Proof. Exactly one of the following must be true:
(1) $b < 0$ (2) $b = 0$ (3) $b > 0$.

Case (1): Since $b < 0$, we get: $\text{sgn}_b = -1$ and $b/2 < 0$.
We have $|a| \leqslant |b|/2 = -b/2$, so $b/2 \leqslant a \leqslant -b/2$.
Since $a \leqslant -b/2$, we get $b + a \leqslant b - (b/2)$.
Since $b + a \leqslant b - (b/2) = b/2 < 0$, we get $\text{sgn}_{b+a} = -1$.
Then $\text{sgn}_{b+a} = -1 = \text{sgn}_b$.
End of Case (1).

Case (2): Since $b = 0$, we get: $\text{sgn}_b = 0$ and $b/2 = 0$.
We have $|a| \leqslant |b|/2 = 0$, so $-0 \leqslant a \leqslant 0$.
Then $a = 0$, so $b + a = 0 + 0 = 0$, so $\text{sgn}_{b+a} = 0$.
Then $\text{sgn}_{b+a} = 0 = \text{sgn}_b$.
End of Case (2).

Case (3): Since $b > 0$, we get: $\text{sgn}_b = 1$ and $0 < b/2$.
We have $|a| \leqslant |b|/2 = b/2$, so $-b/2 \leqslant a \leqslant b/2$.
Since $-b/2 \leqslant a$, we get $b - (b/2) \leqslant b + a$.
Since $0 < b/2 = b - (b/2) \leqslant b + a$, we get $\text{sgn}_{b+a} = 1$.
Then $\text{sgn}_{b+a} = 1 = \text{sgn}_b$.
End of Case (3). □

DEFINITION 4.10.5. *Let* S *be a set,* f *a functional,* $b \in \mathbb{R}$.
By $\boxed{\text{on } S,\, f < b}$, *we mean:* $\forall x \in S, \quad f_x < b$.
By $\boxed{\text{on } S,\, f > b}$, *we mean:* $\forall x \in S, \quad f_x > b$.
By $\boxed{\text{on } S,\, f \leqslant b}$, *we mean:* $\forall x \in S, \quad f_x \leqslant b$.
By $\boxed{\text{on } S,\, f \geqslant b}$, *we mean:* $\forall x \in S, \quad f_x \geqslant b$.
By $\boxed{\text{on } S,\, b < f}$, *we mean:* $\forall x \in S, \quad b < f_x$.
By $\boxed{\text{on } S,\, b > f}$, *we mean:* $\forall x \in S, \quad b > f_x$.

By $\boxed{\text{on } S,\, b \leqslant f}$, *we mean:* $\forall x \in S, \quad b \leqslant f_x$.

By $\boxed{\text{on } S,\, b \geqslant f}$, *we mean:* $\forall x \in S, \quad b \geqslant f_x$.

DEFINITION 4.10.6. *Let S be a set and let f, g be functionals.*

By $\boxed{\text{on } S,\, f < g}$, *we mean:* $\forall x \in S, \quad f_x < g_x$.

By $\boxed{\text{on } S,\, f > g}$, *we mean:* $\forall x \in S, \quad f_x > g_x$.

By $\boxed{\text{on } S,\, f \leqslant g}$, *we mean:* $\forall x \in S, \quad f_x \leqslant g_x$.

By $\boxed{\text{on } S,\, f \geqslant g}$, *we mean:* $\forall x \in S, \quad f_x \geqslant g_x$.

There are many theorems like the next one.
All are unassigned HW, and may be used without comment, in proofs.

THEOREM 4.10.7. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $S \subseteq \mathbb{R}$, $a \in \mathbb{D}_f$, $b \in \mathbb{R}$.*
Then: $[\text{ on } S, \quad f < b \,]$
$\Leftrightarrow \quad [\text{ on } S - a, \quad f_a^{\mathbb{T}} < b - f_a \,]$.

THEOREM 4.10.8. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{D}_f$, $\phi := f_a^{\mathbb{T}}$.*
Then $\mathrm{LINS}_a f \;=\; \mathrm{LINS}_0 \phi$.

Proof. **Want:** $\forall L \in \mathcal{L}$, $(\, L \in \mathrm{LINS}_a f \,) \Leftrightarrow (\, L \in \mathrm{LINS}_0 \phi \,)$.
Given $L \in \mathcal{L}$. **Want:** $(\, L \in \mathrm{LINS}_a f \,) \Leftrightarrow (\, L \in \mathrm{LINS}_0 \phi \,)$.
Since $a \in \mathbb{D}_f$, we get $(f_a)_0^{\mathbb{T}} = 0$. Then $\phi_0 = 0$. Then $\phi_0^{\mathbb{T}} = \phi$.
Then $f_a^{\mathbb{T}} = \phi = \phi_0^{\mathbb{T}}$.
Then: $(\, L \in \mathrm{LINS}_a f \,) \Leftrightarrow (\, f_a^{\mathbb{T}} - L \in \mathcal{o}(1) \,)$
$\Leftrightarrow (\, \phi_0^{\mathbb{T}} - L \in \mathcal{o}(1) \,) \Leftrightarrow (\, L \in \mathrm{LINS}_0 \phi \,)$. □

THEOREM 4.10.9. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{D}_f$, $\phi := f_a^{\mathbb{T}}$.*
Then: $D_a f \;=\; D_0 \phi$ *and* $f'_a \;=\; \phi'_0$.

Proof. From the preceding theorem, $\mathrm{LINS}_a f = \mathrm{LINS}_0 \phi$.
Then: $D_a f \;=\; \mathrm{UE}(\mathrm{LINS}_a f) \;=\; \mathrm{UE}(\mathrm{LINS}_0 \phi) \;=\; D_0 \phi$.
Want: $f'_a \;=\; \phi'_0$.
We have: $f'_a \;=\; [D_a f] \;=\; [D_0 \phi] \;=\; \phi'_0$. □

DEFINITION 4.10.10. *Let f be a functional, $a \in \mathbb{D}_f$.*

By f has a **global semi-maximum** *at a, we mean:*
$\forall x \in \mathbb{D}_f, \quad f_x \leqslant f_a$.

By f has a **global strict-maximum** *at a, we mean:*
$\forall x \in (\mathbb{D}_f)_a^{\times}, \quad f_x < f_a$.

By f has a **global semi-minimum** *at a, we mean:*
$\forall x \in \mathbb{D}_f, \quad f_x \geqslant f_a$.

By f has a **global strict-minimum** *at a, we mean:*

$$\forall x \in (\mathbb{D}_f)_a^\times, \quad f_x > f_a.$$

DEFINITION 4.10.11. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{D}_f$.*

By f has a **global semi-extremum** *at a, we mean:*

f has a global semi-maximum at a

or f has a global semi-minimum at a.

By f has a **global strict-extremum** *at a, we mean:*

f has a global strict-maximum at a

or f has a global strict-minimum at a.

DEFINITION 4.10.12. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{D}_f$.*

By f has a **local semi-maximum** *at a, we mean:*

$$\exists B \in \mathcal{B}(a) \text{ s.t.}, \forall x \in B, \quad f_x \leqslant f_a.$$

By f has a **local strict-maximum** *at a, we mean:*

$$\exists B \in \mathcal{B}(a) \text{ s.t.}, \forall x \in B, \quad f_x \geqslant f_a.$$

By f has a **local semi-minimum** *at a, we mean:*

$$\exists B \in \mathcal{B}(a) \text{ s.t.}, \forall x \in B_a^\times, \quad f_x < f_a.$$

By f has a **local strict-minimum** *at a, we mean:*

$$\exists B \in \mathcal{B}(a) \text{ s.t.}, \forall x \in B_a^\times, \quad f_x > f_a.$$

DEFINITION 4.10.13. *Let f be a functional, $a \in \mathbb{D}_f$.*

By f has a **local semi-extremum** *at a, we mean:*

f has a local semi-maximum at a

or f has a local semi-minimum at a.

By f has a **local strict-extremum** *at a, we mean:*

f has a local strict-maximum at a

or f has a local strict-minimum at a.

THEOREM 4.10.14. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{D}_f$.*

Assume: f has a local strict-maximum at a.

Then: $f_a^{\mathbb{T}}$ has a local strict-maximum at 0.

Proof. This is HW#12-4. □

The preceding theorem is one of many:

You can change “local” to “global”.

You can change "strict" to "semi".

You can change "maximum" to "minimum" or to "extremum".

Thus there are $2 \cdot 2 \cdot 3 = 12$ different results.

There are also 12 converses:

THEOREM 4.10.15. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{D}_f$.

Assume: $f_a^{\mathbb{T}}$ *has a local strict-maximum at* 0.

Then: f *has a local strict-maximum at* a.

Proof. Unassigned HW. □

The preceding theorem is one of many:

You can change "local" to "global".

You can change "strict" to "semi".

You can change "maximum" to "minimum" or to "extremum".

Thus there are $2 \cdot 2 \cdot 3 = 12$ different results.

THEOREM 4.10.16. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{D}'_f$. *Assume* $f'_a > 0$.

Then $\exists \delta > 0$ *s.t.* $[\text{ on } (a - \delta\,;\, a),\quad f < f_a \,]$

and $[\text{ on } (a\,;\, a + \delta),\quad f > f_a \,]$.

Proof. Let $L := D_a f$, $m := [L]$. Then $m = [D_a f] = f'_a > 0$, so $m > 0$.

We have: $\forall h \in \mathbb{R}$, $L_h = L_{h \cdot 1} = h \cdot L_1 = h \cdot [L] = h \cdot m = m \cdot h$.

Let $R := f_a^{\mathbb{T}} - L$. Then $L + R = f_a^{\mathbb{T}}$.

Since $L \in \text{LINS}_a f$, we get $L \in \mathcal{L}$ and $R \in \mathcal{O}(1)$.

Since $R \in \mathcal{O}(1)$, choose $\delta > 0$ s.t., $\forall h \in \mathbb{R}$,

$$(\,|h| < \delta\,) \Rightarrow (\,|R_h| \leqslant (1/2) \cdot m \cdot |h|^1\,).$$

Then $\delta > 0$. **Want:** $[\text{ on } (a - \delta\,;\, a),\quad f < f_a \,]$

and $[\text{ on } (a\,;\, a + \delta),\quad f > f_a \,]$.

Want: $[\text{ on } (a - \delta\,;\, a) - a,\quad f_a^{\mathbb{T}} < f_a - f_a \,]$

and $[\text{ on } (a\,;\, a + \delta) - a,\quad f_a^{\mathbb{T}} > f_a - f_a \,]$.

Want: $[\text{ on } (-\delta\,;\, 0),\quad f_a^{\mathbb{T}} < 0 \,]$

and $[\text{ on } (0\,;\, \delta),\quad f_a^{\mathbb{T}} > 0 \,]$.

Want: $[\,\forall h \in (-\delta\,;\, 0),\quad (f_a^{\mathbb{T}})_h < 0 \,]$

and $[\,\forall h \in (0\,;\, \delta),\quad (f_a^{\mathbb{T}})_h > 0 \,]$.

Want: $[\,\forall h \in (-\delta\,;\, 0),\quad \text{sgn}((f_a^{\mathbb{T}})_h) = -1 \,]$

and $[\,\forall h \in (0\,;\, \delta),\quad \text{sgn}((f_a^{\mathbb{T}})_h) = 1 \,]$.

Want: $[\,\forall h \in (-\delta\,;\, 0),\quad \text{sgn}((f_a^{\mathbb{T}})_h) = \text{sgn}(h) \,]$

and $[\ \forall h \in (0\,;\,\delta),\ \ \mathrm{sgn}((f_a^{\mathbb{T}})_h) = \mathrm{sgn}(h)\ ]$.

Want: $\forall h \in (-\delta\,;\,\delta),\ \ \mathrm{sgn}((f_a^{\mathbb{T}})_h) = \mathrm{sgn}(h)$.
Given $h \in (-\delta\,;\,\delta)$. **Want:** $\mathrm{sgn}((f_a^{\mathbb{T}})_h) = \mathrm{sgn}(h)$.
We have $|h| < \delta$, so, by the choice of δ, we get: $|R_h| \leqslant (1/2) \cdot m \cdot |h|^1$.
So, since $|L_h| = |m \cdot h| = |m| \cdot |h| = |m| \cdot |h|^1$, we have: $|R_h| \leqslant (1/2) \cdot |L_h|$.
Let $b := L_h$ and $a := R_h$. Then $|a| \leqslant |b|/2$.
So, by Theorem 4.10.4, we get: $\mathrm{sgn}_{b+a} = \mathrm{sgn}_b$.
That is, $\mathrm{sgn}(L_h + R_h) = \mathrm{sgn}(L_h)$. Since $m > 0$, we get: $\mathrm{sgn}_m = 1$.
Then $\mathrm{sgn}((f_a^{\mathbb{T}})_h) = \mathrm{sgn}((L+R)_h) = \mathrm{sgn}(L_h + R_h) = \mathrm{sgn}(L_h)$
$= \mathrm{sgn}(m \cdot h) = \mathrm{sgn}_m \cdot \mathrm{sgn}_h = 1 \cdot \mathrm{sgn}_h = \mathrm{sgn}(h)$. □

THEOREM 4.10.17. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{D}'_f$. Assume $f'_a < 0$.*
Then $\exists \delta > 0$ s.t. $[\ \textit{on}\ (a-\delta\,;\,a),\ \ f > f_a\ ]$
and $[\ \textit{on}\ (a\,;\,a+\delta),\ \ f < f_a\ ]$.

Proof. Let $L := D_a f$, $m := [L]$. Then $m = [D_a f] = f'_a < 0$, so $m < 0$.
We have: $\forall h \in \mathbb{R}$, $L_h = L_{h \cdot 1} = h \cdot L_1 = h \cdot [L] = h \cdot m = m \cdot h$.
Let $R := f_a^{\mathbb{T}} - L$. Then $L + R = f_a^{\mathbb{T}}$.
Since $L \in \mathrm{LINS}_a f$, we get $L \in \mathcal{L}$ and $R \in \mathcal{o}(1)$.
Since $R \in \mathcal{o}(1)$, choose $\delta > 0$ s.t., $\forall h \in \mathbb{R}$,

$$(\,|h| < \delta\,) \Rightarrow (\,|R_h| \leqslant (1/2) \cdot m \cdot |h|^1\,).$$

Then $\delta > 0$. **Want:** $[\ \text{on}\ (a-\delta\,;\,a),\ \ f > f_a\ ]$
and $[\ \text{on}\ (a\,;\,a+\delta),\ \ f < f_a\ ]$.

Want: $[\ \text{on}\ (a-\delta\,;\,a) - a,\ \ f_a^{\mathbb{T}} > f_a - f_a\ ]$
and $[\ \text{on}\ (a\,;\,a+\delta) - a,\ \ f_a^{\mathbb{T}} < f_a - f_a\ ]$.
Want: $[\ \text{on}\ (-\delta\,;\,0),\ \ f_a^{\mathbb{T}} > 0\ ]$
and $[\ \text{on}\ (0\,;\,\delta),\ \ f_a^{\mathbb{T}} < 0\ ]$.
Want: $[\ \forall h \in (-\delta\,;\,0),\ \ (f_a^{\mathbb{T}})_h > 0\ ]$
and $[\ \forall h \in (0\,;\,\delta),\ \ (f_a^{\mathbb{T}})_h < 0\ ]$.
Want: $[\ \forall h \in (-\delta\,;\,0),\ \ \mathrm{sgn}((f_a^{\mathbb{T}})_h) = 1\ ]$
and $[\ \forall h \in (0\,;\,\delta),\ \ \mathrm{sgn}((f_a^{\mathbb{T}})_h) = -1\ ]$.
Want: $[\ \forall h \in (-\delta\,;\,0),\ \ \mathrm{sgn}((f_a^{\mathbb{T}})_h) = -\mathrm{sgn}(h)\ ]$
and $[\ \forall h \in (0\,;\,\delta),\ \ \mathrm{sgn}((f_a^{\mathbb{T}})_h) = -\mathrm{sgn}(h)\ ]$.

Want: $\forall h \in (-\delta\,;\,\delta),\ \ \mathrm{sgn}((f_a^{\mathbb{T}})_h) = -\mathrm{sgn}(h)$.
Given $h \in (-\delta\,;\,\delta)$. **Want:** $\mathrm{sgn}((f_a^{\mathbb{T}})_h) = -\mathrm{sgn}(h)$.
We have $|h| < \delta$, so, by the choice of δ, we get: $|R_h| \leqslant (1/2) \cdot m \cdot |h|^1$.

So, since $|L_h| = |m \cdot h| = |m| \cdot |h| = |m| \cdot |h|^1$, we have: $|R_h| \leqslant (1/2) \cdot |L_h|$.
Let $b := L_h$ and $a := R_h$. Then $|a| \leqslant |b|/2$.
So, by Theorem 4.10.4, we get: $\mathrm{sgn}_{b+a} = \mathrm{sgn}_b$.
That is, $\mathrm{sgn}(L_h + R_h) = \mathrm{sgn}(L_h)$. Since $m > 0$, we get: $\mathrm{sgn}_m = -1$.
Then $\mathrm{sgn}((f_a^{\mathbb{T}})_h) = \mathrm{sgn}((L+R)_h) = \mathrm{sgn}(L_h + R_h) = \mathrm{sgn}(L_h)$
$= \mathrm{sgn}(m \cdot h) = \mathrm{sgn}_m \cdot \mathrm{sgn}_h = -1 \cdot \mathrm{sgn}_h = -\mathrm{sgn}(h)$. □

THEOREM 4.10.18. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{D}'_f$.*
Assume: *f has a local semi-maximum at a.*
Then: $f'_a = 0$.

Proof. Assume $f'_a \neq 0$. **Want:** Contradiction.
Exactly one of the following is true:
(1) $f'_a > 0$ or (2) $f'_a < 0$.

Case (1):
By Theorem 4.10.16, choose $\delta > 0$ s.t.
[on $(a - \delta\,;\, a)$, $f < f_a$] and [on $(a\,;\, a + \delta)$, $f > f_a$].
Since f has a local semi-maximum at a, choose $B \in \mathcal{B}(a)$ s.t.
$\forall x \in B,\ \ f_x \leqslant f_a$.
Choose $\rho > 0$ s.t. $B = B(a, \rho)$. Let $\mu := \min\{\delta, \rho\}$.
Since $\mu \leqslant \delta$, we get: $(a\,;\, a + \mu) \subseteq (a\,;\, a + \delta)$.
Since $\mu \leqslant \rho$, we get: $B(a, \mu) \subseteq B(a, \rho)$.
Let $x := a + (\mu/2)$. Then $a < x < a + \mu$. Then $x \in (a\,;\, a + \mu)$.
Since $x \in (a\,;\, a + \mu) \subseteq (a\,;\, a + \delta)$,
by choice of δ, we have $f_x > f_a$, and so $f_a < f_x$.
Since $x \in (a\,;\, a + \mu) \subseteq B(a, \mu) \subseteq B(a, \rho) = B$,
by choice of B, we have $f_x \leqslant f_a$.
Then $f_a < f_x \leqslant f_a$, so $f_a < f_a$. Contradiction.
End of Case (1).

Case (2):
By Theorem 4.10.17, choose $\delta > 0$ s.t.
[on $(a - \delta\,;\, a)$, $f > f_a$] and [on $(a\,;\, a + \delta)$, $f < f_a$].
Since f has a local semi-maximum at a, choose $B \in \mathcal{B}(a)$ s.t.
$\forall x \in B,\ \ f_x \leqslant f_a$.
Choose $\rho > 0$ s.t. $B = B(a, \rho)$. Let $\mu := \min\{\delta, \rho\}$.
Since $\mu \leqslant \delta$, we get: $(a - \mu\,;\, a) \subseteq (a - \delta\,;\, a)$.
Since $\mu \leqslant \rho$, we get: $B(a, \mu) \subseteq B(a, \rho)$.

