MATH 4603 Fall 2022, Midterm #2
Handout date: Thursday 3 November 2022
Instructor: Scot Adams

PRINT YOUR NAME:

SOLUTIONS

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Thirty minute exam.

Closed book, closed notes, no calculators/PDAs; no reference materials of any kind.

Turn off all mobile electronic devices.

I. Definitions

A. (5 pts) Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and $a \in \mathbb{R}$.
Then $f_a^{\mathbb{T}} : \mathbb{R} \dashrightarrow \mathbb{R}$ is defined by: $\forall h \in \mathbb{R}$, $(f_a^{\mathbb{T}})_h = \cdots$.

$$f_{a+h} - f_a.$$

B. (5 pts) Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$.
By f is **uniformly continuous**, we mean: ...

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ s.t., } \forall w, x \in \mathbb{D}_f,$$
$$(\,|w - x| < \delta\,) \Rightarrow (\,|f_w - f_x| < \varepsilon\,).$$

C. (5 pts) Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $S \subseteq \mathbb{R}$.
By f is **continuous** on S, we mean: ...

$$\forall a \in S, \quad f \text{ is continuous at } a.$$

D. (5 pts) Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a, z \in \mathbb{R}$.
By as $x \to a$, $f_x \to z$, we mean:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t.}, \forall x \in \mathbb{D}_f,$$
$$(\, 0 < |x - a| < \delta \,) \Rightarrow (\, |f_x - z| < \varepsilon \,).$$

E. (5 pts) Let S be a set. By S is **countably infinite**, we mean: ...

$$\exists S \hookrightarrow\!\!> \mathbb{N}.$$

F. (5 pts) Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$. By f is **Lipschitz**, we mean: ...

$$\exists L \geqslant 0 \text{ s.t. } f \text{ is } L\text{-Lipschitz}.$$

II. True or false (no partial credit):

a. (5 pts) $\forall x \in \mathbb{R}, \quad (x^2, x^2, x^2, \ldots) \to x^2$.

True. [[Theorem 2.1.6]]

b. (5 pts) Let $p, q \in \mathbb{R}$. Assume $p \neq q$.
Then: $\exists \varepsilon > 0$ s.t. $(B(p, \varepsilon)) \cap (B(q, \varepsilon)) = \varnothing$.

False. [[Theorem 1.33.1]]

c. (5 pts) $\exists 2^{\mathbb{R}} \hookrightarrow \mathbb{R}$.

False. [[Theorem 1.28.20]]

d. (5 pts) Let $s, t \in \mathbb{R}^{\mathbb{N}}$, $a, b \in \mathbb{R}$.
Assume $s \to a$ and $t \to b$. Then $s \cdot t + s + t \to a \cdot b + a + b$.

True. [Theorems 2.1.10 and 2.1.11]]

e. (5 pts) Let $f : \mathbb{R} \to \mathbb{R}$. Assume f is continuous. Then f is Lipschitz.

False. [[Theorems 3.2.7 and 3.2.15]]

III. Hand-graded problems. Show work.

1. (10 pts) Find $f, g : \mathbb{R} \to \mathbb{R}$ s.t.

$$\begin{array}{lll} \text{as } x \to 1, & f_x \to 5 & \text{and} \\ \text{as } y \to 5, & g_y \to 9 & \text{and} \\ \text{as } x \to 1, & (g \circ f)_x \to 0. & \end{array}$$

(No need to prove that your answer is correct; just giving the correct answer will earn you full credit. However, if you give an incorrect answer, an explanation might be worth some partial credit.)

(NOTE: Avoid unbound variables. To bind f and g, say,

"Define $f : \mathbb{R} \to \mathbb{R}$ by: $\forall x \in \mathbb{R}$, $f_x = \cdots$"

and

"Define $g : \mathbb{R} \to \mathbb{R}$ by: $\forall y \in \mathbb{R}$, $g_y = \cdots$" .)

Solution:

Define $f : \mathbb{R} \to \mathbb{R}$ by: $\forall x \in \mathbb{R}$, $f_x = 5$.

Define $g : \mathbb{R} \to \mathbb{R}$ by: $\forall y \in \mathbb{R}$, $g_x = \begin{cases} 9, & \text{if } y \neq 5 \\ 0, & \text{if } y = 5. \end{cases}$

Then: $\forall x \in \mathbb{R}$, $(g \circ f)_x = g_{f_x} = g_5 = 0$.

2. (10 pts) Show: Let $s, t \in \mathbb{R}^{\mathbb{N}}$, $a \in \mathbb{R}$.
Assume $s \to -\infty$ and $t \to \infty$. Then $s - t \to -\infty$.

