MATH 4603 Fall 2022, Midterm #1
Handout date: Thursday 13 October 2022
Instructor: Scot Adams

PRINT YOUR NAME:

**SOLUTIONS**

PRINT YOUR @umn.edu EMAIL ADDRESS:

Thirty minute exam.

Closed book, closed notes, no calculators/PDAs; no reference materials of any kind.

Turn off all mobile electronic devices.

I. Definitions

A. (5 pts) Let $f$ be a function. By $f$ is **1-1**, we mean: ...

$$\forall w, x \in \mathbb{D}_f, \quad (\, f_w = f_x \,) \Rightarrow (\, w = x \,).$$

---

B. (5 pts) Let $f$ be an injective function.
Then $f^{-1} : \mathbb{I}_f \to \mathbb{D}_f$ is defined by: $\forall y \in \mathbb{I}_f,\ f_y^{-1} = \cdots$.

$$\mathrm{UE}(f^*\{y\}).$$

---

C. (5 pts) Let $S \subseteq \mathbb{R}^*$. Then $\mathrm{UB}_S := \cdots$.

$$\{x \in \mathbb{R}^* \,|\, S \leqslant x\}.$$

D. (5 pts) Let $S \subseteq \mathbb{R}^*$. Then $\sup_S := \cdots$.

$\min(\mathrm{UB}_S)$.

---

E. (5 pts) Let $f$ and $g$ be functions.
Then $g \circ f$ is the function defined by: $\forall x$, $(g \circ f)_x = \cdots$.

$g_{f_x}$.

---

F. (5 pts) Let $A$ be a set.
Then $\mathrm{id}^A : A \to A$ is the function defined by: $\forall x \in A$, $\mathrm{id}^A_x = \cdots$.

$x$.

II. True or false (no partial credit):

a. (5 pts) Let $A, B$ be sets, $f : A \to B$, $g : B \to A$.
Assume $g \circ f = \mathrm{id}^A$. Then $f : A \hookrightarrow B$.

True. [[(1) of Theorem 1.23.7]]

b. (5 pts) $\min \mathbb{Z} = ☹$.

True. [[Theorem 1.10.15]]

c. (5 pts) $\sup \varnothing = ☹$.

False. [[Theorem 1.10.25]]

d. (5 pts) Let $A$, $B$ and $C$ be sets. Let $f : A \to\!> B$ and $g : B \to\!> C$.
Then $g \circ f : A \to\!> C$.

True. [[(4) of Theorem 1.22.4]]

e. (5 pts) $\forall x, y \in \mathbb{R}$, $\Big|\, |y| - |x| \,\Big| \leqslant \Big|\, y - x \,\Big|$.

True. [[HW 3-5]]

III. Hand-graded problems. Show work.

1. (10 pts) Let $S := \{2, 4, 6, 8, \ldots\}$. Compute: $\max S$ and $\sup S$.

(No need to prove that your answers are correct; just giving the correct answers will earn you full credit. However, if you give an incorrect answer, an explanation might be worth some partial credit.)

$$\max S = ☹ \qquad \text{and} \qquad \sup S = \infty.$$

2. (10 pts) Using the 0-Principle of Mathematical Induction,

show: $\forall j \in \mathbb{N}_0, \quad \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^j} = 2 - \frac{1}{2^j}.$

**Proof:** Let $S := \left\{ j \in \mathbb{N}_0 \,\middle|\, \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^j} = 2 - \frac{1}{2^j} \right\}.$

**Want:** $S = \mathbb{N}_0$.

Since $\frac{1}{1} = 1 = 2 - 1 = 2 - \frac{1}{1} = 2 - \frac{1}{2^0}$, we get: $0 \in S$.

By the 0-PMI, we wish to show: $\forall j \in S$, $j + 1 \in S$.

Given $j \in S$. **Want:** $j + 1 \in S$.