Let $x := a - (\mu/2)$. Then $a - \mu < x < a$. Then $x \in (a - \mu\,;\,a)$.
Since $x \in (a - \mu\,;\,a) \subseteq (a - \delta\,;\,a)$,
by choice of δ, we have $f_x > f_a$, and so $f_a < f_x$.
Since $x \in (a\,;\,a + \mu) \subseteq B(a, \mu) \subseteq B(a, \rho) = B$,
by choice of B, we have $f_x \leqslant f_a$.
Then $f_a < f_x \leqslant f_a$, so $f_a < f_a$. Contradiction.
End of Case (2). □

The following is called **Fermat's Theorem**:

THEOREM 4.10.19. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{D}'_f$.*
Assume: *f has a local semi-extremum at a.*
Then: $f'_a = 0$.

Proof. At least one of the following is true:
(1) f has a local semi-maximum at a.
(2) f has a local semi-minimum at a.

Case (1):
By Theorem 4.10.18, we get: $f'_a = 0$.
End of Case (1).

Case (2):
Since f has a local semi-minimum at a,
it follows that $-f$ has a local semi-maximum at a,
so, by Theorem 4.10.18, we get: $(-f)'_a = 0$.
Since $a \in \mathbb{D}'_f$, we get: $(-f)'_a = -f'_a$.
Then $f'_a = -(-f'_a) = -((-f)'_a) = -0 = 0$.
End of Case (2). □

The following theorem does not require differentiability of f:

THEOREM 4.10.20. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$ $b > a$.*
Assume f is continuous on $[a; b]$. Assume $f_a = f_b$.
Then $\exists c \in (a; b)$ s.t. f has a local semi-extremum at c.

Proof. Let $V := f_*[a; b]$, $y := \min V$, $z := \max V$.
By the EVT, $y \neq ☹ \neq z$.
Then $\forall x \in [a; b]$, we have $y \leqslant f_x \leqslant z$.
In particular, $y \leqslant f_a \leqslant z$.
At least one of the following must be true:

$$(1)\ y = f_a = z \quad \text{or} \quad (2)\ y \neq f_a \quad \text{or} \quad (3)\ z \neq f_a.$$

Case (1): Let $c := (a+b)/2$. Since $b > a$, we get: $c \in (a;b)$.
Want: f has a local semi-extremum at c.
Want: f has a local semi-maximum at c.
Want: $\exists \delta > 0$ s.t., $\forall x \in B(c,\delta)$, $f_x \leqslant f_c$.
Let $\delta := (b-a)/2$. Since $b > a$, we get: $\delta > 0$.
Want: $\forall x \in B(c,\delta)$, $f_x \leqslant f_c$.
Given $x \in B(c,\delta)$. **Want:** $f_x \leqslant f_c$.
We have $c - \delta = a$ and $c + \delta = b$,
so $B(c,\delta) = (a;b)$. Then $B(c,\delta) \subseteq [a;b]$.
Since f is continuous on $[a;b]$, we get: $[a;b] \subseteq \mathbb{D}_f$.
We have $x, c \in B(c,\delta) \subseteq [a;b]$, so $x, c \in [a;b]$.
Then $x, c \in [a;b] \subseteq \mathbb{D}_f$, so $x, c \in \mathbb{D}_f$.
Since $x, c \in [a;b]$ and $x, c \in \mathbb{D}_f$, we get $f_x, f_c \in f_*[a;b]$.
Since $f_x, f_c \in f_*[a;b] = V$, it follows that:
$\min V \leqslant f_x \leqslant \max V$ and $\min V \leqslant f_c \leqslant \max V$.
Then: $y \leqslant f_x \leqslant z$ and $y \leqslant f_c \leqslant z$.
So, since $y = f_a = z$, we get: $f_a \leqslant f_x \leqslant f_a$ and $f_a \leqslant f_c \leqslant f_a$.
Then $f_x = f_a$ and $f_c = f_a$. Then $f_x = f_c$. Then $f_x \leqslant f_c$.
End of Case (1).

Case (2): Since $y \in V = f_*[a;b]$, choose $c \in [a;b]$ s.t. $f_c = y$.
Since $f_c = y \neq f_a$, we get $f_c \neq f_a$, and so $c \neq a$.
By hypothesis, $f_a = f_b$.
Since $f_c = y \neq f_a = f_b$, we get $f_c \neq f_b$, and so $c \neq b$.
Then $c \in [a;b] \backslash \{a,b\} = (a;b)$.
Want: f has a local semi-extremum at c.
Want: f has a local semi-minimum at c.
Want: $\exists \delta > 0$ s.t., $\forall x \in B(c,\delta)$, $f_x \geqslant f_c$.
Since $(a;b)$ is open and $c \in (a;b)$, choose $\delta > 0$ s.t. $B(c,\delta) \subseteq (a;b)$.
Then $\delta > 0$. **Want:** $\forall x \in B(c,\delta)$, $f_x \geqslant f_c$.
Given $x \in B(c,\delta)$. **Want:** $f_x \geqslant f_c$.
Since $x \in [a;b]$ and since $[a;b] \subseteq \mathbb{D}_f$, we get: $x \in \mathbb{D}_f$.
Since $x \in [a;b]$ and $x \in \mathbb{D}_f$, we get: $f_x \in f_*[a;b]$.
Since $f_x \in f_*[a;b] = V$ and $y = \min V$, we get: $f_x \geqslant y$.
Then $f_x \geqslant y = f_c$.
End of Case (2).

Case (3):
Since $z \in V = f_*[a;b]$, choose $c \in [a;b]$ s.t. $f_c = z$.
Since $f_c = z \neq f_a$, we get $f_c \neq f_a$, and so $c \neq a$.
By hypothesis, $f_a = f_b$.
Since $f_c = z \neq f_a = f_b$, we get $f_c \neq f_b$, and so $c \neq b$.
Then $c \in [a;b] \backslash \{a, b\} = (a;b)$.
Want: f has a local semi-extremum at c.
Want: f has a local semi-maximum at c.
Want: $\exists \delta > 0$ s.t., $\forall x \in B(c, \delta)$, $f_x \leqslant f_c$.
Since $(a;b)$ is open and $c \in (a;b)$, choose $\delta > 0$ s.t. $B(c, \delta) \subseteq (a;b)$.
Then $\delta > 0$. **Want:** $\forall x \in B(c, \delta)$, $f_x \leqslant f_c$.
Given $x \in B(c, \delta)$. **Want:** $f_x \leqslant f_c$.
Since $x \in [a;b]$ and since $[a;b] \subseteq \mathbb{D}_f$, we get: $x \in \mathbb{D}_f$.
Since $x \in [a;b]$ and $x \in \mathbb{D}_f$, we get: $f_x \in f_*[a;b]$.
Since $f_x \in f_*[a;b] = V$ and $z = \max V$, we get: $f_x \leqslant z$.
Then $f_x \leqslant z = f_c$.
End of Case (3). □

DEFINITION 4.10.21. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $S \subseteq \mathbb{R}$.*
By f is **c/d** *on S, we mean: f is continuous on S and* $\text{Int}\, S \subseteq \mathbb{D}'_f$.

Let $a \in \mathbb{R}$ and let $b \geqslant a$.
Recall: $\text{Int}[a;b] = (a;b)$.
So, for any $f : \mathbb{R} \dashrightarrow \mathbb{R}$, we have: f is c/d on $[a;b]$ iff
f is continuous on $[a;b]$ and f is differentiable on $(a;b)$.

Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $S \subseteq \mathbb{R}$.
If $S \subseteq \mathbb{D}'_f$, then f is continuous on S and $\text{Int}\, S \subseteq S \subseteq \mathbb{D}'_f$,
and so f is c/d on S. However, the converse is not necessarily true:
A function might be
continuous on $[0;\infty)$ and differentiable on $(0;\infty)$
but NOT differentiable at 0.

The following is **Rolle's Theorem**:

THEOREM 4.10.22. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, let $a \in \mathbb{R}$ and let $b > a$.*
Assume: f *is c/d on* $[a;b]$ *and* $f_a = f_b$.
Then: $\exists c \in (a;b)$ *s.t.* $f'_c = 0$.

Proof. By Theorem 4.10.20,

choose $c \in (a;b)$ s.t. f has a local semi-extremum at c.

Then $c \in (a;b)$. **Want:** $f'_c = 0$.
By assumption, f is c/d on $[a;b]$, and so $\mathrm{Int}[a;b] \subseteq \mathbb{D}'_f$.
Then $c \in (a;b) = \mathrm{Int}[a;b] \subseteq \mathbb{D}'_f$.
Then, by Fermat's Theorem, we have $f'_c = 0$. □

The following is the **Mean Value Theorem** or **MVT**:

THEOREM 4.10.23. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$, $b > a$.
Assume: f *is c/d on* $[a;b]$.
Then: $\exists c \in (a;b)$ *s.t.* $f'_c = \mathrm{DQ}_f(a,b)$.

Proof. Let $m := \mathrm{DQ}_f(a,b)$, $L := m \cdot (\bullet)$.
Then $L' = C_m^{\mathbb{R}}$. In particular, $\mathbb{D}'_L = \mathbb{R}$, and so $[a;b] \subseteq \mathbb{D}'_L$.
It follows that L is c/d on $[a;b]$.
By assumption, f is also c/d on $[a;b]$.
Then $f - L$ is c/d on $[a;b]$.
Let $g := f - L$. Then g is c/d on $[a;b]$.
By HW#6-3, we have: $g_a = g_b$.
Then, by Rolle's Theorem, choose $c \in (a;b)$ s.t. $g'_c = 0$.
Then $c \in (a;b)$. **Want:** $f'_c = \mathrm{DQ}_f(a,b)$.
Since $L' = C_m^{\mathbb{R}}$, we get $L_c = m$.
Since f is c/d on $[a;b]$, we get: $\mathrm{Int}[a;b] \subseteq \mathbb{D}'_f$.
Since $c \in (a;b) = \mathrm{Int}[a;b] \subseteq \mathbb{D}'_f$ and $c \in \mathbb{R} = \mathbb{D}'_L$, we get:

$$(f-L)'_c = f'_c - L'_c.$$

Then $0 = g'_c = (f-L)'_c = f'_c - L'_c$, and so $f'_c = L'_c$.
Then $f'_c = L'_c = m = \mathrm{DQ}_f(a,b)$. □

DEFINITION 4.10.24. *Let* $I \subseteq \mathbb{R}$. *By* I *is an* **interval**, *we mean:*

$$\forall a,b \in I, \quad [a|b] \subseteq I.$$

THEOREM 4.10.25. ($[1;3] \cup [5;7]$ *is not an interval*)
& ($(4;9]$ *is an interval*)
& ($[0;\infty)$ *is an interval*)
& ($\varnothing$ *is an interval*)
& ($\mathbb{R}$ *is an interval*)
& ($\mathbb{R}_0^\times$ *is not an interval*).

THEOREM 4.10.26. *Let $I \subseteq \mathbb{R}$ be a nonempty interval.*
Let $a := \inf I$, $b := \sup I$.
Then: $(I = [a;b]) \vee (I = [a;b)) \vee (I = (a;b]) \vee (I = (a;b))$.

DEFINITION 4.10.27. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $S \subseteq \mathbb{D}_f$.*
Then: $\boxed{\mathrm{DQ}_f^S} := \{\mathrm{DQ}_f(a,b) \,|\, a,b \in S,\ a \neq b\}$.

The set DQ_f^S represents the set of "secant slopes for f over S", or the set of "slopes of secant lines for f over S".

DEFINITION 4.10.28. *Let f be a function. By f is* **constant**, *we mean:* $\forall a,b \in \mathbb{D}_f$, $f_a = f_b$.

THEOREM 4.10.29. $\varnothing$ *is constant.*

THEOREM 4.10.30. *Let f be a function.*
Assume: $\#\mathbb{D}_f = 1$. *Then: f is constant.*

THEOREM 4.10.31. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$. Then:*
$(\#\mathbb{I}_f \leqslant 1) \Leftrightarrow (f \text{ is constant}) \Leftrightarrow (\mathrm{DQ}_f^{\mathbb{D}_f} \subseteq \{0\})$.

The preceding theorem shows, for functions $\mathbb{R} \dashrightarrow \mathbb{R}$, that
a precalculus concept such as "constant"
is equivalent to
a statement about secant slopes.
The following theorem contains six such equivalences:

THEOREM 4.10.32. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $S \subseteq \mathbb{D}_f$. Then:*
$[(f|S \text{ is constant}) \Leftrightarrow (\mathrm{DQ}_f^S \subseteq \{0\})]$
$\&\ [(f|S \text{ is one-to-one}) \Leftrightarrow (0 \notin \mathrm{DQ}_f^S)]$
$\&\ [(f|S \text{ is strictly-increasing}) \Leftrightarrow (\mathrm{DQ}_f^S > 0)]$
$\&\ [(f|S \text{ is semi-increasing}) \Leftrightarrow (\mathrm{DQ}_f^S \geqslant 0)]$
$\&\ [(f|S \text{ is strictly-decreasing}) \Leftrightarrow (\mathrm{DQ}_f^S < 0)]$
$\&\ [(f|S \text{ is semi-decreasing}) \Leftrightarrow (\mathrm{DQ}_f^S \leqslant 0)]$.

THEOREM 4.10.33. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $S \subseteq \mathbb{D}_f$, $m \in \mathrm{DQ}_f^S$.*
Then: $\exists a,b \in S$ *s.t.* $(a < b)$ & $(m = \mathrm{DQ}_f(a,b))$.

Proof. Since $m \in \mathrm{DQ}_f^S$, choose $\alpha, \beta \in S$ s.t.
$(\alpha \neq \beta)\,\&\,(m = \mathrm{DQ}_f(\alpha,\beta))$.
Let $a := \min\{\alpha,\beta\}$, $b := \max\{\alpha,\beta\}$. Then $a,b \in S$.
Want: $(a < b)$ & $(m = \mathrm{DQ}_f(a,b))$.
Since $\alpha \neq \beta$, we get: either $\alpha < \beta$ or $\beta < \alpha$.

Then: either $(\,(\,\alpha<\beta\,)\,\&\,(\,a=\alpha\,)\,\&\,(\,b=\beta\,)\,)$
or $(\,(\,\beta<\alpha\,)\,\&\,(\,a=\beta\,)\,\&\,(\,b=\alpha\,)\,)$.
Then $a<b$. **Want:** $m=\mathrm{DQ}_f(a,b)$.
Since $m=\mathrm{DQ}_f(\alpha,\beta)$ and since $\mathrm{DQ}_f(\alpha,\beta)=\mathrm{DQ}_f(\beta,\alpha)$,
it follows that $m=\mathrm{DQ}_f(a,b)$. □

THEOREM 4.10.34. *Let $f:\mathbb{R}\dashrightarrow\mathbb{R}$ and let I be an interval.*
Assume: f is c/d on I.
Then: $\mathrm{DQ}_f^I\subseteq f'_*(\mathrm{Int}\,I)$.

Proof. **Want:** $\forall m\in\mathrm{DQ}_f^I,\ m\in f'_*(\mathrm{Int}\,I)$.
Given $m\in\mathrm{DQ}_f^I$. **Want:** $m\in f'_*(\mathrm{Int}\,I)$.
By Theorem 4.10.33, choose $a,b\in I$ s.t. $(\,a<b\,)\quad\&\quad(\,m=\mathrm{DQ}_f(a,b)\,)$.
Since $a,b\in I$ and since I is an interval, it follows that $[a|b]\subseteq I$.
Since $a<b$, it follows that $[a|b]=[a;b]$. Then $[a;b]\subseteq I$.
By assumption, f is c/d on I. Then f is c/d on I.
Then, by the MVT, choose $c\in(a;b)$ s.t. $f'_c=\mathrm{DQ}_f(a,b)$.
Since $[a;b]\subseteq I$, we get $\mathrm{Int}[a;b]\subseteq\mathrm{Int}\,I$.
We have $c\in(a;b)=\mathrm{Int}[a;b]\subseteq\mathrm{Int}\,I$, so $c\in\mathrm{Int}\,I$.
Since f is c/d on I, it follows that $\mathrm{Int}\,I\subseteq\mathbb{D}'_f$.
We have $c\in\mathrm{Int}\,I\subseteq\mathbb{D}'_f=\mathbb{D}_{f'}$, so $c\in\mathbb{D}_{f'}$.
Since $c\in\mathrm{Int}\,I$ and $c\in\mathbb{D}_{f'}$, we get: $f'_c\in f'_*(\mathrm{Int}\,I)$.
Then $m=\mathrm{DQ}_f(a,b)=f'_c\in f'_*(\mathrm{Int}\,I)$. □

Combining Theorem 4.10.32 with Theorem 4.10.34,
we get six applications to the MVT,
as follows:

THEOREM 4.10.35. *Let $f:\mathbb{R}\dashrightarrow\mathbb{R}$. Let I be an interval.*
Assume f is c/d on I. Then:
(i) $[\,(\,f|I$ *is constant*$\,)\Leftarrow(\,f'_*(\mathrm{Int}\,I)\subseteq\{0\}\,)\,]$
& *(ii)* $[\,(\,f|I$ *is one-to-one*$\,)\Leftarrow(\,0\notin f'_*(\mathrm{Int}\,I)\,)\,]$
& *(iii)* $[\,(\,f|I$ *is strictly-increasing*$\,)\Leftarrow(\,f'_*(\mathrm{Int}\,I)>0\,)\,]$
& *(iv)* $[\,(\,f|I$ *is semi-increasing*$\,)\Leftarrow(\,f'_*(\mathrm{Int}\,I)\geqslant 0\,)\,]$
& *(v)* $[\,(\,f|I$ *is strictly-decreasing*$\,)\Leftarrow(\,f'_*(\mathrm{Int}\,I)<0\,)\,]$
& *(vi)* $[\,(\,f|I$ *is semi-decreasing*$\,)\Leftarrow(\,f'_*(\mathrm{Int}\,I)\leqslant 0\,)\,]$.

Let $f:=(\bullet)^3$ and let $I:=\mathbb{R}$.
Then f is one-to-one and strictly-increasing.
Also, $0=f'_0\in f'_*(\mathbb{R})=f'_*(\mathrm{Int}\,I)$, so $\neg(\,f'_*(\mathrm{Int}\,I)>0\,)$.
This provides a counterexample to the converse of (ii) in Theorem 4.10.35

and provides a counterexample to the converse of (iii) in Theorem 4.10.35.

Let $f := -(\bullet)^3$ and let $I := \mathbb{R}$.
Then f is strictly-decreasing.
Also, $0 = f'_0 \in f'_*(\mathbb{R}) = f'_*(\text{Int}\, I)$, so $\neg(\, f'_*(\text{Int}\, I) < 0\,)$.
This provides a counterexample to the converse of (v) in Theorem 4.10.35.

Unassigned HW:
The converses to (i), (iv) and (vi) in Theorem 4.10.35 are all true:

THEOREM 4.10.36. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$. Let I be an interval. Assume f is c/d on I. Then:*

$$[\,(\, f|I \text{ is constant}\,) \Leftrightarrow (\, f'_*(\text{Int}\, I) \subseteq \{0\}\,)\,]$$
$$\&\ [\,(\, f|I \text{ is semi-increasing}\,) \Leftrightarrow (\, f'_*(\text{Int}\, I) \geqslant 0\,)\,]$$
$$\&\ [\,(\, f|I \text{ is semi-decreasing}\,) \Leftrightarrow (\, f'_*(\text{Int}\, I) \leqslant 0\,)\,].$$

4.11. Taylor's Theorem.

Here is another form of the MVT:

THEOREM 4.11.1. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a, b \in \mathbb{R}$.*
Assume: $[a|b] \subseteq \mathbb{D}'_f$.
Then: $\exists c \in [a|b]$ *s.t.* $f_b - f_a = f'_c \cdot (b - a)$.