Proof: **Want:** $\forall N \in \mathbb{R}$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$(j \geqslant K) \Rightarrow ((s - t)_j < N).$$

Given $N \in \mathbb{R}$. **Want:** $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$(j \geqslant K) \Rightarrow ((s - t)_j < N).$$

Since $s \to -\infty$, choose $A \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$(j \geqslant A) \Rightarrow (s_j < N).$$

Since $t \to \infty$, choose $B \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$(j \geqslant B) \Rightarrow (t_j > 0).$$

Let $K := \max\{A, B\}$. Then $K \in \mathbb{N}$.
Want: $\forall j \in \mathbb{N}$, $(j \geqslant K) \Rightarrow ((s - t)_j < N)$.
Given $j \in \mathbb{N}$ Assume: $j \geqslant K$. Want: $(s - t)_j < N$.
Since $j \geqslant K \geqslant A$, by choice of A, we get: $s_j < N$.
Since $j \geqslant K \geqslant B$, by choice of B, we get: $t_j > 0$.
Since $t_j > 0$, we get: $s_j - t_j < s_j$.
Then $(s - t)_j = s_j - t_j < s_j < N$. QED

3. (10 pts) Show: Define $f : \mathbb{R} \to \mathbb{R}$ by: $\forall x \in \mathbb{R}$, $f_x = |x|$.
Then f is uniformly continuous.

Proof: **Want:** $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall w, x \in \mathbb{D}_f$,
$$(\, |w - x| < \delta \,) \Rightarrow (\, |f_w - f_x| < \varepsilon \,).$$
Given $\varepsilon > 0$. **Want:** $\exists \delta > 0$ s.t., $\forall w, x \in \mathbb{D}_f$,
$$(\, |w - x| < \delta \,) \Rightarrow (\, |f_w - f_x| < \varepsilon \,).$$
Let $\delta := \varepsilon$. Then $\delta = \varepsilon > 0$.
Want: $\forall w, x \in \mathbb{D}_f$, $(\, |w - x| < \delta \,) \Rightarrow (\, |f_w - f_x| < \varepsilon \,)$.
Given $w, x \in \mathbb{D}_f$. Assume: $|w - x| < \delta$. **Want:** $|f_w - f_x| < \varepsilon$.
By HW#3-5, we have: $\Big|\, |w| - |x| \,\Big| \leqslant \Big|\, w - x \,\Big|$.
So, since $f_w = |w|$ and since $f_x = |x|$, we get: $|f_w - f_x| \leqslant |w - x|$.
Then $|f_w - f_x| \leqslant |w - x| < \delta = \varepsilon$. QED

4. (15 pts) Show: Let $s \in \mathbb{R}^{\mathbb{N}}$.
Assume $s \to -\infty$. Then: $s^2 + s \to \infty$.

Proof: **Want:** $\forall M \in \mathbb{R}$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\, j \geqslant K \,) \Rightarrow (\, (s^2 + s)_j > M \,).$$
Given $M \in \mathbb{R}$. **Want:** $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\, j \geqslant K \,) \Rightarrow (\, (s^2 + s)_j > M \,).$$
Let $A := \max\{M, 0\}$. Then $A \geqslant M$ and $A \geqslant 0$.
Let $N := -\sqrt{A} - 1$. Then $N + 1 = -\sqrt{A}$.
Since $s \to -\infty$, choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\, j \geqslant K \,) \Rightarrow (\, s_j < N \,).$$
Then $K \in \mathbb{N}$. **Want:** $\forall j \in \mathbb{N}$, $(\, j \geqslant K \,) \Rightarrow (\, (s^2 + s)_j > M \,)$.
Given $j \in \mathbb{N}$. Assume: $j \geqslant K$. **Want:** $(s^2 + s)_j > M$.
Since $j \geqslant K$, by choice of K, we get: $s_j < N$.
Then $s_j + 1 < N + 1 = -\sqrt{A}$, so $s_j + 1 < -\sqrt{A}$.
Then $s_j < s_j + 1 < -\sqrt{A}$, so $s_j < -\sqrt{A}$.
Since $-(s_j + 1) > \sqrt{A} \geqslant 0$ and $-s_j > \sqrt{A} \geqslant 0$,
we get: $[-(s_j + 1)] \cdot [-s_j] > [\sqrt{A}] \cdot [\sqrt{A}]$.
Then $(s_j + 1) \cdot s_j > A$.
Then $(s^2 + s)_j = s_j^2 + s_j = (s_j + 1) \cdot s_j > A \geqslant M$. QED