MATH 4603 Fall 2022, Final Exam
Handout date: Tuesday 13 December 2022
Due: Tuesday 20 December 2022 at 11:59pm
Instructor: Scot Adams

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SOLUTIONS

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NO COLLABORATION.

Hand-graded problems. Show work.

1. (20 pts) Show:
Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $k \in \mathbb{N}_0$. Assume: $(\, f_0 = 0\,) \,\&\, (\, f' \in \mathcal{o}(k)\,)$.
Then: $f \in \mathcal{o}(k+1)$.

Proof: **Want:** $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall h \in \mathbb{R}$,

$$(\, |h| < \delta \,) \Rightarrow (\, |f_h| \leqslant \varepsilon \cdot |h|^{k+1}\,).$$

Given $\varepsilon > 0$. **Want:** $\exists \delta > 0$ s.t., $\forall h \in \mathbb{R}$,

$$(\, |h| < \delta \,) \Rightarrow (\, |f_h| \leqslant \varepsilon \cdot |h|^{k+1}\,).$$

Since $f' \in \mathcal{o}(k)$, choose $\delta > 0$ s.t., $\forall h \in \mathbb{R}$,

$$(\, |h| < \delta \,) \Rightarrow (\, |f'_h| \leqslant \varepsilon \cdot |h|^{k}\,).$$

Then $(-\delta; \delta \subseteq \mathbb{D}'_f$. Also, $\delta > 0$.
Want: $\forall h \in \mathbb{R}$, $(\, |h| < \delta \,) \Rightarrow (\, |f_h| \leqslant \varepsilon \cdot |h|^{k+1}\,)$.
Given $h \in \mathbb{R}$. Assume: $|h| < \delta$. **Want:** $|f_h| \leqslant \varepsilon \cdot |h|^{k+1}$.
We have: $0, h \in [\, -|h|\, ; |h|\,]$.
So, since $[\, -|h|\, ; |h|\,]$ is an interval, we get: $[0|h] \subseteq [\, -|h|\, ; |h|\,]$.
Since $|h| < \delta$, we get: $[\, -|h|\, ; |h|\,] \subseteq (-\delta; \delta)$.
Since $[0|h] \subseteq [\, -|h|\, ; |h|\,] \subseteq (-\delta; \delta)$, we get: $[0|h] \subseteq (-\delta; \delta)$.
Since $[0|h] \subseteq (-\delta; \delta) \subseteq \mathbb{D}'_f$, by the quotient-free MVT,

choose $c \in [0|h]$ s.t. $f_h - f_0 = f'_c \cdot (h - 0)$.

By hypothesis, $f_0 = 0$. Then $f_h = f'_c \cdot h$. Then $|f_h| = |f'_c| \cdot |h|$.
Since $c \in [0|h] \subseteq [\, -|h|\, ; |h|\,]$, we get: $|c| \leqslant |h|$.
Since $|c| \leqslant |h| < \delta$, by choice of δ, we have: $|f'_c| \leqslant \varepsilon \cdot |c|^k$.
Then $|f_h| = |f'_c| \cdot |h| \leqslant \varepsilon \cdot |c|^k \cdot |h| \leqslant \varepsilon \cdot |h|^k \cdot |h| = \varepsilon \cdot |h|^{k+1}$. QED

2. (20 pts) Show:
Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$. Assume $f_0 = f'_0 = f''_0 = 0$. Then: $f \in \mathcal{O}(2)$.

Proof: Let $g := f'$. Then $g_0 = f'_0 = 0$ and $g'_0 = f''_0 = 0$.
Since $g_0 = g'_0 = 0$, by the First-Order Vanishing Taylor Theorem,
we get: $g \in \mathcal{O}(1)$.
Since $f_0 = 0$ and since $f' = g \in \mathcal{O}(1)$, by Problem 1,
we get: $f \in \mathcal{O}(2)$. QED

3. (20 pts) Show:
Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{D}''_f$. Assume: $(\, f'_a = 0\,) \,\&\, (\, f''_a > 0\,)$.
Then: f has a local strict-minimum at a.