Proof. We have $a, b \in [a|b] \subseteq \mathbb{D}'_f \subseteq \mathbb{D}_f$. Then $f_a, f_b \in \mathbb{I}_f \subseteq \mathbb{R}$.
Exactly one of the following is true:
(1) $a < b$ or (2) $a = b$ or (3) $a > b$.

Case (1): Since $a < b$, we have $[a|b] = [a;b]$.
Then $[a;b] = [a|b] \subseteq \mathbb{D}'_f$, and so f is c/d on $[a;b]$.
By Theorem 4.10.23, choose $c \in (a;b)$ s.t. $f'_c = \text{DQ}_f(a,b)$.
Then $c \in (a;b) \subseteq [a;b] = [a|b]$. **Want:** $f_b - f_a = f'_c \cdot (b - a)$.
By defnition of $\text{DQ}_f(a,b)$, we have $f_b - f_a = (\text{DQ}_f(a,b)) \cdot (b - a)$.
Then $f_b - f_a = (\text{DQ}_f(a,b)) \cdot (b - a) = f'_c \cdot (b - a)$.
End of Case (1).

Case (2):
Since $a = b$, we get $b - a = 0$ and $f_b - f_a = 0$.
Let $c := a$. Then $c \in [a|b]$. **Want:** $f_b - f_a = f'_c \cdot (b - a)$.
We have $c = a \in [a|b] \subseteq \mathbb{D}'_f$, so $f'_c \in \mathbb{I}_{f'} \subseteq \mathbb{R}$, so $0 = f'_c \cdot 0$.
$f_b - f_a = 0 = f'_c \cdot 0 = f'_c \cdot (b - a)$.

End of Case (2).

Case (3):
Since $a < b$, we have $[a|b] = [b;a]$.
Then $[b;a] = [a|b] \subseteq \mathbb{D}'_f$, and so f is c/d on $[b;a]$.
By Theorem 4.10.23, choose $c \in (b;a)$ s.t. $f'_c = \mathrm{DQ}_f(b,a)$.
Then $c \in (b;a) \subseteq [b;a] = [a|b]$. **Want:** $f_b - f_a = f'_c \cdot (b-a)$.
By defnition of $\mathrm{DQ}_f(b,a)$, we have $f_b - f_a = (\mathrm{DQ}_f(b,a)) \cdot (b-a)$.
Then $f_b - f_a = (\mathrm{DQ}_f(b,a)) \cdot (b-a) = f'_c \cdot (b-a)$.
End of Case (3). □

The following is Unassigned HW:

THEOREM 4.11.2. $\forall x, h \in \mathbb{R}$, $(\, x \in [0|h]\,) \Rightarrow (\, |x| < |h|\,)$.

THEOREM 4.11.3. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $k \in \mathbb{N}_0$.
Assume: $(\, f_0 = 0\,) \,\&\, (\, f' \in \mathcal{O}(k)\,)$.
Then: $f \in \mathcal{O}(k+1)$.

Proof. This is Problem 1 from the Final Exam. □

THEOREM 4.11.4. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $k \in \mathbb{N}_0$.
Assume: $f_0 = f'_0 = f''_0 = 0$.
Then: $f \in \mathcal{O}(2)$.

Proof. This is Problem 2 from the Final Exam. □

The following is Unassigned HW:

THEOREM 4.11.5. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{D}'_f$. *Then* $D_a f = f'_a \cdot (\bullet)$.

DEFINITION 4.11.6. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$, $k \in \mathbb{N}_0$. *Then*
$$\boxed{D^k_a f} := \frac{1}{k!} \cdot f^{(k)}_a \cdot (\bullet)^k.$$

Note: $D^0_a f = C^{\mathbb{R}}_{f_a}$, $D^1_a f = D_a f$, $D^2_a f = \frac{1}{2} \cdot f''_a \cdot (\bullet)^2$.
Note that $D^k_a f \ {}^*\!\in\ \mathcal{H}(k)$.

Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $k \in \mathbb{N}_0$, $a \in \mathbb{D}^{(k)}_f$.
The philosophy of Taylor's theorem is that the best
approximation of $f^{\mathbb{T}}_a$ by
a general (not necessarily homogeneous) kth order polynomial
is $(D_a f) + (D^2_a f) + (D^3_a f) + \cdots + (D^k_a f)$,

and the error

$$f_a^{\mathbb{T}} \;-\; (\,(D_af) + (D_a^2f) + (D_a^3f) + \cdots + (D_a^kf)\,),$$

is "sub-k", or, in other words, the error is an element of $\mathcal{o}(k)$.
We will only prove the Taylor Theorem at second order, *i.e.*, for $k = 2$, but the proof we give easily generalizes to all k.
Note: The first order Taylor Theorem is just the assertion that
$f_a^{\mathbb{T}} \;-\; (D_af) \;\in\; \mathcal{o}(1)$,
which follows from the definition of D_af.

DEFINITION 4.11.7. *Let X be a set, $f : \mathbb{R} \dashrightarrow X$, $a \in \mathbb{R}$. Then $\boxed{f_{a+\bullet}} : \mathbb{R} \dashrightarrow X$ is defined by: $\forall h \in \mathbb{R}$,*

$$(f_{a+\bullet})_h \;=\; f_{a+h}.$$

THEOREM 4.11.8. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$. Then $f_a^{\mathbb{T}} = f_{a+\bullet} - C_{f_a}^{\mathbb{R}}$.*

THEOREM 4.11.9. *Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $C \in \mathcal{C}$.*
Assume: $f = g - C$.
Then: $f' = g'$.

Proof. **Want:** $\forall h \in \mathbb{R}$, $f'_h \;=\; g'_h$.
Given $h \in \mathbb{R}$. **Want:** $f'_h \;=\; g'_h$.
Since $C \in \mathcal{C}$, we get $C' = \mathbf{0}$. Then $C'_h = 0$.
Since $f = g - C$, we get $f'_h \;=^*\; g'_h - C'_h = g'_h - 0 = g'_h$.
Want: $g'_h \;=^*\; f'_h$.
Since $C \in \mathcal{C}$, we get $-C + C = \mathbf{0}$.
We have $f + C = g - C + C = g + \mathbf{0} = g$.
Since $g = f + C$, we get $g'_h \;=^*\; f'_h + C'_h = f'_h + 0 = f'_h$. □

THEOREM 4.11.10. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a, h \in \mathbb{R}$.*
Then: $(f_{a+\bullet})_h^{\mathbb{T}} = f_{a+h}^{\mathbb{T}}$.

Proof. **Want:** $\forall k \in \mathbb{R}$, $((f_{a+\bullet})_h^{\mathbb{T}})_k \;=\; (f_{a+h}^{\mathbb{T}})_k$.
Given $k \in \mathbb{R}$. **Want:** $((f_{a+\bullet})_h^{\mathbb{T}})_k \;=\; (f_{a+h}^{\mathbb{T}})_k$.
We have $((f_{a+\bullet})_h^{\mathbb{T}})_k \;=\; (f_{a+\bullet})_{h+k} - (f_{a+\bullet})_k$
$\;=\; f_{a+h+k} - f_{a+k} \;=\; (f_{a+h}^{\mathbb{T}})_k$. □

We can express the next theorem by saying
differentiation commutes with (horizontal) translation.
It implies that $f'_{a+\bullet}$ is unambiguous; in principle, it might mean

$$(f_{a+\bullet})' \quad \text{or} \quad (f')_{a+\bullet},$$

but, according to the theorem, these are equal.

THEOREM 4.11.11. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$.
Then $(f_{a+\bullet})' = (f')_{a+\bullet}$.

Proof. **Want:** $\forall h \in \mathbb{R}$, $((f_{a+\bullet})')_h \;=\; ((f')_{a+\bullet})_h$.
Given $h \in \mathbb{R}$. **Want:** $((f_{a+\bullet})')_h \;=\; ((f')_{a+\bullet})_h$.
Want: $((f_{a+\bullet})')_h \;=\; (f')_{a+h}$. **Want:** $(f_{a+\bullet})'_h \;=\; f'_{a+h}$.
By Theorem 4.11.10, $(f_{a+\bullet})^{\mathbb{T}}_h = f^{\mathbb{T}}_{a+h}$.
Then $\mathrm{LINS}_h f_{a+\bullet} = \{L \in \mathcal{L} \,|\, (f_{a+\bullet})^{\mathbb{T}}_h - L \in \mathcal{O}(1)\}$
$= \{L \in \mathcal{L} \,|\, f^{\mathbb{T}}_{a+h} - L \in \mathcal{O}(1)\} = \mathrm{LINS}_{a+h} f$,
so $\mathrm{LINS}_h f_{a+\bullet} = \mathrm{LINS}_{a+h} f$.
Then $D_h f_{a+\bullet} \;=\; \mathrm{UE}(\mathrm{LINS}_h\, f_{a+\bullet}) \;=\; \mathrm{UE}(\mathrm{LINS}_{a+h}\, f) \;=\; D_{a+h} f$,
so $D_h f_{a+\bullet} \;=\; D_{a+h} f$.
Then $(f_{a+\bullet})'_h \;=\; [D_h f_{a+\bullet}] \;=\; [D_{a+h} f] \;=\; f'_{a+h}$. □

THEOREM 4.11.12. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{D}_{f'}$.
Then $(f^{\mathbb{T}}_a)' = f'_{a+\bullet}$.

Proof. $(f^{\mathbb{T}}_a)' = (f_{a+\bullet} - C^{\mathbb{R}}_{f_a})' = f'_{a+\bullet} - \mathbf{0} = f'_{a+\bullet}$. □

THEOREM 4.11.13. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{D}_{f'}$.
Then $(f^{\mathbb{T}}_a)'' = f''_{a+\bullet}$.

Proof. $(f^{\mathbb{T}}_a)'' = ((f^{\mathbb{T}}_a)')' = (f'_{a+\bullet})' = f''_{a+\bullet}$. □

The next theorem is Unassigned HW:

THEOREM 4.11.14.
$(\ \forall m \in \mathbb{R},\quad (m \cdot (\bullet))' = C^{\mathbb{R}}_m\ )$ *and*
$(\ \forall c \in \mathbb{R},\quad (c \cdot (\bullet)^2)' = 2 \cdot c \cdot (\bullet)\ )$.

We have a quantified equivalence for $g \supseteq f$:

THEOREM 4.11.15. *Let* f *and* g *be functions.*
Then: $(\ g \supseteq f\ ) \Leftrightarrow (\ \forall x,\ \ g_x \ =^*\ f_x\ )$.

When a superdomain for f is known,
we have another quantified equivalence for $g \supseteq f$:

THEOREM 4.11.16. *Let* f *and* g *be functions. Let* S *be a set.*
Assume: $\mathbb{D}_f \subseteq S$.
Then: $(\ g \supseteq f\ ) \Leftrightarrow (\ \forall x \in S,\ \ g_x \ =^*\ f_x\ )$.

THEOREM 4.11.17. *Let* $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$.
Assume: *near* a, $f = g$.
Then: $f'_a \;=\; g'_a$.

Proof. We have: near 0, $f_a^{\mathbb{T}} = g_a^{\mathbb{T}}$.
Then, $\forall L \in \mathcal{L}$, we have: near 0, $f_a^{\mathbb{T}} - L = g_a^{\mathbb{T}} - L$,
and so $(\, f_a^{\mathbb{T}} - L \in \mathcal{O}(1)\,) \Leftrightarrow (\, g_a^{\mathbb{T}} - L \in \mathcal{O}(1)\,)$.
Then $\mathrm{LINS}_a f = \mathrm{LINS}_a g$.
Then $f'_a = [D_a f] = [\mathrm{UE}(\mathrm{LINS}_a f)]$
$\stackrel{*}{=} [\mathrm{UE}(\mathrm{LINS}_a g)] = [D_a g] = g'_a$. □

THEOREM 4.11.18. *Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$.*
Assume: $(\, g \supseteq f\,)$ & $(\, f$ is defined near $a\,)$.
Then: $g = f$ near a.

Proof. Unassigned HW. □

THEOREM 4.11.19. *Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$.*
Assume: $g \supseteq f$.
Then: $g' \supseteq f'$.

Proof. **Want:** $\forall a \in \mathbb{R}$, $g'_a \ \supseteq\ f'_a$.
Given $a \in \mathbb{R}$. **Want:** $g'_a \ \supseteq\ f'_a$.
Want: $(\, f'_a \neq ☹\,) \Rightarrow (\, g'_a = f'_a\,)$.
Assume $f'_a \neq ☹$. **Want:** $g'_a = f'_a$.
Since $f'_a \neq ☹$, we get $a \in \mathbb{D}_{f'}$, and so f is defined near a.
So, since $g \supseteq f$, by Theorem 4.11.18, we conclude: $g = f$ near a.
Then, by Theorem 4.11.17, $g'_a \ \supseteq\ f'_a$.
So, since $f'_a \neq ☹$, we conclude that $g'_a = f'_a$. □

THEOREM 4.11.20. *Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$. Then $(f+g)' \supseteq f' + g'$.*

Proof. We have: $\forall x \in \mathbb{R}$, $(f+g)'_x \ =^* \ f'_x + g'_x = (f'+g')_x$.
Then: $(f+g)' \supseteq f' + g'$. □

THEOREM 4.11.21. *Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$. Then $(f+g)'' \supseteq f'' + g''$.*

Proof. $(f+g)'' = ((f+g)')' \supseteq (f'+g')' \supseteq f'' + g''$. □

THEOREM 4.11.22. *Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$.*
Then $(f+g)''_a \ =^* \ f''_a + g''_a$.

Proof. Since $(f+g)'' \supseteq f'' + g''$,
it follows that $(f+g)''_a \ =^* \ (f''+g'')_a$.
Then $(f+g)''_a \ =^* \ (f''+g'')_a = f''_a + g''_a$. □

We can now state and prove the **Taylor Theorem, second order**:

THEOREM 4.11.23. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{D}''_f$, $L := D_a f$, $Q := D^2_a f$.

Then: $\quad f^{\mathbb{T}}_a - L - Q \in \mathcal{O}(2)$.

Proof. Let $R := f^{\mathbb{T}}_a - L - Q$. **Want:** $R \in \mathcal{O}(2)$.
By Final Exam Problem 2, we want: $R_0 = R'_0 = R''_0 = 0$.
By hypothesis, $a \in \mathbb{D}''_f$. Then $a \in \mathbb{D}''_f \subseteq \mathbb{D}'_f$. Then, $\forall h \in \mathbb{R}$,

$$L_h = L_{h\cdot 1} = h \cdot L_1 = h \cdot [L] = h \cdot [D_a f] \stackrel{*}{=} h \cdot f'_a = f'_a \cdot h = (f'_a \cdot (\bullet))_h.$$

Then $L = f'_a \cdot (\bullet)$. Then $L' = C^{\mathbb{R}}_{f'_a}$. Then $L'' = 0$.
Then $L_0 = 0$ and $L'_0 = f'_a$ and $L''_0 = 0$.
By hypothesis, $a \in \mathbb{D}''_f$.
We have $Q = D^2_a f = (1/2) \cdot f''_a \cdot (\bullet)^2$.
Then $Q = (1/2) \cdot f''_a \cdot (\bullet)^2$. Then $Q' = (1/2) \cdot f''_a \cdot 2 \cdot (\bullet) = f''_a \cdot (\bullet)$.
Then $Q' = f''_a \cdot (\bullet)$. Then $Q'' = C^{\mathbb{R}}_{f''_a}$.
Then $Q_0 = 0$ and $Q'_0 = 0$ and $Q''_0 = f''_a$.
By hypothesis, $a \in \mathbb{D}''_f$. Then $a \in \mathbb{D}''_f \subseteq \mathbb{D}_f$.
Then $(f^{\mathbb{T}}_a)_0 = 0$. So, since $L_0 = Q_0 = 0$, we get $(f^{\mathbb{T}}_a - L - Q)_0 = 0$.
Then $R_0 = (f^{\mathbb{T}}_a - L - Q)_0 = 0$. **Want:** $R'_0 = R''_0 = 0$.
By hypothesis, $a \in \mathbb{D}''_f$. Then $a \in \mathbb{D}''_f \subseteq \mathbb{D}'_f$. Then $(f'_{a+\bullet})_0 = f'_a$.
We have $(f^{\mathbb{T}}_a)' = f'_{a+\bullet}$. Then $(f^{\mathbb{T}}_a)'_0 = (f'_{a+\bullet})_0 = f'_a$, so $(f^{\mathbb{T}}_a)'_0 = f'_a$.
So, since $L'_0 = f'_a$ and $Q'_0 = 0$, we get $(f^{\mathbb{T}}_a - L - Q)'_0 = f'_a - f'_a - 0$.
Then $R'_0 = (f^{\mathbb{T}}_a - L - Q)'_0 = f'_a - f'_a - 0 = 0$. **Want:** $R''_0 = 0$.
By hypothesis, $a \in \mathbb{D}''_f$. Then $(f''_{a+\bullet})_0 = f''_a$.
We have $(f^{\mathbb{T}}_a)'' = f''_{a+\bullet}$. Then $(f^{\mathbb{T}}_a)''_0 = (f''_{a+\bullet})_0 = f''_a$, so $(f^{\mathbb{T}}_a)''_0 = f''_a$.
So, since $L'_0 = 0$ and $Q'_0 = f''_a$, we get $(f^{\mathbb{T}}_a - L - Q)''_0 = f''_a - 0 - f''_a$.
Then $R''_0 = (f^{\mathbb{T}}_a - L - Q)''_0 = f''_a - 0 - f''_a = 0$. □

4.12. **The Second Derivative Test.**

THEOREM 4.12.1. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{D}''_f$.

Assume: $(\, f'_a = 0 \,) \,\&\, (\, f''_a > 0 \,)$.
Then: f *has a local strict-minimum at* a.

Proof. This is Problem 3 on the Final Exam. □

THEOREM 4.12.2. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{D}''_f$.

Assume: $(\, f'_a = 0 \,) \,\&\, (\, f''_a < 0 \,)$.
Then: f *has a local strict-maximum at* a.

Proof. Unassigned HW. □

4.13. **The Inverse Function Theorem.**

THEOREM 4.13.1. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$ *be one-to-one. Let* $g := f^{-1}$.
Assume: $(\, f_0 = 0 \,) \,\&\, (\, f'_0 = 3 \,) \,\&\, (\, g_0 \in \mathcal{O}(0) \,)$.
Then: $g'_0 = 1/3$.

Proof. This is Problem 4 on the Final Exam. □

THEOREM 4.13.2. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$ *be one-to-one. Let* $g := f^{-1}$.
Assume: $(\, f_0 = 0 \,) \,\&\, (\, f'_0 \neq 0 \,) \,\&\, (\, g_0 \in \mathcal{O}(0) \,)$.
Then: $g'_0 = 1/(f'_0)$.

Proof. Unassigned HW. □

The next theorem is the **Precalculus Inverse Function Theorem**:

THEOREM 4.13.3. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$ *be one-to-one. Let* $g := f^{-1}$.
Let $a \in \mathbb{D}_f$. *Let* $b := f_a$.
Then: $f_a^{\mathbb{T}}$ *is one-to-one and* $(f_a^{\mathbb{T}})^{-1} = g_b^{\mathbb{T}}$.

Proof. Unassigned HW. □

The next theorem is called **Invariance of Domain,** $\mathbb{R} \dashrightarrow \mathbb{R}$.

THEOREM 4.13.4. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$ *be one-to-one. Let* $g := f^{-1}$.
Let $a \in \mathbb{D}_f$. *Let* $b := f_a$.
Assume: f *is continuous near* a.
Then: g *is continuous near* b.

Proof. To be proved in spring semester. □

Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ be one-to-one. Let $g := f^{-1}$.
Let $a \in \mathbb{D}_f$. Let $b := f_a$.
Then the following are all equivalent:
$f_a^{\mathbb{T}}$ is continuous near 0.
f is continuous near a.
g is continuous near b.
$g_b^{\mathbb{T}}$ is continuous near 0.
There are many theorems called the Inverse Function Theorem.
There's the Precalculus Inverse Function Theorem mentioned above.
There are topological inverse function theorems,
but they're usually called "Open Mapping Theorems"
and we covered one of them above, Theorem 3.11.1.
Finally, there are a variety of differentiable inverse function theorems.