Know: $\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^j} \quad = \quad 2 - \frac{1}{2^j}.$

**Want:** $\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^j} + \frac{1}{2^{j+1}} \quad = \quad 2 - \frac{1}{2^{j+1}}.$

We have:

$$
\begin{aligned}
\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^j} + \frac{1}{2^{j+1}} \quad &= \quad 2 - \frac{1}{2^j} + \frac{1}{2^{j+1}} \\
&= \quad 2 - \frac{1}{2^j} + \frac{1}{2 \cdot 2^j} \quad = \quad 2 - 1 \cdot \frac{1}{2^j} + \frac{1}{2} \cdot \frac{1}{2^j} \\
&= \quad 2 - \left(1 - \frac{1}{2}\right) \cdot \frac{1}{2^j} \quad = \quad 2 - \frac{1}{2} \cdot \frac{1}{2^j} \\
&= \quad 2 - \frac{1}{2 \cdot 2^j} \quad = \quad 2 - \frac{1}{2^{j+1}}.
\end{aligned}
$$

**QED**

3. (10 pts) Show: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\delta^5 + 3\delta^3 - 2\delta < \varepsilon$.

**Proof:** Given $\varepsilon > 0$. **Want:** $\exists \delta > 0$ s.t. $\delta^5 + 3\delta^3 - 2\delta < \varepsilon$.

Let $\delta := 0.1$. Then $\delta > 0$. **Want:** $\delta^5 + 3\delta^3 - 2\delta < \varepsilon$.

We have
$$\begin{aligned} \delta^5 + 3\delta^3 - 2\delta &= 0.00001 + 3 \cdot 0.001 - 2 \cdot 0.1 \\ &= 0.00301 - 0.2 < 0 < \varepsilon. \end{aligned}$$
**QED**

4. (15 pts) Show: $\forall \varepsilon > 0$, $\exists j \in \mathbb{N}$ s.t. $(1/\sqrt[3]{j}) + (1/2^j) < \varepsilon$.

**Proof:** Given $\varepsilon > 0$. **Want:** $\exists j \in \mathbb{N}$ s.t. $(1/\sqrt[3]{j}) + (1/2^j) < \varepsilon$.

Let $A := \max\left\{ \dfrac{8}{\varepsilon^3} , \dfrac{2}{\varepsilon} \right\}$. Then $A \geqslant \dfrac{8}{\varepsilon^3}$ and $A \geqslant \dfrac{2}{\varepsilon}$.

By the AP, choose $j \in \mathbb{N}$ s.t. $j > A$.

Then $j \in \mathbb{N}$. **Want:** $(1/\sqrt[3]{j}) + (1/2^j) < \varepsilon$.

Since $j > A \geqslant \dfrac{8}{\varepsilon^3}$, we get: $j > \dfrac{8}{\varepsilon^3}$. Then $\sqrt[3]{j} > \dfrac{2}{\varepsilon}$.

Since $\sqrt[3]{j} > \dfrac{2}{\varepsilon} > 0$, we get: $\dfrac{1}{\sqrt[3]{j}} < \dfrac{\varepsilon}{2}$.

Since $j \in \mathbb{N} \subseteq \mathbb{N}_0$, by a class theorem, we get: $2^j \geqslant j + 1$.

Since $2^j \geqslant j + 1 > j > A \geqslant \dfrac{2}{\varepsilon}$, we get: $2^j > \dfrac{2}{\varepsilon}$.

Since $2^j > \dfrac{2}{\varepsilon} > 0$, we get: $\dfrac{1}{2^j} < \dfrac{\varepsilon}{2}$.

So, since $\dfrac{1}{\sqrt[3]{j}} < \dfrac{\varepsilon}{2}$, we get: $\dfrac{1}{\sqrt[3]{j}} + \dfrac{1}{2^j} < \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2}$.

Then $(1/\sqrt[3]{j}) + (1/2^j) = \dfrac{1}{\sqrt[3]{j}} + \dfrac{1}{2^j} < \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2} = \varepsilon$. **QED**