Proof: **Want:** $f^{\mathbb{T}}_a$ has a local strict-minimum at 0.
Let $L = D_a f$. Then $L = f'_a \cdot (\bullet) = 0 \cdot (\bullet) = \mathbf{0}$, so $L = \mathbf{0}$.
Let $\varepsilon := f''_a/4$ and $Q := D^2_a f$.
Then $Q = f''_a \cdot (\bullet)^2/2 = 2 \cdot \varepsilon \cdot (\bullet)^2$, so $Q = 2 \cdot \varepsilon \cdot (\bullet)^2$.
Let $R := f^{\mathbb{T}}_a - L - Q$.
By the Second-Order Taylor Theorem, we get: $R \in \mathcal{O}(2)$.
Let $g := f^{\mathbb{T}}_a$. **Want:** g has a local strict-minimum at 0.
Want: $\exists B \in \mathcal{B}(0)$ s.t., $\forall h \in B^\times_0$, $g_h > g_0$.
Since $R := f^{\mathbb{T}}_a - L - Q = g - \mathbf{0} - Q = g - Q$, we get: $R = g - Q$.
Since $R \in \mathcal{O}(2)$, choose $\delta > 0$ s.t., $\forall h \in \mathbb{R}$,
$$(\, |h| < \delta\,) \Rightarrow (\, |R_h| \leqslant \varepsilon \cdot |h|^2\,).$$
Let $B := B(0, \delta)$. Then $B \in \mathcal{B}(0)$. **Want:** $\forall h \in B^\times_0$, $g_h > g_0$.
Given $h \in B^\times_0$. **Want:** $g_h > g_0$.
Since $h \in B^\times_0$, we get: $h \in B$ and $h \neq 0$.
Since $\varepsilon > 0$ and $h \neq 0$, we get $2 \cdot \varepsilon \cdot h^2 > 0$.
Since $h \in B = B(0, \delta)$, we get $|h| < \delta$.
So, by choice of δ, we get: $|R_h| \leqslant \varepsilon \cdot |h|^2$.
Let $a := R_h$ and $b := Q_h$.
Since $R = g - Q$, we get $R_h = g_h - Q_h$, so $a = g_h - b$, so $g_h = b + a$.
Since $Q = 2 \cdot \varepsilon \cdot (\bullet)^2$, we get $Q_h = 2 \cdot \varepsilon \cdot h^2$, so $b = 2 \cdot \varepsilon \cdot h^2$.
We have $|a| = |R_h| \leqslant \varepsilon \cdot |h|^2 = (2 \cdot \varepsilon \cdot |h|^2)/2 = |b|/2$, so $|a| \leqslant |b|/2$.
So, by robustness of sgn, we get: $\mathrm{sgn}_{b+a} = \mathrm{sgn}_b$.
Recall: $2 \cdot \varepsilon \cdot h^2 > 0$.
Since $b = 2 \cdot \varepsilon \cdot h^2 > 0$, we get $b > 0$, so $\mathrm{sgn}_b = 1$.
Then $\mathrm{sgn}_{b+a} = \mathrm{sgn}_b = 1$, so $b + a > 0$. Recall: $g_h = b + a$.
We have $g_0 = (f^{\mathbb{T}}_a)_0 = f_{a+0} - f_a = f_a - f_a = 0$.
Then $g_h = b + a > 0 = g_0$. QED

4. (20 pts) Show:
Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$. Assume f is one-to-one. Let $g := f^{-1}$.
Assume: $(f_0 = 0) \,\&\, (f'_0 = 3) \,\&\, (g \in \mathcal{O}(0))$. Then: $g'_0 = 1/3$.