We will call the following theorem

the **Inverse Function Theorem, first order,** $\mathbb{R} \to \mathbb{R}$.

THEOREM 4.13.5. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$ *be one-to-one. Let* $g := f^{-1}$.

Let $a \in \mathbb{D}'_f$. *Let* $b := f_a$.

Assume: $(\, f'_a \neq 0 \,) \,\&\, (\, f$ *is continuous near* $a \,)$.

Then: $g'_b = 1/(f'_a)$.

Proof. Unassigned HW. Uses Theorem 4.13.2.
Uses Theorem 4.13.3, the Precalculus Inverse Function Theorem.
Uses Theorem 4.13.4, Invariance of Domain, $\mathbb{R} \dashrightarrow \mathbb{R}$. □

The disadvantage of Theorem 4.13.2 is:

it assumes that $g \in \mathcal{O}(0)$.

In a typical inverse problem, much is known about the function f,

and the GOAL is to understand g.

In that context, an assumption about g might be difficult to verify.
Theorem 4.13.5 uses Invariance of Domain to trade in

an assumption that g is continuous near b

for

an assumption that f is continuous near a.

5. Integrability of functions $\mathbb{R} \dashrightarrow \mathbb{R}$

5.1. Outer measure.

We define $\boxed{\infty + \infty} := \infty$ and $\boxed{\infty - (-\infty)} := \infty$.
For all $x \in \mathbb{R}$, we define

$\boxed{x + \infty} := \infty$ and $\boxed{\infty + x} := \infty$ and $\boxed{\infty - x} := \infty$.

For all $x \in \mathbb{R}$, we define

$\boxed{-\infty + x} := -\infty$ and $\boxed{-\infty - x} := -\infty$.

We define $\boxed{\infty + (-\infty)} :=$ ☹ and $\boxed{(-\infty) + \infty} :=$ ☹.
For all $c \in (0; \infty]$, we define $\boxed{c \cdot \infty} := \infty$ and $\boxed{\infty \cdot c} := \infty$.
For all $c \in (0; \infty]$, we define $\boxed{c \cdot (-\infty)} := -\infty$ and $\boxed{(-\infty) \cdot c} := -\infty$.
For all $c \in [-\infty; 0)$, we define $\boxed{c \cdot \infty} := -\infty$ and $\boxed{\infty \cdot c} := -\infty$.
For all $c \in [-\infty; 0)$, we define $\boxed{c \cdot (-\infty)} := \infty$ and $\boxed{(-\infty) \cdot c} := \infty$.
We define $\boxed{0 \cdot \infty} :=$ ☹ and $\boxed{\infty \cdot 0} :=$ ☹.
We define $\boxed{0 \cdot (-\infty)} :=$ ☹ and $\boxed{(-\infty) \cdot 0} :=$ ☹.
For all $a \in \mathbb{R}^*$, we define $\boxed{-a} := (-1) \cdot a$.
For all $a, b \in \mathbb{R}^*$, we defein $\boxed{a - b} := a + (-b)$.

DEFINITION 5.1.1. *Let $a \in [0;\infty]^{\mathbb{N}}$. Then we define:*

$$\boxed{\sum_{j\in\mathbb{N}} a} := \sup\left\{\sum_{j=1}^{k} a_j \,\middle|\, k \in \mathbb{N}\right\}.$$

DEFINITION 5.1.2. $\boxed{\mathcal{OI}} := \{(a;b) \,|\, a,b \in \mathbb{R}\}$.

That is, $\mathcal{OI}$ is the set of bounded open intervals.
Note that $\varnothing \in \mathcal{OI}$. We consider $\varnothing$ to be an interval.
We define the **length** of any interval I:

DEFINITION 5.1.3. *Let I be an interval.*
If $I = \varnothing$, then $\boxed{\mathrm{L}_I} := 0$.
If $I \neq \varnothing$, then $\boxed{\mathrm{L}_I} := (\sup I) - (\inf I)$.

THEOREM 5.1.4. $\mathrm{L}_{[1;3)} = 3 - 1 = 2$ *and*
$\mathrm{L}_{(-\infty;4]} = 4 - (-\infty) = \infty$.

We define the
total length TL_U and **support** $\operatorname{supp} U$
of any sequence U of bounded open intervals:

DEFINITION 5.1.5. *Let $U \in \mathcal{OI}^{\mathbb{N}}$. Then*

$$\boxed{TL_U} := \sum_{j\in\mathbb{N}} \mathrm{L}_{U_j}$$

and

$$\boxed{\operatorname{supp} U} := \{j \in \mathbb{N} \,|\, U_j \neq \varnothing\}.$$

We next define various kinds of **open cover**s of S:

DEFINITION 5.1.6. *Let $S \subseteq \mathbb{R}$. Then:*

$$\boxed{\mathrm{LOC}_S} := \{\, U \in \mathcal{OI}^{\mathbb{N}} \,|\, \bigcup \mathbb{I}_U \supseteq S \,\} \quad \textit{and}$$

$$\boxed{\mathrm{JOC}_S} := \{\, U \in \mathcal{OI}^{\mathbb{N}} \,|\, (\, \bigcup \mathbb{I}_U \supseteq S \,) \,\&\, (\, \#\mathrm{supp}\, U < \infty \,) \,\}.$$

Also, $\forall k \in \mathbb{N}$,

$$\boxed{\mathrm{JOC}_S^k} := \{\, U \in \mathcal{OI}^{\mathbb{N}} \,|\, (\, \bigcup \mathbb{I}_U \supseteq S \,) \,\&\, (\, \#\mathrm{supp}\, U \leqslant k \,) \,\}.$$

We next define various kinds of **overmeasure**s of S:

DEFINITION 5.1.7. *Let $S \subseteq \mathbb{R}$. Then:*

$$\boxed{\mathrm{TLOC}_S} := \{\, \mathrm{TL}_U \,|\, U \in \mathrm{LOC}_S \,\} \quad \textit{and}$$

$$\boxed{\mathrm{TJOC}_S} := \{\, \mathrm{TL}_U \,|\, U \in \mathrm{JOC}_S \,\}.$$

Also, $\forall k \in \mathbb{N}$,

$$\boxed{\mathrm{TJOC}_S^k} := \{\, \mathrm{TL}_U \,|\, U \in \mathrm{JOC}_S^k \,\}.$$

For any $S \subseteq \mathbb{R}$, we now define

the **Lebesgue outer measure** LO_S of S and

the **Jordan outer measure** JO_S of S and

for any $k \in \mathbb{N}$, the **Jordan outer k-measure** JO_S^k of S

as follows:

DEFINITION 5.1.8. *Let $S \subseteq \mathbb{R}$. Then:*

$$\boxed{\mathrm{LO}_S} := \inf \mathrm{TLOC}_S \quad \textit{and}$$

$$\boxed{\mathrm{JO}_S} := \inf \mathrm{TJOC}_S.$$

Also, $\forall k \in \mathbb{N}$, we define:

$$\boxed{\mathrm{JO}_S^k} := \inf \mathrm{TJOC}_S^k.$$

THEOREM 5.1.9. *Let $S \subseteq \mathbb{R}$, $V \in \mathcal{OI}^{\mathbb{N}}$.*

Assume: $\bigcup \mathbb{I}_V \supseteq S$.

Then: $TL_V \geqslant \mathrm{LO}_S$.

THEOREM 5.1.10. *Let $S \subseteq \mathbb{R}$, $V \in \mathcal{OI}^{\mathbb{N}}$.*

Assume: $\bigcup \mathbb{I}_V \supseteq S$ *and* $\#\mathrm{supp}\, V < \infty$.

Then: $TL_V \geqslant \mathrm{JO}_S$.

THEOREM 5.1.11. *Let $S \subseteq \mathbb{R}$, $V \in \mathcal{OI}^{\mathbb{N}}$, $k \in \mathbb{N}$.*

Assume: $\bigcup \mathbb{I}_V \supseteq S$ *and* $\#\mathrm{supp}\, V \leqslant k$.

Then: $TL_V \geqslant \mathrm{JO}_S^k$.

THEOREM 5.1.12. *Let $S \subseteq \mathbb{R}$ be unbounded. Then $\mathrm{JO}_S = \infty$.*

Proof. We have: $\forall I \in \mathcal{OI}$, I is bounded.
Then $\forall U \in \mathcal{OI}^{\mathbb{N}}$, if $\#\mathrm{supp}\, U < \infty$, then $\bigcup \mathbb{I}_U$ is bounded.
So, since S is unbounded, we get $\mathrm{JOC}_S = \varnothing$, and so Then $\mathrm{TJOC}_S = \varnothing$.
Then $\mathrm{JO}_S = \inf \mathrm{TJOC}_S = \inf \varnothing = \infty$. □

The preceding theorem shows, in particular, that $\mathrm{JO}_{\mathbb{Z}} = \infty$.
Unassigned HW: Show that $\mathrm{LO}_{\mathbb{Z}} = 0$.
Thus JO and LO are different.

THEOREM 5.1.13.

$$\forall a, b \in \mathbb{R}, \quad \mathrm{JO}^1_{[a|b]} \geqslant |b - a|.$$

Proof. Unassigned HW. □

THEOREM 5.1.14. *$\forall k \in \mathbb{N}$,*

$$\forall a, b \in \mathbb{R}, \quad \mathrm{JO}^k_{[a|b]} \geqslant |b - a|.$$

Proof. Let $S := \{k \in \mathbb{N} \,|\, \forall a, b \in \mathbb{R},\ \mathrm{JO}^k_{[a|b]} \geqslant |b-a|\}$.
Want: $S = \mathbb{N}$. By Theorem 5.1.13, $1 \in S$.
By the PMI, want: $\forall k \in S,\ k+1 \in S$.
Given $k \in S$. **Want:** $k+1 \in S$.
Know: $\forall a, b \in \mathbb{R}, \quad \mathrm{JO}^k_{[a|b]} \geqslant |b-a|$.
Want: $\forall a, b \in \mathbb{R}, \quad \mathrm{JO}^{k+1}_{[a|b]} \geqslant |b-a|$.
Know: $\forall y, z \in \mathbb{R}, \quad \mathrm{JO}^k_{[y|z]} \geqslant |y-z|$.
Want: $\forall \alpha, \beta \in \mathbb{R}, \quad \mathrm{JO}^{k+1}_{[\alpha|\beta]} \geqslant |\beta-\alpha|$.
Given $\alpha, \beta \in \mathbb{R}$. **Want:** $\mathrm{JO}^{k+1}_{[\alpha|\beta]} \geqslant |\beta-\alpha|$.
Let $a := \min\{\alpha, \beta\}$, $b := \max\{\alpha, \beta\}$. **Want:** $\mathrm{JO}^{k+1}_{[a;b]} \geqslant b-a$.
Let $Q := \mathrm{TJOC}^{k+1}_{[a;b]}$. Then $\inf Q = \mathrm{JO}^{k+1}_{[a;b]}$.
Want: $\inf Q \geqslant b-a$. **Want:** $Q \geqslant b-a$.
Want: $\forall q \in Q,\ q \geqslant b-a$.
Given $q \in Q$. **Want:** $q \geqslant b-a$.
Since $q \in Q = \mathrm{TJOC}^{k+1}_{[a;b]}$, choose $U \in \mathrm{JOC}^{k+1}_{[a;b]}$ s.t. $q = \mathrm{TL}_U$.
Since $U \in \mathrm{JOC}^{k+1}_{[a;b]}$, we know: $\bigcup \mathbb{I}_U \supseteq [a;b]$ and $\#\mathrm{supp}\, U \leqslant k+1$.
Since $b \in [a;b] \subseteq \bigcup \mathbb{I}_U = \bigcup_{j \in \mathbb{N}} U_j$, choose $j \in \mathbb{N}$ s.t. $b \in U_j$.
Since $U \in \mathcal{OI}^{\mathbb{N}}$, we get $U_j \in \mathcal{OI}$. Choose $s, t \in \mathbb{R}$ s.t. $U_j = (s;t)$.
Since $b \in U_j = (s;t)$, it follows that $(s;t) \neq \varnothing$. Then $s < t$.
Since $b \in U_j = (s;t)$, we conclude that $s < b < t$.
Recall: $\forall y, z \in \mathbb{R}$, $\mathrm{JO}^k_{[y|z]} \geqslant |y-z|$. Then: $\mathrm{JO}^k_{[a;s]} \geqslant |s-a|$.
So, since $|s-a| \geqslant s-a$, we get: $\mathrm{JO}^k_{[a;s]} \geqslant s-a$.
Define $V \in \mathcal{OI}^{\mathbb{N}}$ by: $\forall i \in \mathbb{N}, \quad V_i = \begin{cases} U_i, & \text{if } i \neq j \\ \varnothing, & \text{if } i = j. \end{cases}$
Then $\mathrm{TL}_V = \mathrm{TL}_U - \mathrm{L}_{U_j}$ and $\bigcup \mathbb{I}_V \supseteq [a;s]$
and $\#\mathrm{supp}\, V = (\#\mathrm{supp}\, U) - 1 \leqslant (k+1) - 1 = k$.
Since $\bigcup \mathbb{I}_V \supseteq [a;s]$ and $\#\mathrm{supp}\, V \leqslant k$, we get: $V \in \mathrm{JOC}^k_{[a;s]}$.
Since $V \in \mathrm{JOC}^k_{[a;s]}$, we get $\mathrm{TL}_V \in \mathrm{TJOC}^k_{[a;s]}$, so $\mathrm{TL}_V \geqslant \inf \mathrm{TJOC}^k_{[a;s]}$.
Then $\mathrm{TL}_V \geqslant \inf \mathrm{TJOC}^k_{[a;s]} = \mathrm{JO}^k_{[a;s]}$, so $\mathrm{TL}_V \geqslant \mathrm{JO}^k_{[a;s]}$.
Since $\mathrm{TL}_V = \mathrm{TL}_U - \mathrm{L}_{U_j}$, we get: $\mathrm{TL}_U = \mathrm{L}_{U_j} + \mathrm{TL}_V$.
Since $U_j = (s;t)$ and $s < t$, we get: $\mathrm{L}_{U_j} = t-s$.
So, since $\mathrm{TL}_V \geqslant \mathrm{JO}^k_{[a;s]} \geqslant s-a$, we get: $\mathrm{L}_{U_j} + \mathrm{TL}_V \geqslant (t-s)+(s-a)$.
Since $b < t$, we get: $t-a > b-a$.
Then $\mathrm{TL}_U = \mathrm{L}_{U_j} + \mathrm{TL}_V \geqslant (t-s)+(s-a) = t-a > b-a$,

so $\mathrm{TL}_U > b - a$, so $\mathrm{TL}_U \geqslant b - a$.
Then $q = \mathrm{TL}_U \geqslant b - a$. □

THEOREM 5.1.15. *Let $a, b \in \mathbb{R}$. Then $\mathrm{JO}_{[a|b]} \geqslant |b - a|$.*

Proof. Let $Q := \mathrm{TJOC}_{[a|b]}$. Then $\inf Q = \mathrm{JO}_{[a|b]}$.
Want: $\inf Q \geqslant |b - a|$. **Want:** $Q \geqslant |b - a|$.
Want: $\forall q \in Q$, $q \geqslant b - a$.
Given $q \in Q$. **Want:** $q \geqslant b - a$.
Since $q \in Q = \mathrm{TJOC}_{[a|b]}$, choose $U \in \mathrm{JOC}_{[a|b]}$ s.t. $q = \mathrm{TL}_U$.
Since $U \in \mathrm{JOC}_{[a|b]}$, we get $\#\mathrm{supp}\, U < \infty$, so $\#\mathrm{supp}\, U \in \mathbb{N}_0$.
Let $k := \#\mathrm{supp}\, U$. Then $k \in \mathbb{N}_0$.
Since $U \in \mathrm{JOC}_{[a|b]}$, we get: $\bigcup \mathbb{I}_U \supseteq [a|b]$.
Since $a \in [a|b]$, we get: $[a|b] \neq \varnothing$.
So, since $\bigcup \mathbb{I}_U \supseteq [a|b]$, we get $\bigcup \mathbb{I}_U \neq \varnothing$.
Since $\displaystyle\bigcup_{j \in \mathbb{N}} U_j = \bigcup \mathbb{I}_U \neq \varnothing$, we conclude: $\exists j \in \mathbb{N}$ s.t. $U_j \neq \varnothing$.
Then $\mathrm{supp}\, U \neq \varnothing$. Then $\#\mathrm{supp}\, U \neq 0$.
Since $k \in \mathbb{N}_0$ and since $k = \#\mathrm{supp}\, U \neq 0$, we get: $k \in \mathbb{N}$.
So, since $k = \#\mathrm{supp}\, U$ and since $\bigcup \mathbb{I}_U \supseteq [a|b]$, we get: $U \in \mathrm{JOC}^k_{[a|b]}$.
Then $\mathrm{TL}_U \in \mathrm{TJOC}^k_{[a|b]}$. Then $\mathrm{TL}_U \geqslant \inf \mathrm{TJOC}^k_{[a|b]}$.
So, since $\mathrm{JO}^k_{[a|b]} = \inf \mathrm{TJOC}^k_{[a|b]}$, we get $\mathrm{TL}_U \geqslant \mathrm{JO}^k_{[a|b]}$.
By Theorem 5.1.14, we get: $\mathrm{JO}^k_{[a|b]} \geqslant |b - a|$.
Then $\mathrm{TL}_U \geqslant \mathrm{JO}^k_{[a|b]} \geqslant |b - a|$. □

THEOREM 5.1.16. *Let $a \in \mathbb{R}$, $b \geqslant a$. Then $\mathrm{JO}_{[a;b]} = b - a$.*

Proof. By Theorem 5.1.15, $\mathrm{JO}_{[a|b]} \geqslant |b - a|$.
Then $\mathrm{JO}_{[a;b]} = \mathrm{JO}_{[a|b]} \geqslant |b - a| = b - a$. **Want:** $\mathrm{JO}_{[a;b]} \leqslant b - a$.
Want: $\forall \varepsilon > 0$, $\mathrm{JO}_{[a;b]} \leqslant b - a + \varepsilon$.
Given $\varepsilon > 0$. **Want:** $\mathrm{JO}_{[a;b]} \leqslant b - a + \varepsilon$.
Let $I := (\, a - (\varepsilon/2)\, ;\, b + (\varepsilon/2)\,)$. Then $I \supseteq [a; b]$.
Let $U := (\, I\, ,\, \varnothing\, ,\, \varnothing\, ,\, \varnothing\, ,\, \varnothing\, ,\, \varnothing\, ,\, \ldots\,)$.
Then $U \in \mathcal{OI}^{\mathbb{N}}$ and $\bigcup \mathbb{I}_U = I \cup \varnothing \cup \varnothing \cup \cdots = I \supseteq [a; b]$.
Also, since $\mathrm{supp}\, U = \{1\}$, we get $\#\mathrm{supp}\, U < \infty$. Then $U \in \mathrm{JOC}_{[a;b]}$.
Then $\mathrm{TL}_U \in \mathrm{TJOC}_{[a;b]}$. Then $\mathrm{TL}_U \geqslant \inf \mathrm{TJOC}_{[a;b]}$.
So, since $\mathrm{JO}_{[a;b]} \inf \mathrm{TJOC}_{[a;b]}$, we get $\mathrm{TL}_U \geqslant \mathrm{JO}_{[a;b]}$, and so $\mathrm{JO}_{[a;b]} \leqslant \mathrm{TL}_U$.
Then $\mathrm{JO}_{[a;b]} \leqslant \mathrm{TL}_U = b - a + \varepsilon$. □

THEOREM 5.1.17. *Let $a \in \mathbb{R}$. Then $\mathrm{JO}_{\{a\}} = 0$.*

Proof. By the preceding theorem, $\mathrm{JO}_{[a;a]} = a - a$.
Then $\mathrm{JO}_{\{a\}} = \mathrm{JO}_{[a;a]} = a - a = 0$. □

We next show that JO is monotonic:

THEOREM 5.1.18. *Let* $S, T \subseteq \mathbb{R}$.
Assume: $T \supseteq S$. *Then:* $\mathrm{JO}_T \geqslant \mathrm{JO}_S$.