Proof: Since $f_0 = 0$ and since $g = f^{-1}$, we get: $0 = g_0$.
Since $f_0 = 0$, we get: $f_0^{\mathbb{T}} = f$. Since $g_0 = 0$, we get: $g_0^{\mathbb{T}} = g$.
Since $f'_0 = 3$, we get: $D_0 f = 3 \cdot (\bullet)$.
Let $L := (\bullet)$. Then $L = \mathrm{id}^{\mathbb{R}}$. Then: $\forall h \in \mathbb{R}$, $L_h = h$.
We have: $\forall h \in \mathbb{R}$,

$$\begin{aligned}(D_0 f)_h &= (D_0 f)_{h \cdot 1} = h \cdot ((D_0 f)_1) = h \cdot [D_0 f] \\ &= h \cdot f'_0 = h \cdot 3 = 3 \cdot h = 3 \cdot L_h = (3 \cdot L)_h.\end{aligned}$$

Then: $D_0 f = 3 \cdot L$. Then: $f_0^{\mathbb{T}} - 3 \cdot L \in \mathcal{o}(1)$.
Let $R := f - 3 \cdot L$. Then: $R = f_0^{\mathbb{T}} - 3 \cdot L$. Then: $R \in \mathcal{o}(1)$.
We have $[(1/3) \cdot L] = ((1/3) \cdot L)_1 = (1/3) \cdot L_1 = (1/3) \cdot 1 = 1/3$.
Also, $g'_0 = [D_0 g]$. **Want:** $[D_0 g] = [(1/3) \cdot L]$.
It suffices to show: $D_0 g = (1/3) \cdot L$.
By uniqueness of linearization, **want:** $(1/3) \cdot L \in \mathrm{LINS}_0 g$.
Want: $g_0^{\mathbb{T}} - (1/3) \cdot L \in \mathcal{o}(1)$.
Recall: $g_0^{\mathbb{T}} = g$. **Want:** $g - (1/3) \cdot L \in \mathcal{o}(1)$.
We have: $L \circ g = \mathrm{id}^{\mathbb{R}} \circ g = g$.
Then: $(3 \cdot L) \circ g = 3 \cdot (L \circ g) = 3 \cdot g$.
Since $R = f - 3 \cdot L$, we get: $f = 3 \cdot L + R$.
Then: $\mathrm{id}^{\mathbb{D}_g} = f \circ g = ((3 \cdot L) \circ g) + (R \circ g) = 3 \cdot g + (R \circ g)$.
Then: $\mathrm{id}^{\mathbb{D}_g} = 3 \cdot g + (R \circ g)$.
Since $g_0 = 0$, we get $0 \in \mathbb{D}'_g$, so g is defined near 0.
Then $\mathrm{id}^{\mathbb{R}} = \mathrm{id}^{\mathbb{D}_g}$ near 0.
Then $\mathrm{id}^{\mathbb{R}} = 3 \cdot g + (R \circ g)$ near 0.
Recall: $L = \mathrm{id}^{\mathbb{R}}$. Then $L = 3 \cdot g + (R \circ g)$ near 0.
Then: $(1/3) \cdot L = g + (1/3) \cdot (R \circ g)$ near 0.
Then: $(1/3) \cdot L - g = (g - g) + (1/3) \cdot (R \circ g)$ near 0.
Since g is defined near 0, we get: $g - g = \mathbf{0}$ near 0.
Then: $(1/3) \cdot L - g = (1/3) \cdot (R \circ g)$ near 0.
Then: $g - (1/3) \cdot L = -(1/3) \cdot (R \circ g)$ near 0.
Want: $-(1/3) \cdot (R \circ g) \in \mathcal{o}(1)$. **Want:** $R \circ g \in \mathcal{o}(1)$.
Since $R \in \mathcal{o}(1)$, **it suffices to show:** $g \in \widehat{\mathcal{O}}(1)$.
Want: $\exists C \geqslant 0$, $\exists \varepsilon > 0$ s.t., $\forall k \in \mathbb{R}$, $(|k| < \varepsilon) \Rightarrow (|g_k| \leqslant C \cdot |k|^1)$.
Let $C := 1/2$. Then $C \geqslant 0$.
Want: $\exists \varepsilon > 0$ s.t., $\forall k \in \mathbb{R}$, $(|k| < \varepsilon) \Rightarrow (|g_k| \leqslant C \cdot |k|^1)$.