Proof. Since $T \supseteq S$, it follows that $\mathrm{JOC}_T \subseteq \mathrm{JOC}_S$, so $\mathrm{TJOC}_T \subseteq \mathrm{TJOC}_S$.
We have $\mathrm{TJOC}_T \subseteq \mathrm{TJOC}_S \geqslant \inf \mathrm{TJOC}_S = \mathrm{JO}_S$.
Since $\mathrm{TJOC}_T \geqslant \mathrm{JO}_S$, it follows that $\inf \mathrm{TJOC}_T \geqslant \mathrm{JO}_S$.
Then $\mathrm{JO}_T = \inf \mathrm{TJOC}_T \geqslant \mathrm{JO}_S$. □

We next show that JO is subadditive:

THEOREM 5.1.19. *Let* $S, T \subseteq \mathbb{R}$. *Then* $\mathrm{JO}_{S \cup T} \leqslant \mathrm{JO}_S + \mathrm{JO}_T$.

Proof. This is HW#13-1. □

The next two theorems are Unassigned HW:

THEOREM 5.1.20. *Let* $I \in \mathcal{OI}$, $a := \inf I$, $b := \sup I$. *Then:*
$(I = \varnothing) \Rightarrow ((a = \infty) \& (b = -\infty))$ *and*
$(I \neq \varnothing) \Rightarrow (-\infty < a < b < \infty)$ *and*
$I = (a; b)$ *and* $\mathrm{Cl}_I = [a; b]$.

THEOREM 5.1.21. *Let* $I \in \mathcal{OI}$, $a := \inf I$, $b := \sup I$, $\gamma > 0$.
Let $J := (a - \gamma; b + \gamma)$. *Then:*
$(I = \varnothing) \Leftrightarrow (J = \varnothing)$ *and*
$\mathrm{Cl}_I \subseteq J$ *and* $\mathrm{L}_J \leqslant L_I + 2 \cdot \gamma$.

THEOREM 5.1.22. *Let* $U \in \mathcal{OI}^{\mathbb{N}}$, $\varepsilon > 0$.
Then $\exists V \in \mathcal{OI}^{\mathbb{N}}$ *s.t.*
$\mathrm{supp}\, U = \mathrm{supp}\, V$ *and*
$\forall j \in \mathbb{N}$, $\mathrm{Cl}_{U_j} \subseteq V_j$ *and*
$\mathrm{TL}_V \leqslant \mathrm{TL}_U + \varepsilon$.

Proof. Define $a, b \in [-\infty; \infty]^{\mathbb{N}}$ by: $\forall j \in \mathbb{N}$,
$a_j := \inf U_j$ and $b_j := \sup U_j$.
Define $s, t \in [-\infty; \infty]^{\mathbb{N}}$ by: $\forall j \in \mathbb{N}$,
$s_j := a_j - \dfrac{\varepsilon/2}{2^j}$ and $t_j := b_j + \dfrac{\varepsilon/2}{2^j}$.
Then, $\forall j \in \mathbb{N}$, we have: $(U_j = \varnothing) \Rightarrow ((s_j = \infty) \& (t_j = -\infty))$.

Then, $\forall j \in \mathbb{N}$, we have: $(\, U_j = \varnothing\,) \Rightarrow (\,(s_j;t_j) = \varnothing\,)$.
Define $V \in \mathcal{OI}^{\mathbb{N}}$ by: $\forall j \in \mathbb{N}$, $V_j = (s_j;t_j)$. Then $V \in \mathcal{OI}^{\mathbb{N}}$.
Want: $\operatorname{supp} U = \operatorname{supp} V$ and
$\forall j \in \mathbb{N}$, $\mathrm{Cl}_{U_j} \subseteq V_j$ and
$\mathrm{TL}_V \leqslant \mathrm{TL}_U + \varepsilon$.
We have: $\forall j \in \mathbb{N}$,

$$(\, U_j = \varnothing\,) \Leftrightarrow (\, V_j = \varnothing\,)$$

and

$$\mathrm{Cl}_{U_j} = \mathrm{Cl}(\,(\inf U_j; \sup U_j)\,) = \mathrm{Cl}(\,(a_j;b_j)\,) = [a_j;b_j] \subseteq (s_j;t_j) = V_j$$

and

$$\mathrm{L}_{V_j} \leqslant \mathrm{L}_{U_j} + 2 \cdot \frac{\varepsilon/2}{2^j}.$$

Then: $\operatorname{supp} U = \operatorname{supp} V$ and
$\forall j \in \mathbb{N}$, $\mathrm{Cl}_{U_j} \subseteq V_j$.
Want: $\mathrm{TL}_V \leqslant \mathrm{TL}_U + \varepsilon$.

$$\text{We have } \mathrm{TL}_V \;\leqslant\; \mathrm{TL}_U + 2 \cdot \left(\sum_{j \in \mathbb{N}} \frac{\varepsilon/2}{2^j} \right) \;=\; \mathrm{TL}_U + 2 \cdot (\varepsilon/2) \;=\; \mathrm{TL}_U + \varepsilon.$$

□

THEOREM 5.1.23. *Let* $U \in \mathcal{OI}^{\mathbb{N}}$. *Assume* $\#\operatorname{supp} U < \infty$.
Define $\overline{U} \in (2^{\mathbb{R}})^{\mathbb{N}}$ *by:* $\forall j \in \mathbb{N}$, $\overline{U}_j = \mathrm{Cl}\, U_j$.
Then $\bigcup \mathbb{I}_{\overline{U}}$ *is closed.*

Proof. We have: $\forall i \in \mathbb{N}$, $\overline{U}_i$ is closed.
Let $F := \operatorname{supp} U$. Then F is finite.
Then $\bigcup_{i \in F} \overline{U}_i$ is closed.
Since $F = \operatorname{supp} U$, we get: $\forall i \in \mathbb{N} \backslash F$, $U_i = \varnothing$.
Then $\forall i \in \mathbb{N} \backslash F$, $\overline{U}_i = \mathrm{Cl}\, U_i = \mathrm{Cl}\, \varnothing = \varnothing$.
Then $\bigcup_{i \in \mathbb{N}} \overline{U}_i = \bigcup_{i \in F} \overline{U}_i$.
Since $\bigcup \mathbb{I}_{\overline{U}} = \bigcup_{i \in \mathbb{N}} \overline{U}_i = \bigcup_{i \in F} \overline{U}_i$, and since $\bigcup_{i \in F} \overline{U}_i$ is closed,
we conclude that $\bigcup \mathbb{I}_{\overline{U}}$ is closed. □

THEOREM 5.1.24. *Let* $S \subseteq \mathbb{R}$. *Then* $\mathrm{JO}_S = JO_{\mathrm{Cl}\, S}$.

Proof. Let $\overline{S} := \mathrm{Cl}\, S$. **Want:** $\mathrm{JO}_S = \mathrm{JO}_{\overline{S}}$.
Since $S \subseteq \mathrm{Cl}\, S = \overline{S}$, we get $\mathrm{JO}_S \leqslant \mathrm{JO}_{\overline{S}}$.
Want: $\mathrm{JO}_{\overline{S}} \leqslant \mathrm{JO}_S$. We have: $\mathrm{JO}_S = \inf \mathrm{TJOC}_S$.
Want: $\mathrm{JO}_{\overline{S}} \leqslant \inf \mathrm{TJOC}_S$. **Want:** $\mathrm{JO}_{\overline{S}} \leqslant \mathrm{TJOC}_S$.
Want: $\forall a \in \mathrm{TJOC}_S$, $\mathrm{JO}_{\overline{S}} \leqslant a$.
Given $a \in \mathrm{TJOC}_S$. **Want:** $\mathrm{JO}_{\overline{S}} \leqslant a$.
Since $a \in \mathrm{TJOC}_S$, choose $U \in \mathrm{JOC}_S$ s.t. $a = \mathrm{TL}_U$.

Want: $\mathrm{JO}_{\overline{S}} \leqslant \mathrm{TL}_U$. **Want:** $\forall \varepsilon > 0$, $\mathrm{JO}_{\overline{S}} \leqslant \mathrm{TL}_U + \varepsilon$.
Given $\varepsilon > 0$. **Want:** $\mathrm{JO}_{\overline{S}} \leqslant \mathrm{TL}_U + \varepsilon$.
By Theorem 5.1.22, choose $V \in \mathcal{OI}^{\mathbb{N}}$ s.t. $(\,\mathrm{supp}\, U = \mathrm{supp}\, V\,)$
and $(\,\forall j \in \mathbb{N},\ \mathrm{Cl}\, U_j \subseteq V_j\,)$ and $(\,\mathrm{TL}_V \leqslant \mathrm{TL}_U + \varepsilon\,)$.
Want: $\mathrm{JO}_{\overline{S}} \leqslant \mathrm{TL}_V$.
Define $\overline{U} \in (2^{\mathbb{R}})^{\mathbb{N}}$ by: $\forall j \in \mathbb{N}$, $\overline{U}_j = \mathrm{Cl}\, U_j$.
Since $U \in \mathrm{JOC}_S$, we get: $\bigcup \mathbb{I}_U \supseteq S$ and $\#\mathrm{supp}\, U < \infty$.
By Theorem 5.1.23, $\bigcup \mathbb{I}_{\overline{U}}$ is closed, and so $\mathrm{Cl}\,(\bigcup \mathbb{I}_{\overline{U}}) = \bigcup \mathbb{I}_{\overline{U}}$.
Since $\forall j \in \mathbb{N}$, $U_j \subseteq \mathrm{Cl}\, U_j = \overline{U}_j$, we get: $\bigcup \mathbb{I}_U \subseteq \bigcup \mathbb{I}_{\overline{U}}$.
Since $\forall j \in \mathbb{N}$, $\overline{U}_j = \mathrm{Cl}\, U_j \subseteq V_j$, we get: $\bigcup \mathbb{I}_{\overline{U}} \subseteq \bigcup \mathbb{I}_V$.
Since $S \subseteq \bigcup \mathbb{I}_U \subseteq \bigcup \mathbb{I}_{\overline{U}}$, we get: $\mathrm{Cl}\, S \subseteq \mathrm{Cl}\,(\bigcup \mathbb{I}_{\overline{U}})$.
Then $\overline{S} = \mathrm{Cl}\, S \subseteq \mathrm{Cl}\,(\bigcup \mathbb{I}_{\overline{U}}) = \bigcup \mathbb{I}_{\overline{U}} \subseteq \bigcup \mathbb{I}_V$.
So, since $\#\mathrm{supp}\, V = \#\mathrm{supp}\, U < \infty$, we get: $V \in \mathrm{JOC}_{\overline{S}}$.
Then $\mathrm{TL}_V \in \mathrm{TJOC}_{\overline{S}}$. Then $\mathrm{TL}_V \geqslant \inf \mathrm{TJOC}_{\overline{S}}$.
Then $\mathrm{JO}_{\overline{S}} = \inf \mathrm{TJOC}_{\overline{S}} \leqslant \mathrm{TL}_V$. □

For any sets A and S,
$\{\ S \cap A\ ,\ S\backslash A\ \}$ is a partition of S,
and this would lead us to expect that
$\mathrm{JO}_{S\cap A} + \mathrm{JO}_{S\backslash A} = \mathrm{JO}_S$,
but we will soon see that this is not always true.
We capture the equation $\mathrm{JO}_{S\cap A} + \mathrm{JO}_{S\backslash A} = \mathrm{JO}_S$ in a definition:

DEFINITION 5.1.25. *Let $A, S \subseteq \mathbb{R}$.*
*By A **splits** S **well**, we mean:* $\mathrm{JO}_{S\cap A} + \mathrm{JO}_{S\backslash A} = \mathrm{JO}_S$.

Our focus here is on Jordan measure theory,
but we can do something similar for Lebesgue measure:

DEFINITION 5.1.26. *Let $A, S \subseteq \mathbb{R}$.*
*By A **splits** S **L-well**, we mean:* $\mathrm{LO}_{S\cap A} + \mathrm{LO}_{S\backslash A} = \mathrm{LO}_S$.

Since $\mathrm{Cl}([0;1] \cap \mathbb{Q}) = [0;1]$,
from the preceding theorem, we get: $\mathrm{JO}_{[0;1]\cap\mathbb{Q}} = \mathrm{JO}_{[0;1]}$.
So, since $\mathrm{JO}_{[0;1]} = 1 - 0$, we conclude: $\mathrm{JO}_{[0;1]\cap\mathbb{Q}} = 1$.
Since $\mathrm{Cl}([0;1]\backslash\mathbb{Q}) = [0;1]$,
from the preceding theorem, we get: $\mathrm{JO}_{[0;1]\backslash\mathbb{Q}} = \mathrm{JO}_{[0;1]}$.
So, since $\mathrm{JO}_{[0;1]} = 1 - 0$, we conclude: $\mathrm{JO}_{[0;1]\backslash\mathbb{Q}} = 1$.
Then $\{\ [0;1] \cap \mathbb{Q}\ ,\ [0;1]\backslash\mathbb{Q}\}$ is a partition of $[0;1]$, but, paradoxically,
$\mathrm{JO}_{[0;1]\cap\mathbb{Q}}\ +\ \mathrm{JO}_{[0;1]\backslash\mathbb{Q}}\ \neq\ \mathrm{JO}_{[0;1]}$.
So $\mathbb{Q}$ does NOT split $[0;1]$ well.

In this sense, $\mathbb{Q}$ is not a good set
from the point of view of Jordan measure.
Note that $\mathbb{Q} \cap [7;8]$ splits $[0;1]$ well, because
$[0;1] \cap (\mathbb{Q} \cap [7;8]) = \varnothing$ and $[0;1] \backslash (\mathbb{Q} \cap [7;8]) = [0;1]$.
However, there's a *different* set that $\mathbb{Q} \cap [7;8]$ does NOT split well.
Specifically, $\mathbb{Q} \cap [7;8]$ does NOT split $[7;8]$ well.
In this sense, $\mathbb{Q} \cap [7;8]$ is another bad set
from the point of view of Jordan measure.
In the next definition, we will formalize the idea that a "good" set,
from the point of view of Jordan measure,
is one which splits *every* subset of $\mathbb{R}$ well.

DEFINITION 5.1.27. *Let* $A \subseteq \mathbb{R}$.
By A *is* **Carathéodory-Jordan measurable** *or* **CJ-measurable**,
we mean: $\forall S \subseteq \mathbb{R}$, A *splits* S *well.*

DEFINITION 5.1.28. *Let* $A \subseteq \mathbb{R}$.
By A *is* **Carathéodory-Lebesgue measurable** *or* **CL-measurable**,
we mean: $\forall S \subseteq \mathbb{R}$, A *splits* S *L-well.*

We won't be developing Lebesgue measure theory,
but we comment that,
every CJ-measurable set is CL-measurable,
so the Lebesgue theory
has fewer paradoxical decompositions than the Jordan theory,
and is, in that sense, a better theory. However, it is not perfect:
there do exist subsets of $\mathbb{R}$ that are not CL-measurable,
but proof of their existence is known to require the Axiom of Choice.
So subsets of $\mathbb{R}$ that are not CL-measurable are very obscure.
By contrast, it is not hard to describe
subsets of $\mathbb{R}$ (like $\mathbb{Q}$) that are not CJ-measurable.
However, we will eventually show that
there is a broad enough collection of CJ-measurable sets
to suffice for most applications in the natural sciences.
In that sense, Jordan measure theory
and the corresponding integration theory,
which we will soon be describing
are good enough for government work.

We develop some of the theory of CJ-measurable sets,

then use it in showing that every interval is CJ-measurable.
We begin by showing that
Jordan outer measure is pairwise-additive on CJ-measurable sets:

THEOREM 5.1.29. *Let $A, B \subseteq \mathbb{R}$.*
Assume A is CJ-measurable.
Assume $A \cap B = \varnothing$. *Then* $\mathrm{JO}_{A \cup B} = \mathrm{JO}_A + \mathrm{JO}_B$.

The preceding theorem expresses that Jordan measure is **pairwise additive**.

Proof. Let $S := A \cup B$.
Then, because $A \cap B = \varnothing$, it follows that $S \cap A = A$ and $S \backslash A = B$.
Since A is CJ-measurable, A splits S well, so $\mathrm{JO}_S = \mathrm{JO}_{S \cap A} + \mathrm{JO}_{S \backslash A}$.
Then $\mathrm{JO}_{A \cup B} = \mathrm{JO}_S = \mathrm{JO}_{S \cap A} + \mathrm{JO}_{S \backslash A} = \mathrm{JO}_A + \mathrm{JO}_B$. □

Unassigned HW: Use Theorem 5.1.29 and induction on $\#\mathcal{Q}$ to prove the following.

THEOREM 5.1.30. *Let $S \subseteq \mathbb{R}$.*
Let $\mathcal{Q}$ be a finite partition of S by CJ-measurable sets.
Then $$\mathrm{JO}_S = \sum_{A \in \mathcal{Q}} \mathrm{JO}_A.$$

The preceding theorem expresses that Jordan measure is **finitely additive**.

DEFINITION 5.1.31. $\forall A \subseteq \mathbb{R}$, $\boxed{A^c} := \mathbb{R} \backslash A$.

For any $A, S \subseteq \mathbb{R}$, we have: $S \backslash A = S \cap A^c$, and so
A splits S well iff $\mathrm{JO}_S = \mathrm{JO}_{S \cap A} + \mathrm{JO}_{S \cap A^c}$.

THEOREM 5.1.32. *Let $A \subseteq \mathbb{R}$ be CJ-measurable.*
Then A^c is CJ-measurable.

Proof. **Want:** $\forall S \subseteq \mathbb{R}$, A^c splits S well.
Given $S \subseteq \mathbb{R}$. **Want:** A^c splits S well.
Want: $\mathrm{JO}_S = \mathrm{JO}_{S \cap A^c} + \mathrm{JO}_{S \cap A^{cc}}$.
Since A is CJ-measurable,
we conclude that A splits S well,
and so $\mathrm{JO}_S = \mathrm{JO}_{S \cap A} + \mathrm{JO}_{S \cap A^c}$.
So, since $A = A^{cc}$, we get $\mathrm{JO}_S = \mathrm{JO}_{S \cap A^{cc}} + \mathrm{JO}_{S \cap A^c}$.
Then $\mathrm{JO}_S = \mathrm{JO}_{S \cap A^c} + \mathrm{JO}_{S \cap A^{cc}}$. □

Unassigned HW. Show: Let $A, B \subseteq \mathbb{R}$. Then

$$\{ A \cap B \ , \ A \cap B^c \ , \ A^c \cap B \ , \ A^c \cap B^c \}$$

is a partition of $\mathbb{R}$.