Since $R \in \mathcal{O}(1)$, choose $\delta > 0$ s.t., $\forall h \in \mathbb{R}$,

$$(|h| < \delta) \Rightarrow (|R_h| \leqslant 1 \cdot |h|^1).$$

Since $g \in \mathcal{O}(0)$, choose $\varepsilon > 0$ s.t., $\forall k \in \mathbb{R}$,

$$(|k| < \varepsilon) \Rightarrow (|g_k| \leqslant (1/2) \cdot \delta \cdot |k|^0).$$

Then $\varepsilon > 0$. **Want:** $\forall k \in \mathbb{R}$, $(|k| < \varepsilon) \Rightarrow (|g_k| \leqslant C \cdot |k|^1)$.

Given $k \in \mathbb{R}$. Assume: $|k| < \varepsilon$. **Want:** $|g_k| \leqslant C \cdot |k|^1$.

Recall: $C = 1/2$. **Want:** $|g_k| \leqslant (1/2) \cdot |k|$.

Since $|k| < \varepsilon$, by choice of ε, we get: $|g_k| \leqslant (1/2) \cdot \delta \cdot |k|^0$.

Let $h := g_k$. Then $|h| = |g_k| \leqslant (1/2) \cdot \delta \cdot |k|^0 = (1/2) \cdot \delta < \delta$.

Then $|h| < \delta$. So, by choice of δ, we get: $|R_h| \leqslant 1 \cdot |h|^1$.

We have $R_h = (f - 3 \cdot L)_h$ and $L_h = h$, so $R_h = f_h - 3 \cdot h$.

Then $|f_h - 3 \cdot h| = |R_h| \leqslant 1 \cdot |h|^1 = |h|$, so $|f_h - 3 \cdot h| \leqslant |h|$.

Since $| \bullet |$ is distance-semidecreasing, we get:

$$|\,|f_h| - |3 \cdot h|\,| \leqslant |\,f_h - 3 \cdot h\,|.$$

Since $|\,|f_h| - |3 \cdot h|\,| \leqslant |\,f_h - 3 \cdot h\,| \leqslant |\,h\,|$,

we get: $|3 \cdot h| - |h| \leqslant |f_h| \leqslant |3 \cdot h| + |h|$.

Then: $3 \cdot |h| - |h| \leqslant |f_h| \leqslant 3 \cdot |h| + |h|$. Then: $2 \cdot |h| \leqslant |f_h| \leqslant 4 \cdot |h|$.

Then $2 \cdot |h| \leqslant |f_h|$. Recall: $h = g_k$. Then: $2 \cdot |g_k| \leqslant |f_h|$.

Since $h = g_k$ and since $g = f^{-1}$, we get: $f_h = k$.

Then $2 \cdot |g_k| \leqslant |k|$. Then $|g_k| \leqslant (1/2) \cdot |k|$. QED

5. (20 pts) Show:
Let $a \in \mathbb{R}$, $b \geqslant a$, $I := [a;b]$, $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$. Assume: $I \subseteq \mathbb{D}_f \bigcap \mathbb{D}_g$.
Assume: on I, $f \leqslant g$. Then: $\mathrm{JL}_I f \leqslant \mathrm{JL}_I g$.

Proof:
Let $S := \{$ simple functions $s : I \to \mathbb{R} \mid s \leqslant f$ on I $\}$.
Let $T := \{$ simple functions $s : I \to \mathbb{R} \mid s \leqslant g$ on I $\}$.
Then: $\mathrm{JL}_f = \sup\{\mathrm{JI}_s \mid s \in S\}$ and $\mathrm{JL}_g = \sup\{\mathrm{JI}_s \mid s \in T\}$.
Since $f \leqslant g$ on I, we get: $S \subseteq T$.
Then: $\{\mathrm{JI}_s \mid s \in S\} \subseteq \{\mathrm{JI}_s \mid s \in T\}$.
Then: $\sup\{\mathrm{JI}_s \mid s \in S\} \leqslant \sup\{\mathrm{JI}_s \mid s \in T\}$.
Then: $\mathrm{JL}_f = \sup\{\mathrm{JI}_s \mid s \in S\} \leqslant \sup\{\mathrm{JI}_s \mid s \in T\} = \mathrm{JL}_g$. QED