THEOREM 5.1.33. *Let $A, B \subseteq \mathbb{R}$ both be CJ-measurable.*
Let $W' := A \cap B, \quad X' := A \cap B^c,$
$Y' := A^c \cap B, \quad Z' := A^c \cap B^c.$
Let $W \subseteq W', \quad X \subseteq X', \quad Y \subseteq Y', \quad Z \subseteq Z'.$
Then $\mathrm{JO}_{W \cup X \cup Y \cup Z} = \mathrm{JO}_W + \mathrm{JO}_X + \mathrm{JO}_Y + \mathrm{JO}_Z.$

Proof. Let $S := W \cup X \cup Y \cup Z$. **Want:** $\mathrm{JO}_S = \mathrm{JO}_W + \mathrm{JO}_X + \mathrm{JO}_Y + \mathrm{JO}_Z$.
Since $W \subseteq W'$, we get $W \cap W' = W$.
We have: $\{ W' \ , \ X' \ , \ Y' \ , \ Z' \}$ is pairwise-disjoint.
So, since $W \subseteq W'$, we get: $W \cap X' = W \cap Y' = W \cap Z' = \varnothing$.
Then

$$\begin{aligned} S \cap W' &= (W \cup X \cup Y \cup Z) \cap W' \\ &= (W \cap W') \cup (W \cap X') \cup (W \cap Y') \cup (W \cap Z') \\ &= W \cup \varnothing \cup \varnothing \cup \varnothing = W. \end{aligned}$$

Similarly, $S \cap X' = X$ and $S \cap Y' = Y$ and $S \cap Z' = Z$.
Want: $\mathrm{JO}_S = \mathrm{JO}_{S \cap W'} + \mathrm{JO}_{S \cap X'} + \mathrm{JO}_{S \cap Y'} + \mathrm{JO}_{S \cap Z'}$.
Because A is CJ-measurable, A splits S well,
so $\mathrm{JO}_S = \mathrm{JO}_{S \cap A} + \mathrm{JO}_{S \cap A^c}$.
Because B is CJ-measurable, B splits $S \cap A$ well,
so $\mathrm{JO}_{S \cap A} = \mathrm{JO}_{S \cap A \cap B} + \mathrm{JO}_{S \cap A \cap B^c}$.
Then $\mathrm{JO}_{S \cap A} = \mathrm{JO}_{S \cap W'} + \mathrm{JO}_{S \cap X'}$.
Because B is CJ-measurable, B splits $S \cap A^c$ well,
so $\mathrm{JO}_{S \cap A^c} = \mathrm{JO}_{S \cap A^c \cap B} + \mathrm{JO}_{S \cap A^c \cap B^c}$.
Then $\mathrm{JO}_{S \cap A^c} = \mathrm{JO}_{S \cap Y'} + \mathrm{JO}_{S \cap Z'}$.
Then

$$\begin{aligned} \mathrm{JO}_S &= \mathrm{JO}_{S \cap A} + \mathrm{JO}_{S \cap A^c} \\ &= \mathrm{JO}_{S \cap W'} + \mathrm{JO}_{S \cap X'} + \mathrm{JO}_{S \cap Y'} + \mathrm{JO}_{S \cap Z'}. \end{aligned}$$

□

THEOREM 5.1.34. *Let $A, B \subseteq \mathbb{R}$ both be CJ-measurable.*
Then $A \cap B$ is CJ-measurable.

Proof. Let $W' := A \cap B, \quad X' := A \cap B^c,$
$Y' := A^c \cap B, \quad Z' := A^c \cap B^c.$
Want: W' is CJ-measurable.
Want: $\forall S \subseteq \mathbb{R}$, W' splits S well.
Given $S \subseteq \mathbb{R}$. **Want:** W' splits S well.
Want: $\mathrm{JO}_S = \mathrm{JO}_{S \cap W'} + \mathrm{JO}_{S \backslash W'}$.

Let $W := S \cap W'$, $X := S \cap X'$,
$Y := S \cap Y'$, $Z := S \cap Z'$.
Since $\{ W', X', Y', Z' \}$ is a partition of $\mathbb{R}$,
we get: $\{ W, X, Y, Z \}$ is a partition of S.
Then $S \backslash W = X \cup Y \cup Z$. Also, $S \backslash W = S \backslash (S \cap W') = S \backslash W'$.
We get: $\mathrm{JO}_{S \cap W'} = \mathrm{JO}_W$ and $\mathrm{JO}_{S \backslash W'} = \mathrm{JO}_{S \backslash W} = \mathrm{JO}_{X \cup Y \cup Z}$.
Want: $\mathrm{JO}_S = \mathrm{JO}_W + \mathrm{JO}_{X \cup Y \cup Z}$.
By Theorem 5.1.33, we get:
both $\mathrm{JO}_{W \cup X \cup Y \cup Z} = \mathrm{JO}_W + \mathrm{JO}_X + \mathrm{JO}_Y + \mathrm{JO}_Z$
and $\mathrm{JO}_{\varnothing \cup X \cup Y \cup Z} = \mathrm{JO}_{\varnothing} + \mathrm{JO}_X + \mathrm{JO}_Y + \mathrm{JO}_Z$.
Then: both $\mathrm{JO}_S = \mathrm{JO}_W + \mathrm{JO}_X + \mathrm{JO}_Y + \mathrm{JO}_Z$
and $\mathrm{JO}_{X \cup Y \cup Z} = 0 + \mathrm{JO}_X + \mathrm{JO}_Y + \mathrm{JO}_Z$.
Then $\mathrm{JO}_S = \mathrm{JO}_W + \mathrm{JO}_X + \mathrm{JO}_Y + \mathrm{JO}_Z$
$= \mathrm{JO}_W + (0 + \mathrm{JO}_X + \mathrm{JO}_Y + \mathrm{JO}_Z)$
$= \mathrm{JO}_W + \mathrm{JO}_{X \cup Y \cup Z}$. □

THEOREM 5.1.35. *Let $A, B \subseteq \mathbb{R}$ both be CJ-measurable.*
Then $A \cap B$, $A \cup B$, $A \backslash B$ are all CJ-measurable.

Proof. By Theorem 5.1.34, $A \cap B$ is CJ-measurable.
Want: $A \cup B$, $A \backslash B$ are both CJ-measurable.
By Theorem 5.1.32, we get: A^c, B^c are both CJ-measurable.
Since A, B^c are both CJ-measurable, by Theorem 5.1.34,
$A \cap B^c$ is CJ-measurable.
So, since $A \backslash B = A \cap B^c$, we see that $A \backslash B$ is CJ-measurable.
Want: $A \cup B$ is CJ-measurable.
Since A^c and B^c are both CJ-measurable, by Theorem 5.1.34,
$A^c \cap B^c$ is CJ-measurable.
Then, by Theorem 5.1.32, $(A^c \cap B^c)^c$ is CJ-measurable.
So, since $A \cup B = (A^c \cap B^c)^c$, we see that $A \cup B$ is CJ-measurable. □

THEOREM 5.1.36. *Let $A, B \subseteq \mathbb{R}$.*
Assume: A is CJ-measurable and $\mathrm{JO}_A < \infty$.
Then: $\mathrm{JO}_{B \backslash A} = \mathrm{JO}_B - \mathrm{JO}_A$.

Proof. Since A is CJ-measurable, A splits B well,
so $\mathrm{JO}_B = \mathrm{JO}_{B \cap A} + \mathrm{JO}_{B \backslash A}$.
Since $A \subseteq B$, we get $B \cap A = A$. Then $\mathrm{JO}_B = \mathrm{JO}_A + \mathrm{JO}_{B \backslash A}$.
So, since $\mathrm{JO}_A < \infty$, we get: $\mathrm{JO}_B - \mathrm{JO}_A = \mathrm{JO}_{B \backslash A}$.
Then $\mathrm{JO}_{B \backslash A} = \mathrm{JO}_B - \mathrm{JO}_A$. □

We have developed a certain amount of theory about CJ-measurable sets.
However, we have yet to produce examples.
Our next goal is to show that all intervals are CJ-measurable.
We begin by showing, that: $\forall a \in \mathbb{R}$, $\forall U \in \mathcal{OI}$,
the sets $(-\infty; a)$ and $(a; \infty)$ split U well, in length:

THEOREM 5.1.37. *Let $U \in \mathcal{OI}$, $a \in \mathbb{R}$, $A := (-\infty; a)$, $B := (a; \infty)$. Then* $(\, U \cap A, U \cap B \in \mathcal{OI}\,)$ & $(\, L_U = L_{U\cap A} + L_{U\cap B}\,)$.

Idea of proof:
There are two cases: $(\, U = \varnothing\,) \vee (\, U \neq \varnothing\,)$.
The case where $U = \varnothing$ is an exercise for the reader.
We concentrate on the case where $U \neq \varnothing$.
Choose $s, t \in \mathbb{R}$ s.t. $s < t$ and $U = (s; t)$.
We have $L_U = L_{(s;t)} = \sup_{(s;t)} - \inf_{(s;t)} = t - s$, so $L_U = t - s$.
There are three subcases: $(\, a \leqslant s\,) \vee (\, s < a < t\,) \vee (\, t \leqslant a\,)$.
The subcases where $a \leqslant s$ or $t \leqslant a$ are exercises for the reader.
We concentrate on the case where $s < a < t$.
Then $U \cap A = (s; a)$ and $U \cap B = (a; t)$.
Then $U \cap A, U \cap B \in \mathcal{OI}$. **It remains to show:** $L_U = L_{U\cap A} + L_{U\cap B}$.
We have $L_{U\cap A} = L_{(s;a)} = \sup_{(s;a)} - \inf_{(s;a)} = a - s$, so $L_{U\cap A} = a - s$.
We have $L_{U\cap B} = L_{(a;t)} = \sup_{(a;t)} - \inf_{(a;t)} = t - a$, so $L_{U\cap B} = t - a$.
Then $L_{U\cap B} + L_{U\cap A} = (t - a) + (a - s) = t - s = L_U$.
Then $L_U = L_{U\cap A} + L_{U\cap B}$. QED

THEOREM 5.1.38. *Let $A, S \subseteq \mathbb{R}$. Assume* $\mathrm{JO}_S = \infty$. *Then A splits S well.*

Proof. By subadditivity of JO, we get: $\mathrm{JO}_S \leqslant \mathrm{JO}_{S\cap A} + \mathrm{JO}_{S\backslash A}$.
Want: $\mathrm{JO}_{S\cap A} + \mathrm{JO}_{S\backslash A} \leqslant \mathrm{JO}_S$.
We have $\mathrm{JO}_{S\cap A} + \mathrm{JO}_{S\backslash A} \in \mathbb{R}^* \leqslant \infty = \mathrm{JO}_S$. □

Because of the preceding theorem,
to show that a set A is CJ-measurable,
it suffices to show that it splits all
sets of finite outer Jordan measure
well; the sets of infinite measure are split well for free.

THEOREM 5.1.39. *Let $S \subseteq \mathbb{R}$. Then: S is CJ-measurable iff* $\forall S \subseteq \mathbb{R}$, $(\, \mathrm{JO}_S < \infty\,) \Rightarrow (\, A$ *splits S well*$)$.

THEOREM 5.1.40. *Let $a \in \mathbb{R}$. Then $(-\infty; a)$ is CJ-measurable.*

Proof. Let $A := (-\infty; a)$, $B := (a; \infty)$. **Want:** A is CJ-measurable.
Want: $\forall S \subseteq \mathbb{R}$, ($\mathrm{JO}_S < \infty$) $\Rightarrow$ (A splits S well).
Given $S \subseteq \mathbb{R}$. Assume $\mathrm{JO}_S < \infty$. **Want:** A splits S well.
Want: $\mathrm{JO}_S = \mathrm{JO}_{S \cap A} + \mathrm{JO}_{S \backslash A}$.
By HW 13-1, $\mathrm{JO}_S \leqslant \mathrm{JO}_{S \cap A} + \mathrm{JO}_{S \backslash A}$.
Want: $\mathrm{JO}_{S \cap A} + \mathrm{JO}_{S \backslash A} \leqslant \mathrm{JO}_S$.
Want: $\forall \varepsilon > 0$, $\mathrm{JO}_{S \cap A} + \mathrm{JO}_{S \backslash A} \leqslant \mathrm{JO}_S + \varepsilon$.
Given $\varepsilon > 0$. **Want:** $\mathrm{JO}_{S \cap A} + \mathrm{JO}_{S \backslash A} \leqslant \mathrm{JO}_S + \varepsilon$.
Since $\mathrm{JO}_S = \inf \mathrm{TJOC}_S$, we get $\neg(\mathrm{JO}_S + \varepsilon \leqslant \mathrm{TJOC}_S)$.
Choose $c \in \mathrm{TJOC}_S$ s.t. $\mathrm{JO}_S + \varepsilon > c$.
Want: $\mathrm{JO}_{S \cap A} + \mathrm{JO}_{S \backslash A} \leqslant c$.
Since $c \in \mathrm{TJOC}_S$, choose $U \in \mathrm{JOC}_S$ s.t. $c = \mathrm{TL}_U$.
Since $U \in \mathrm{JOC}_S$, we get:
$U \in \mathcal{OI}^{\mathbb{N}}$ and $\#\mathrm{supp}\, U < \infty$ and $\bigcup \mathbb{I}_U \supseteq S$.
Define $V, W \in \mathcal{OI}^{\mathbb{N}}$ by: $\forall j \in \mathbb{N}$,
$$V_j = U_j \cap A \quad \text{and} \quad W_j = U_j \cap B.$$
Since $\mathrm{supp}\, V \subseteq \mathrm{supp}\, U$, we get $\#\mathrm{supp}\, V \leqslant \#\mathrm{supp}\, U$.
We have $\bigcup \mathbb{I}_V = \bigcup_{j \in \mathbb{N}} V_j = \bigcup_{j \in \mathbb{N}} (U_j \cap A)$
$$= \left(\bigcup_{j \in \mathbb{N}} U_j\right) \cap A = (\bigcup \mathbb{I}_U) \cap A \supseteq S \cap A.$$
Then $\bigcup \mathbb{I}_V \supseteq S \cap A$. We also have $\#\mathrm{supp}\, V \leqslant \#\mathrm{supp}\, U < \infty$,
and so we get: $V \in \mathrm{JOC}_{S \cap A}$.
Then $\mathrm{TL}_V \in \mathrm{TJOC}_{S \cap A} \geqslant \inf \mathrm{TJOC}_{S \cap A} = \mathrm{JO}_{S \cap A}$, so $\mathrm{TL}_V \geqslant \mathrm{JO}_{S \cap A}$.
Since $\mathrm{supp}\, W \subseteq \mathrm{supp}\, U$, we get $\#\mathrm{supp}\, W \leqslant \#\mathrm{supp}\, U$.
We have $\bigcup \mathbb{I}_W = \bigcup_{j \in \mathbb{N}} W_j = \bigcup_{j \in \mathbb{N}} (U_j \cap B)$
$$= \left(\bigcup_{j \in \mathbb{N}} U_j\right) \cap B = (\bigcup \mathbb{I}_U) \cap B \supseteq S \cap B.$$
Then $\bigcup \mathbb{I}_W \supseteq S \cap B$. We also have, $\#\mathrm{supp}\, W \leqslant \#\mathrm{supp}\, U < \infty$,
and so we get: $W \in \mathrm{JOC}_{S \cap B}$.
Then $\mathrm{TL}_W \in \mathrm{TJOC}_{S \cap B} \geqslant \inf \mathrm{TJOC}_{S \cap B} = \mathrm{JO}_{S \cap B}$, so $\mathrm{TL}_W \geqslant \mathrm{JO}_{S \cap B}$.
We have both $\mathrm{TL}_V \geqslant \mathrm{JO}_{S \cap A}$ and $\mathrm{TL}_W \geqslant \mathrm{JO}_{S \cap B}$,
and so $\mathrm{JO}_{S \cap A} + \mathrm{JO}_{S \cap B} \leqslant \mathrm{TL}_V + \mathrm{TL}_W$.
By Theorem 5.1.37, we get: $\forall j \in \mathbb{N}$, $L_{U_i} = L_{U_i \cap A} + L_{U_i \cap B}$.
Then $\mathrm{TL}_U = \sum_{i \in \mathbb{N}} L_{U_i} = \sum_{i \in \mathbb{N}} (L_{U_i \cap A} + L_{U_i \cap B}) = \sum_{i \in \mathbb{N}} (L_{V_i} + L_{W_i})$
$$= (\textstyle\sum_{i \in \mathbb{N}} L_{V_i}) + (\sum_{i \in \mathbb{N}} L_{W_i}) = \mathrm{TL}_V + \mathrm{TL}_W.$$
We have: $S \backslash A = S \cap A^c = S \cap [a; \infty) = S \cap (\{a\} \cup (a; \infty))$
$$= S \cap (\{a\} \cup B) = (S \cap \{a\}) \cup (S \cap B)$$
$$\subseteq \{a\} \cup (S \cap B).$$

Then $\mathrm{JO}_{S\backslash A} \leqslant \mathrm{JO}_{\{a\}\cup(S\cap B)}$.
So, by subadditivity of JO, we get: $\mathrm{JO}_{S\backslash A} \leqslant \mathrm{JO}_{\{a\}} + \mathrm{JO}_{S\cap B}$.
We have $\mathrm{JO}_{\{a\}} = \mathrm{JO}_{[a;a]} = a - a = 0$. Then $\mathrm{JO}_{S\backslash A} \leqslant \mathrm{JO}_{S\cap B}$.
Then $\mathrm{JO}_{S\cap A}+\mathrm{JO}_{S\backslash A} \leqslant \mathrm{JO}_{S\cap A}+\mathrm{JO}_{S\cap B} \leqslant \mathrm{TL}_V+\mathrm{TL}_W = \mathrm{TL}_U = c$. □

THEOREM 5.1.41. *Let $a \in \mathbb{R}$. Then $[a;\infty)$ is CJ-measurable.*

Proof. By Theorem 5.1.40, $(-\infty;a)$ is CJ-measurable.
Then $(-\infty;a)^c$ is CJ-measurable.
So, since $(-\infty;a)^c = [a;\infty)$, we get: $[a;\infty)$ is CJ-measurable. □

THEOREM 5.1.42. *Let $a \in \mathbb{R}$. Then $(a;\infty)$ is CJ-measurable.*

Proof. Unassigned HW. Similar to Theorem 5.1.40. □

THEOREM 5.1.43. *Let $a \in \mathbb{R}$. Then $(-\infty;a]$ is CJ-measurable.*

Proof. By Theorem 5.1.42, $(a;\infty)$ is CJ-measurable.
Then $(a;\infty)^c$ is CJ-measurable.
So, since $(a;\infty)^c = (-\infty;a]$, we get: $(-\infty;a]$ is CJ-measurable. □

THEOREM 5.1.44. *Let $a, b \in \mathbb{R}$. Then $[a;b]$ is CJ-measurable.*

Proof. Since $(-\infty;b]$ and $[a;\infty)$ are CJ-measurable,
we get: $(-\infty;b] \cap [a;\infty)$ is CJ-measurable.
So, since $(-\infty;b] \cap [a;\infty) = [a;b]$, we get: $[a;b]$ is CJ-measurable. □

THEOREM 5.1.45. *Let $a, b \in \mathbb{R}$. Then $[a;b)$ is CJ-measurable.*

Proof. Since $(-\infty;b)$ and $[a;\infty)$ are CJ-measurable,
we get: $(-\infty;b) \cap [a;\infty)$ is CJ-measurable.
So, since $(-\infty;b) \cap [a;\infty) = [a;b)$, we get: $[a;b)$ is CJ-measurable. □

THEOREM 5.1.46. *Let $a, b \in \mathbb{R}$. Then $(a;b]$ is CJ-measurable.*

Proof. Since $(-\infty;b]$ and $(a;\infty)$ are CJ-measurable,
we get: $(-\infty;b] \cap (a;\infty)$ is CJ-measurable.
So, since $(-\infty;b] \cap (a;\infty) = (a;b]$, we get: $(a;b]$ is CJ-measurable. □

THEOREM 5.1.47. *Let $a, b \in \mathbb{R}$. Then $(a;b)$ is CJ-measurable.*

Proof. Since $(-\infty;b)$ and $(a;\infty)$ are CJ-measurable,
we get: $(-\infty;b) \cap (a;\infty)$ is CJ-measurable.
So, since $(-\infty;b) \cap (a;\infty) = (a;b)$, we get: $(a;b)$ is CJ-measurable. □

Since $\varnothing = (0;0)$, we get: $\varnothing$ is CJ-measurable.
So, since $\varnothing^c = \mathbb{R}$, we get: $\mathbb{R}$ is CJ-measurable.
We have now proved: every interval is CJ-measurable.
A set of sets is called a **ring of sets** if it is closed under
pairwise intersection, pairwise union and set subtraction.
A set is **constructible** if it is in the ring of sets generated by intervals.
Sets of use in the natural sciences are typically constructible,
and we now know: every constructible set is CJ-measurable.

THEOREM 5.1.48. *Let $a \in \mathbb{R}$, $b \geqslant a$. Then:*

$$\mathrm{JO}_{[a;b]} = \mathrm{JO}_{[a;b)} = \mathrm{JO}_{(a;b]} = \mathrm{JO}_{(a;b)} = b - a.$$

Idea of proof: We already proved $\mathrm{JO}_{[a;b]} = b - a$.
Want: $\mathrm{JO}_{[a;b)} = \mathrm{JO}_{(a;b]} = \mathrm{JO}_{(a;b)} = b - a$.
We have $\mathrm{JO}_{\{b\}} = \mathrm{JO}_{[b;b]} = b - b = 0$ and $[a;b) = [a;b]\backslash\{b\}$.
By Theorem 5.1.36, $\mathrm{JO}_{[a;b)} = \mathrm{JO}_{[a;b]} - \mathrm{JO}_{\{b\}}$.
Then $\mathrm{JO}_{[a;b)} = \mathrm{JO}_{[a;b]} - \mathrm{JO}_{\{b\}} = b - a + 0 = b - a$.
Want: $\mathrm{JO}_{(a;b]} = \mathrm{JO}_{(a;b)} = b - a$.
The rest is left as unassigned Homework. QED

5.2. Jordan integration.

Addition is associative, and so we have:

THEOREM 5.2.1. *Let $\mathcal{P}, F$ be finite sets, $\alpha : \mathcal{P} \to \mathbb{R}$, $\beta : \mathcal{P} \to F$.*

$$\textit{Then} \quad \sum_{y \in F} \left(\sum_{P \in \beta^*\{y\}} \alpha_P \right) = \sum_{P \in \mathcal{P}} \alpha_P.$$

In Theorem 5.2.1, for any $y \in F$,
the sum $\displaystyle\sum_{P \in \beta^*\{y\}} \alpha_P$ is the "fiber sum" of α over y.
Also, $\displaystyle\sum_{P \in \mathcal{P}} \alpha_P$ is the "total sum" of α.
Then, informally, Theorem 5.2.1 asserts:
The sum of the fiber sums is equal to the total sum.
That is, the total sum can be "grouped" into fiber sums, and,
by the associative law, the sum is unchanged by that grouping.

DEFINITION 5.2.2. *Let s be a function. Let $\mathcal{Q}$ be a partition of $\mathbb{D}_s$. By s is* **subordinate** *to $\mathcal{Q}$, we mean:*
$\forall Q \in \mathcal{Q}$, $s|Q$ is constant.

THEOREM 5.2.3. *Let s be a function.*
Let $\mathcal{P}$ and $\mathcal{Q}$ be partitions of $\mathbb{D}_s$.
Assume s is subordinate to $\mathcal{Q}$ and $\mathcal{P}$ is a refinement of $\mathcal{Q}$.
Then s is subordinate to $\mathcal{P}$.

DEFINITION 5.2.4. *Let $s : \mathbb{R} \dashrightarrow \mathbb{R}$.*
By s is **J-simple**, *we mean:*
$\exists$finite partition $\mathcal{Q}$ of $\mathbb{D}_s$ s.t. s is subordinate to $\mathcal{Q}$
and s.t. $\forall Q \in Q$, Q is CJ-measurable.

DEFINITION 5.2.5. *Let $s : \mathbb{R} \dashrightarrow \mathbb{R}$.*
By s is **L-simple**, *we mean:*
$\exists$countable partition $\mathcal{Q}$ of $\mathbb{D}_s$ s.t. s is subordinate to $\mathcal{Q}$
and s.t. $\forall Q \in Q$, Q is CL-measurable.

We read "J-simple" as "Jordan simple".
We read "L-simple" as "Lebesgue simple".
Any J-simple function $\mathbb{R} \dashrightarrow \mathbb{R}$ is L-simple.

THEOREM 5.2.6. *Let $s : \mathbb{R} \dashrightarrow \mathbb{R}$ be J-simple. Then $\mathbb{I}_s$ is finite.*

THEOREM 5.2.7. *Let $s : \mathbb{R} \dashrightarrow \mathbb{R}$ be L-simple. Then $\mathbb{I}_s$ is countable.*

We now define
the **simple integral** of a simple function s,
denoted SI_s, as follows:

DEFINITION 5.2.8. *Let $a \in \mathbb{R}$, $b \geqslant a$, $s : [a;b] \to \mathbb{R}$.*
Assume s is J-simple. Then:

$$\boxed{\mathrm{JI}_s} := \sum_{y \in \mathbb{I}_s} (\mathrm{JO}_{s^*\{y\}} \cdot y).$$

THEOREM 5.2.9. *Let $a \in \mathbb{R}$, $b \geqslant a$, $s : [a;b] \to \mathbb{R}$.*
Let $\mathcal{P}$ be a finite partition of $[a;b]$.
Assume: s is subordinate to $\mathcal{P}$.
Assume: $\forall P \in \mathcal{P}$, P is CJ-measurable and nonempty.
Define $\beta : \mathcal{P} \to \mathbb{I}_s$ by: $\forall P \in \mathcal{P}$, $\beta_P = \mathrm{UE}_{s_*P}$.
Then: $\forall y \in \mathbb{I}_s$, $JO_{s^*\{y\}} = \displaystyle\sum_{P \in \beta^*\{y\}} \mathrm{JO}_P$.

Idea of Proof: Given $y \in \mathbb{I}_s$. **Want:** $JO_{s^*\{y\}} = \displaystyle\sum_{P \in \beta^*\{y\}} \mathrm{JO}_P$.

We leave it as an exercise to show that $\beta^*\{y\}$ is a partition of $s^*\{y\}$.

Then, by finite additivity of Jordan measure,
we get the desired result: $JO_{s^*\{y\}} = \sum_{P\in\beta^*\{y\}} \mathrm{JO}_P$. QED

THEOREM 5.2.10. *Let* $a \in \mathbb{R}$, $b \geqslant a$, $s : [a; b] \to \mathbb{R}$.
Let $\mathcal{P}$ *be a finite partition of* $[a; b]$.
Assume: s *is subordinate to* $\mathcal{P}$.
Assume: $\forall P \in \mathcal{P}$, P *is CJ-measurable and nonempty.*
Then: $\mathrm{JI}_s = \sum_{P\in\mathcal{P}} (\mathrm{JO}_P \cdot \mathrm{UE}_{s_*P})$.

Proof. Define $\alpha : \mathcal{P} \to \mathbb{R}$ by: $\forall P \in \mathcal{P}$, $\alpha_P = \mathrm{JO}_P \cdot \mathrm{UE}_{s_*P}$.
Want: $\mathrm{JI}_s = \sum_{P\in\mathcal{P}} \alpha_P$.
Define $\beta : \mathcal{P} \to \mathbb{I}_s$ by: $\forall P \in \mathcal{P}$, $\beta_P = \mathrm{UE}_{s_*P}$.
By Theorem 5.2.9,

$$\forall y \in \mathbb{I}_s, \quad JO_{s^*\{y\}} = \sum_{P\in\beta^*\{y\}} \mathrm{JO}_P.$$

$$\text{Then } \mathrm{JI}_s = \sum_{y\in\mathbb{I}_s} \mathrm{JO}_{s^*\{y\}} \cdot y = \sum_{y\in\mathbb{I}_s} \left(\sum_{P\in\beta^*\{y\}} JO_P \right) \cdot y = \sum_{y\in\mathbb{I}_s} \sum_{P\in\beta^*\{y\}} (JO_P \cdot y).$$

We have: $\forall y \in \mathbb{I}_s$, $\forall P \in \beta^*\{y\}$, $\beta_P \in \{y\}$, so $\beta_P = y$.
Then $\mathrm{JI}_s = \sum_{y\in\mathbb{I}_s} \sum_{P\in\beta^*\{y\}} (JO_P \cdot \beta_P)$.
By definition of β, we have: $\forall P \in \mathcal{P}$, $\beta_P = \mathrm{UE}_{s_*P}$.
Then $\mathrm{JI}_s = \sum_{y\in\mathbb{I}_s} \sum_{P\in\beta^*\{y\}} (JO_P \cdot \mathrm{UE}_{s_*P})$.
By definition of α, we have: $\forall P \in \mathcal{P}$, $\alpha_P = JO_P \cdot \mathrm{UE}_{s_*P}$.
Then $\mathrm{JI}_s = \sum_{y\in\mathbb{I}_s} \sum_{P\in\beta^*\{y\}} \alpha_P$.
Then, by Theorem 5.2.1, we get: $\mathrm{JI}_s = \sum_{P\in\mathcal{P}} \alpha_P$. □

THEOREM 5.2.11. *Let* $a \in \mathbb{R}$, $b > a$.
Let $s, t : [a; b] \to \mathbb{R}$ *both be J-simple.*
Then: $s + t$ *is J-simple* *and*

$$\mathrm{JI}_{s+t} = \mathrm{JI}_s + \mathrm{JI}_t.$$

Proof. Choose a finite partition $\mathcal{P}$ of $[a; b]$ s.t.
$\forall P \in \mathcal{P}$, P is CJ-measurable and

s is subordinate to $\mathcal{P}$.
Choose a finite partition $\mathcal{Q}$ of $[a;b]$ s.t.
$\forall Q \in \mathcal{Q}$, Q is CJ-measurable and
t is subordinate to $\mathcal{Q}$.
Let $\mathcal{R} := \{P \cap Q \,|\, P \in \mathcal{P},\, Q \in \mathcal{Q}\}_{\varnothing}^{\times}$.
Then $\mathcal{R}$ is a finite partition of $[a;b]$ s.t.
$\forall R \in \mathcal{R}$, R is nonempty and CJ-measurable and
both s and t are subordinate to $\mathcal{P}$.
Since both s and t are subordinate to $\mathcal{R}$,
it follows that $s+t$ is subordinate to $\mathcal{R}$.
Then $s+t$ is J-simple. **Want:** $\mathrm{JI}_{s+t} = \mathrm{JI}_s + \mathrm{JI}_t$.
We have: $\forall R \in \mathcal{R}$, $\mathrm{UE}_{(s+t)_*R} = \mathrm{UE}_{s_*R} + \mathrm{UE}_{t_*R}$.
Then: $\forall R \in \mathcal{R}$, $\mathrm{JO}_R \cdot \mathrm{UE}_{(s+t)_*R} = \mathrm{JO}_R \cdot \mathrm{UE}_{s_*R} + \mathrm{JO}_R \cdot \mathrm{UE}_{t_*R}$.
Then:
$$\sum_{R \in \mathcal{R}} (\mathrm{JO}_R \cdot \mathrm{UE}_{(s+t)_*R}) = \left[\sum_{R \in \mathcal{R}} (\mathrm{JO}_R \cdot \mathrm{UE}_{s_*R})\right] + \left[\sum_{R \in \mathcal{R}} (\mathrm{JO}_R \cdot \mathrm{UE}_{t_*R})\right].$$
Then, by Theorem 5.2.10, we get: $\mathrm{JI}_{s+t} = \mathrm{JI}_s + \mathrm{JI}_t$. □

THEOREM 5.2.12. *Let $a \in \mathbb{R}$, $b > a$, $c \in \mathbb{R}$.*
Let $s : [a;b] \to \mathbb{R}$ be J-simple.
Then: $c \cdot s$ is J-simple and $\mathrm{JI}_{c \cdot s} = c \cdot \mathrm{JI}_s$.

Proof. Unassigned HW. □

The preceding two theorems can be summarized by saying:
Jordan simple integration, JI, is algebraically linear.
The next theorem says:
Jordan simple integration, JI, is monotonic.

THEOREM 5.2.13. *Let $a \in \mathbb{R}$, $b > a$, $I := [a;b]$.*
Let $s, t : [a;b] \to \mathbb{R}$ both be J-simple.
Assume: on I, $s \leqslant t$. Then $\mathrm{JI}_s \leqslant \mathrm{JI}_t$.

Proof. By algebraic linearity of JI, we conclude:
$t - s$ is J-simple and $\mathrm{JI}_{t-s} = \mathrm{JI}_t - \mathrm{JI}_s$.
On I, we have $t - s \geqslant 0$. So, since $\mathbb{D}_{t-s} = I$, we see: $\mathbb{I}_{t-s} \geqslant 0$.
Then, by definition of JI_{t-s}, we get: $\mathrm{JI}_{t-s} \geqslant 0$.
Then $\mathrm{JI}_t - \mathrm{JI}_s = \mathrm{JI}_{t-s} \geqslant 0$, so $\mathrm{JI}_t \geqslant \mathrm{JI}_s$, so $\mathrm{JI}_s \leqslant \mathrm{JI}_t$. □

The **upper simple functions** and **lower simple functions**
for f on I are

those that majorize f and those majorized by f on I, as follows:

DEFINITION 5.2.14. *Let $a \in \mathbb{R}$, $b \geqslant a$, $I := [a;b]$.*
Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$. Assume $I \subseteq \mathbb{D}_f$.
Then $\boxed{\mathrm{US}_I^f} := \{$ *J-simple* $t : I \to \mathbb{R} \,|\,$ *on* I, $f \leqslant t \}$.
Also, $\boxed{\mathrm{LS}_I^f} := \{$ *J-simple* $s : I \to \mathbb{R} \,|\,$ *on* I, $s \leqslant f \}$.

The **upper simple integrals** and **lower simple integrals** for f on I are the Jordan integrals of the upper simple functions and lower simple functions for f on I, as follows:

DEFINITION 5.2.15. *Let $a \in \mathbb{R}$, $b \geqslant a$, $I := [a;b]$.*
Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$. Assume $I \subseteq \mathbb{D}_f$.
Then $\boxed{USI_I^f} := \{ \mathrm{JI}_t \,|\, t \in \mathrm{US}_I^f \}$.
Also, $\boxed{LSI_I^f} := \{ \mathrm{JI}_s \,|\, s \in \mathrm{LS}_I^f \}$.

The **Jordan upper integral** and **Jordan lower integral** for f on I are the infimum and supremum of the upper simple integrals and lower simple integrals for f on I, as follows:

DEFINITION 5.2.16. *Let $a \in \mathbb{R}$, $b \geqslant a$, $I := [a;b]$.*
Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$. Assume $I \subseteq \mathbb{D}_f$.
Then $\boxed{\mathrm{JU}_I^f} := \inf \mathrm{USI}_I^f$ *and* $\boxed{\mathrm{JL}_I^f} := \sup \mathrm{LSI}_I^f$.

THEOREM 5.2.17. *Let $A, B \subseteq \mathbb{R}$.*
Assume: $\forall a \in A, \forall b \in B, \quad a \leqslant b$.
Then: $\sup A \leqslant \inf B$.

Proof. We have: $\forall a \in A$, $a \leqslant B$, hence $a \leqslant \inf B$.
Then: $A \leqslant \inf B$, and so $\sup A \leqslant \inf B$. □

THEOREM 5.2.18. *Let $a \in \mathbb{R}$, $b \geqslant a$, $I := [a;b]$.*
Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$. Assume $I \subseteq \mathbb{D}_f$.
Then $\mathrm{JL}_I^f \leqslant \mathrm{JU}_I^f$.

Proof. **Want:** $\sup \mathrm{LSI}_I^f \leqslant \inf \mathrm{USI}_I^f$.
By Theorem 5.2.17, **it suffices to show:**
$\forall a \in \mathrm{LSI}_I^f, \forall b \in \mathrm{USI}_I^f, \quad a \leqslant b$.

Given $a \in \mathrm{LSI}_I^f$, $b \in \mathrm{USI}_I^f$. Want $a \leqslant b$.
Since $a \in \mathrm{LSI}_I^f$, choose $s \in \mathrm{LS}_I^f$ s.t. $a = \mathrm{JI}_s$.
Since $b \in \mathrm{USI}_I^f$, choose $t \in \mathrm{US}_I^f$ s.t. $b = \mathrm{JI}_t$.
Since $s \in \mathrm{LS}_I^f$, we have: on I, $s \leqslant f$.
Since $t \in \mathrm{US}_I^f$, we have: on I, $f \leqslant t$.
Then on I, $s \leqslant t$.
So, by monotonicity of Jordan simple integration, $\mathrm{JI}_s \leqslant \mathrm{JI}_t$.
Then $a = \mathrm{JI}_s \leqslant \mathrm{JI}_t = b$. □

The next theorem says that JL and JU are monontonic:

THEOREM 5.2.19. *Let* $a \in \mathbb{R}$, $b \geqslant a$, $I := [a; b]$, $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$.
Assume: *on* I, $f \leqslant g$.
Thene: $\mathrm{JL}_I^f \leqslant \mathrm{JL}_I^g$ *and* $\mathrm{JU}_I^f \leqslant \mathrm{JU}_I^g$.

Proof. Monotonicity of JL is Problem 5 on the Final exam.
Monotonicity of JU is Unassigned HW. □

When the Jordan upper and lower integrals of f on I agree,
that common integral is called the **Jordan integral** of f on I:

DEFINITION 5.2.20. *Let* $a \in \mathbb{R}$, $b \geqslant a$, $I := [a; b]$.
Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$. *Assume* $I \subseteq \mathbb{D}_f$.
Then: $\boxed{\int_I f} := \mathrm{UE}\,\{\,\mathrm{JL}_I^f\,,\,\mathrm{JU}_I^f\,\}$.

5.3. **Jordan integrability of continuous functions.**

We next show that continuity implies Jordan integrability.
This is one of two deep theorems we will cover in Jordan integration,
the other being the Fundamental Theorem of Calculus,
which will be proved later.

THEOREM 5.3.1. *Let* $a \in \mathbb{R}$, $b > a$, $I := [a; b]$.
Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$. *Assume* f *is continuous on* I.
Then: $\int_I f \neq ☹$.

Proof. **Want:** $\mathrm{JL}_I^f = \mathrm{JU}_I^f$. By Theorem 5.2.18, $\mathrm{JL}_I^f \leqslant \mathrm{JU}_I^f$.
It suffices to show: $\mathrm{JU}_I^f \leqslant \mathrm{JL}_I^f$. **Want:** $\forall \varepsilon > 0$, $\mathrm{JU}_I^f \leqslant \mathrm{JL}_I^f + \varepsilon$.
Given $\varepsilon > 0$. **Want:** $\mathrm{JU}_I^f \leqslant \mathrm{JL}_I^f + \varepsilon$.
Since $I = [a; b]$, we see that I is closed and bounded.
Then I is compact.

Since f is continuous on I, it follows that $f|I$ is continuous.
Let $g := f|I$. Then g is continuous.
So, since I is compact, by Theorem 3.12.1, g is uniformly continuous.
By hypothesis $b > a$. Then $b - a > 0$. Then $\varepsilon/(b-a) > 0$.
By uniform continuity of g, choose $\delta > 0$ s.t., $\forall w, x \in \mathbb{D}_g$,

$$(\,|w-x| < \delta\,) \Rightarrow (\,|g_w - g_x| < \varepsilon/(b-a)\,).$$

By the AP, choose $N \in \mathbb{N}$ s.t. $N > (b-a)/\delta$. Then $(b-a)/N < \delta$.
Let $\gamma := (b-a)/N$. Then $\gamma < \delta$.
We have $a + N \cdot \gamma = a + (b-a) = b$.
For all $j \in [0..N]$, let $x_j := a + j \cdot \gamma$.
Then $x_0 = a$ and $x_N = b$.
Also, $\forall j \in [1..N]$, we have $x_j - x_{j-1} = \gamma$.
For all $j \in [1..N]$, let $K_j := [x_{j-1}; x_j]$.
We have: $\forall j \in [1..N]$, K_j is closed and bounded, hence compact.
Let $Q_1 := [x_0; x_1]$. Also, $\forall j \in [2..N]$, let $Q_j := (x_{j-1}; x_j]$.
By hypothesis, $I = [a; b]$. Then $\{Q_1, \ldots, Q_N\}$ is a partition of I.
Moreover, $\forall j \in [1..N]$, Q_j is an interval, so Q_j is CJ-measurable.
So, by finite additivity of Jordan measure, $\sum_{j=1}^{N} \mathrm{JO}_{Q_j} = \mathrm{JO}_I$.
Also, $\forall j \in [1..N]$, we have: $\mathrm{JO}_{Q_j} = x_j - x_{j-1}$.
Then, $\forall j \in [1..N]$, we have: $\mathrm{JO}_{Q_j} = \gamma$.
We have: $\forall j \in [1..N]$, $\mathrm{Cl}\, Q_j = [x_{j-1}; x_j] = K_j$.
For all $j \in [1..N]$, let $y_j := \min f_* K_j$.
For all $j \in [1..N]$, by the EVT, $y_j \neq ☹$, so choose $u_j \in K_j$ s.t. $f_{u_j} = y_j$.
For all $j \in [1..N]$, let $z_j := \max f_* K_j$.
For all $j \in [1..N]$, by the EVT, $z_j \neq ☹$, so choose $v_j \in K_j$ s.t. $f_{v_j} = z_j$.

Claim 1: $\forall j \in [1..N]$, $z_j - y_j < \varepsilon/(b-a)$.

Proof of Claim 1: Given $j \in [1..N]$. **Want:** $z_j - y_j < \varepsilon/(b-a)$.
We have: $u_j, v_j \in K_j = [x_{j-1}; x_j]$. Then $u_j, v_j \in [x_0; x_N] = [a; b] = I$.
Then $u_j, v_j \in [x_{j-1}; x_j]$ and $u_j, v_j \in I$.
Since $g = f|I$ and since $u_j, v_j \in I$, we get $g_{u_j} = f_{u_j}$ and $g_{v_j} = f_{v_j}$.
Since $y_j = \min f_* K_j \leqslant \max f_* K_j = z_j$, we get $|y_j - z_j| = z_j - y_j$.
Want: $|y_j - z_j| < \varepsilon/(b-a)$.
Since $u_j, v_j \in [x_{j-1}; x_j]$, we get: $|u_j - v_j| \leqslant x_j - x_{j-1}$.
Then $|u_j - v_j| \leqslant x_j - x_{j-1} = \gamma < \delta$.
Also, $u_j, v_j \in I = \mathbb{D}_{f|I} = \mathbb{D}_g$.
Then, by choice of δ, we have: $|g_{u_j} - g_{v_j}| < \varepsilon/(b-a)$.

So, since $y_j = f_{u_j} = g_{u_j}$ and $z_j = f_{v_j} = g_{u_j}$, we get: $|y_j - z_j| < \varepsilon/(b-a)$.
End of proof of Claim 1.

Define $s, t : I \to \mathbb{R}$ by: $\forall j \in [1..N]$, $\forall x \in Q_j$,

$$s_x = y_j \quad \text{and} \quad t_x = z_j.$$

Then s and t are both subordinate to $\{Q_1, \ldots, Q_N\}$.
Recall: $\forall j \in [1..N]$, Q_j is CJ-measurable.
Then s and t are both J-simple.
Also, $\mathrm{JI}_s = \sum_{j=1}^N (\mathrm{JO}_{Q_j} \cdot y_j)$ and $\mathrm{JI}_t = \sum_{j=1}^N (\mathrm{JO}_{Q_j} \cdot z_j)$.
Then $\mathrm{JI}_t - \mathrm{JI}_s = \sum_{j=1}^N (\mathrm{JO}_{Q_j}) \cdot (z_j - y_j)$.
Then, by Claim 1, $\mathrm{JI}_t - \mathrm{JI}_s \leqslant \sum_{j=1}^N ((\mathrm{JO}_{Q_j}) \cdot (\varepsilon/(b-a)))$.
So, since $\sum_{j=1}^N \mathrm{JO}_{Q_j} = \mathrm{JO}_I$, we conclude:
$\mathrm{JI}_t - \mathrm{JI}_s \leqslant (\mathrm{JO}_I) \cdot (\varepsilon/(b-a))$.
So, since $\mathrm{JO}_I = \mathrm{JO}_{[a;b]} = b - a$, we get: $\mathrm{JI}_t - \mathrm{JI}_s \leqslant \varepsilon$.

Claim 2: On I, $s \leqslant f \leqslant t$.

Proof of Claim 2: **Want:** $\forall x \in I$, $s_x \leqslant f_x \leqslant t_x$.
Given $x \in I$. **Want:** $s_x \leqslant f_x \leqslant t_x$.
Since $\{Q_1, \ldots, Q_N\}$ is a partition of I and since $x \in I$,
choose $j \in [1..N]$ s.t. $x \in Q_j$.
Then, by definition of s and t, we get $s_x = y_j$ and $t_x = z_j$.
We have $x \in Q_j \subseteq \mathrm{Cl}\, Q_j = K_j$, so $x \in K_j$.
Since $x \in K_j$, we get $\min f_* K_j \leqslant f_x$.
Then $s_x = y_j = \min f_* K_j \leqslant f_x$, so $s_x \leqslant f_x$. **Want:** $f_x \leqslant t_x$.
Since $x \in K_j$, we get $f_x \leqslant \max f_* K_j$.
Then $f_x \leqslant \max f_* K_j = z_j = t_x$, so $f_x \leqslant t_x$.
End of proof of Claim 2.

By Claim 2, we have: on I, $s \leqslant f$.
So, since s is J-simple, we get $s \in \mathrm{LS}_I^f$.
Then $\mathrm{JI}_s \in \mathrm{LSI}_I^f$, so $\mathrm{JI}_s \leqslant \sup \mathrm{LSI}_I^f$.
Then $\mathrm{JI}_s \leqslant \sup \mathrm{LSI}_I^f = \mathrm{JL}_I^f$.
So, since $\mathrm{JI}_t - \mathrm{JI}_s \leqslant \varepsilon$, we get:
$\mathrm{JI}_s + (\mathrm{JI}_t - \mathrm{JI}_s) \leqslant \mathrm{JL}_I^f + \varepsilon$.
Then $\mathrm{JI}_t \leqslant \mathrm{JL}_I^f + \varepsilon$.
By Claim 2, we have: on I, $f \leqslant t$.

So, since t is J-simple, we get $t \in \mathrm{US}_I^f$.
Then $\mathrm{JI}_t \in \mathrm{USI}_I^f$, so $\mathrm{JI}_t \geqslant \inf \mathrm{USI}_I^f$.
Then $\mathrm{JI}_t \geqslant \inf \mathrm{USI}_I^f = \mathrm{JU}_I^f$. Then $\mathrm{JU}_I^f \leqslant \mathrm{JI}_t$.
So, since $\mathrm{JI}_t \leqslant \mathrm{JL}_I^f + \varepsilon$, we get: $\mathrm{JU}_I^f \leqslant \mathrm{JL}_I^f + \varepsilon$. □

5.4. **Cocycle formulas.**

DEFINITION 5.4.1. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a, b \in \mathbb{R}$, $I := [a|b]$.

$$\textit{Assume } I \subseteq \mathbb{D}_f. \qquad \textit{Then } \boxed{\int_a^b f} := \begin{cases} \displaystyle\int_I f, & \textit{if } a < b \\ 0, & \textit{if } a = b \\ \displaystyle -\int_I f, & \textit{if } a > b \end{cases}.$$

THEOREM 5.4.2. *Let* $a \in \mathbb{R}$, $b \geqslant a$, $f : \mathbb{R} \dashrightarrow \mathbb{R}$.
Assume $[a;b] \subseteq \mathbb{D}_f$. *Let* $s, t, u \in [a;b]$. *Assume* $s \leqslant t \leqslant u$.

$$\textit{Then } \int_s^u f =^* \left(\int_s^t f\right) + \left(\int_t^u f\right).$$

Proof. Unassigned HW. □

The assumption that $s \leqslant t \leqslant u$ can be relaxed:

THEOREM 5.4.3. *Let* $a \in \mathbb{R}$, $b \geqslant a$, $f : \mathbb{R} \dashrightarrow \mathbb{R}$.
Assume $[a;b] \subseteq \mathbb{D}_f$. *Let* $s, t, u \in [a;b]$. *Assume* $s \leqslant u$.

$$\textit{Then } \int_s^u f =^* \left(\int_s^t f\right) + \left(\int_t^u f\right).$$

Proof. Unassigned HW. □

The assumption that $s \leqslant u$ can be removed:

THEOREM 5.4.4. *Let* $a \in \mathbb{R}$, $b \geqslant a$, $f : \mathbb{R} \dashrightarrow \mathbb{R}$.
Assume $[a;b] \subseteq \mathbb{D}_f$. *Let* $s, t, u \in [a;b]$.

$$\textit{Then } \int_s^u f =^* \left(\int_s^t f\right) + \left(\int_t^u f\right).$$

Proof. Unassigned HW. □

THEOREM 5.4.5. *Let* $a \in \mathbb{R}$, $b \geqslant a$, $f : \mathbb{R} \dashrightarrow \mathbb{R}$.
Assume $[a;b] \subseteq \mathbb{D}_f^{\mathrm{con}}$. *Let* $s, t, u \in [a;b]$.

$$\textit{Then } \int_s^u f = \left(\int_s^t f\right) + \left(\int_t^u f\right).$$

Proof. By Theorem 5.4.4, we have:

$$\int_s^u f \;=^*\; \left(\int_s^t f\right) + \left(\int_t^u f\right).$$

By Theorem 5.3.1, $\displaystyle\int_s^t f \;\neq\; ☹ \;\neq\; \int_t^u f$.

Then $\displaystyle\int_s^u f \;=\; \left(\int_s^t f\right) + \left(\int_t^u f\right)$. □

5.5. **The Fundamental Theorem of Calculus.**

THEOREM 5.5.1. *Let $a, b, y \in \mathbb{R}$. Then:* $\displaystyle\int_a^b C_y^{\mathbb{R}} \;=\; (b-a)\cdot y$.

Proof. Unassigned HW. □

DEFINITION 5.5.2. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $y \in \mathbb{R}$. Then* $\boxed{f - y} := f - C_y^{\mathbb{R}}$.

THEOREM 5.5.3. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a, y \in \mathbb{R}$, $b \geqslant a$.*
Assume $[a;b] \subseteq \mathbb{D}_f^{\text{con}}$. *Then* $\displaystyle\int_a^b (f-y) \;=\; \left(\int_a^b f\right) - y\cdot(b-a)$.

Proof. $\displaystyle\int_a^b (f-y) \;=\; \int_a^b (f - C_y^{\mathbb{R}}) \;=\; \left(\int_a^b f\right) - y\cdot(b-a)$. □

THEOREM 5.5.4. *Let $a, \in \mathbb{R}$, $b \geqslant a$, $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$.*
Assume: $(\,[a;b] \subseteq \mathbb{D}_f^{\text{con}} \cap \mathbb{D}_g^{\text{con}}\,)\;\&\;(\,\textit{on } [a;b],\; f \leqslant g\,)$.

Then: $\displaystyle\int_a^b f \;\leqslant\; \int_a^b g$.

Proof. Let $I := [a;b]$.

By Theorem 5.3.1, $\displaystyle\int_a^b f \;\neq\; ☹ \;\neq\; \int_a^b g$.

Then: $\displaystyle\int_a^b f \;=\; \mathrm{JL}_I\, f$ and $\displaystyle\int_a^b g \;=\; \mathrm{JL}_I\, g$.

By Theorem 5.2.19, $\mathrm{JL}_I\, f \;\leqslant\; \mathrm{JL}_I\, g$.

Then $\displaystyle\int_a^b f \;=\; \mathrm{JL}_I\, f \;\leqslant\; \mathrm{JL}_I\, g \;=\; \int_a^b g$. □

The following is the **Fundamental Theorem of Calculus**:

THEOREM 5.5.5. *Let $a \in \mathbb{R}$, $b \geqslant a$, $f : \mathbb{R} \dashrightarrow \mathbb{R}$.*
Assume: $[a;b] \subseteq \mathbb{D}_f^{\text{con}}$.

Define $g : [a;b] \to \mathbb{R}$ *by:* $\quad \forall x \in [a;b],\ g_x = \int_a^x f.$

Then: $\quad \forall x \in (a;b), \quad g'_x = f_x.$

Proof. Given $x \in (a;b)$. **Want:** $g'_x = f_x$.
We have $[D_x g] = g'_x$. **Want:** $[D_x g] = f_x$.
Let $y := f_x$. **Want:** $[D_x g] = y$.
Let $L := y \cdot (\bullet)$. Then $L \in \mathcal{L}$ and $[L] = L_1 = y \cdot 1 = y$.
Want: $[D_x g] = [L]$. **Want:** $D_x g = L$.
We have $\mathrm{UE}(\mathrm{LINS}_x g) = D_x g$. **Want:** $\mathrm{UE}(\mathrm{LINS}_x g) = L$.
Want: $\mathrm{LINS}_x g = \{L\}$.
By uniqueness of linearlization, **it suffices to show:** $L \in \mathrm{LINS}_x g$.
Want: $g_x^{\mathbb{T}} - L \in \mathcal{o}(1)$.
Want: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall h \in \mathbb{R}$,

$$(\,|h| < \delta\,) \Rightarrow (\,|(g_x^{\mathbb{T}} - L)_h| \leqslant \varepsilon \cdot |h|^1\,).$$

Given $\varepsilon > 0$. **Want:** $\exists \delta > 0$ s.t., $\forall h \in \mathbb{R}$,

$$(\,|h| < \delta\,) \Rightarrow (\,|(g_x^{\mathbb{T}} - L)_h| \leqslant \varepsilon \cdot |h|^1\,).$$

Since $x \in (a;b)$ and since $(a;b)$ is open,
choose $\lambda > 0$ s.t. $B(x,\lambda) \subseteq (a;b)$.
By hypothesis, $[a;b] \subseteq \mathbb{D}_f^{\mathrm{con}}$.
Since $x \in (a;b) \subseteq [a;b] \subseteq \mathbb{D}_f^{\mathrm{con}}$, we get: f is continuous at x.
Then choose $\mu > 0$ s.t., $\forall w \in \mathbb{D}_f$,

$$(\,|w - x| < \mu\,) \Rightarrow (\,|f_w - f_x| < \varepsilon\,).$$

Let $\delta := \min\{\lambda, \mu\}$. Then $\delta \leqslant \lambda$ and $\delta \leqslant \mu$ and $\delta > 0$.
Want: $\forall h \in \mathbb{R}$, $\quad (\,|h| < \delta\,) \Rightarrow (\,|(g_x^{\mathbb{T}} - L)_h| \leqslant \varepsilon \cdot |h|^1\,)$.
Given $h \in \mathbb{R}$. Assume $|h| < \delta$. **Want:** $|(g_x^{\mathbb{T}} - L)_h| \leqslant \varepsilon \cdot |h|^1$.
Exactly one of the following is true:
(1) $h > 0$ or (2) $h = 0$ or (3) $h < 0$.

Case (1):
We have $|(x+h) - x| = |h| < \delta$, so $x + h \in B(x,\delta)$.
Then $x, x+h \in B(x,\delta)$.
Since $\delta \leqslant \lambda$ and $\delta \leqslant \mu$, it follows that:

$$B(x,\delta) \subseteq B(x,\lambda) \quad \text{and} \quad B(x,\delta) \subseteq B(x,\mu).$$

Then: $\quad x, x+h \in B(x,\lambda)$ and $x, x+h \in B(x,\mu)$.
We have $x, x+h \in B(x,\lambda) \subseteq (a;b) \subseteq [a;b]$, so $x, x+h \in [a;b]$.
Then $x, x+h \in [a;b] = \mathbb{D}_g$ and $x, x+h \in \mathbb{R} = \mathbb{D}_L$,
so $(g_x^{\mathbb{T}} - L)_h = g_{x+h} - g_x - L_h$.

We have $\displaystyle\int_a^{x+h} f \;=\; \left(\int_a^x f\right) + \left(\int_x^{x+h} f\right)$,

so $\displaystyle\left(\int_a^{x+h} f\right) - \left(\int_a^x f\right) \;=\; \int_x^{x+h} f$.

By Theorem 5.5.1, $\displaystyle\int_x^{x+h} y \;=\; ((x+h)-x)\cdot y$.

Then $\displaystyle\int_x^{x+h} y \;=\; ((x+h)-x)\cdot y \;=\; y\cdot h \;=\; (y\cdot(\bullet))_h \;=\; L_h$.

Then
$$\begin{aligned}(g_x^{\mathbb{T}} - L)_h \;&=\; g_{x+h} - g_x - L_h\\ &=\; \left(\int_a^{x+h} f\right) - \left(\int_a^x f\right) - \left(\int_x^{x+h} y\right)\\ &=\; \left(\int_x^{x+h} f\right) - \left(\int_x^{x+h} y\right) \;=\; \int_x^{x+h}(f-y).\end{aligned}$$

Want: $\displaystyle\left|\int_x^{x+h}(f-y)\right| \;\leqslant\; \varepsilon\cdot|h|^1$.

Since $h>0$, we get $|h| = h$. Then $|h|^1 = |h| = h$.

Want: $\displaystyle\left|\int_x^{x+h}(f-y)\right| \;\leqslant\; \varepsilon\cdot h$.

Want: $\displaystyle -\varepsilon\cdot h \;\leqslant\; \int_x^{x+h}(f-y) \;\leqslant\; \varepsilon\cdot h$.

Want: $\displaystyle\int_x^{x+h}(-\varepsilon) \;\leqslant\; \int_x^{x+h}(f-y) \;\leqslant\; \int_x^{x+h}\varepsilon$.

Want: on $[x;x+h]$, $-\varepsilon \;\leqslant\; f-y \;\leqslant\; \varepsilon$.

Want: $\forall w\in[x;x+h]$, $-\varepsilon \;\leqslant\; (f-y)_w \;\leqslant\; \varepsilon$.

Given $w\in[x;x+h]$. **Want:** $-\varepsilon \;\leqslant\; (f-y)_w \;\leqslant\; \varepsilon$.

Want: $-\varepsilon \;\leqslant\; f_w - y \;\leqslant\; \varepsilon$. **Want:** $|f_w - y| \;\leqslant\; \varepsilon$.

Recall that $x, x+h\in B(x,\delta)$.

Since $B(x,\delta) = (x-\delta;x+\delta)$, we see that $B(x,\delta)$ is an interval.

Then $[x|x+h]\subseteq B(x,\delta)$.

Since $h>0$, we get $[x|x+h] = [x;x+h]$.

Then $w\in[x;x+h] = [x|x+h]\subseteq B(x,\delta)$, so $w\in B(x,\delta)$.

Recall: $B(x,\delta)\subseteq B(x,\lambda)$ and $B(x,\delta)\subseteq B(x,\mu)$.

By choice of λ, we have: $B(x,\lambda)\subseteq(a;b)$.

By assumption, $[a;b]\subseteq\mathbb{D}_f^{\mathrm{con}}$.

Since $w\in B(x,\delta)\subseteq B(x,\lambda)\subseteq(a;b)\subseteq[a;b]\subseteq\mathbb{D}_f^{\mathrm{con}}\subseteq\mathbb{D}_f$,
we concude that $w\in\mathbb{D}_f$.

Since $w\in B(x,\delta)\subseteq B(x,\mu)$, we get: $|w-x|<\mu$.

Then, by the choice of μ, we get: $|f_w - f_x|<\mu$.

So, since $y = f_x$, we get: $|f_w - y| < \mu$.
End of Case (1).

Case (2):
Unassigned HW. *End of Case (2).*

Case (3):
Unassigned HW. *End of Case (3).* $\square$

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