Homework for MATH 4603 (Advanced Calculus I)
Fall 2022

Homework 13: Due on Tuesday 13 December

13-1. Show: Let $S, T \subseteq \mathbb{R}$. Then $\mathrm{JO}_{S\cup T} \leqslant \mathrm{JO}_S + \mathrm{JO}_T$.

13-2. Show: $\forall k \in \mathbb{N}$,

$$\forall S_1, \ldots, S_k \subseteq \mathbb{R}, \quad \mathrm{JO}_{S_1\cup\cdots\cup S_k} \leqslant \mathrm{JO}_{S_1} + \cdots + \mathrm{JO}_{S_k}.$$

13-3. Show: Let $k \in \mathbb{N}$. Assume: $\forall \alpha \in \mathbb{R}, \forall \beta \geqslant \alpha,\ \mathrm{JO}^k_{[\alpha;\beta]} \geqslant \beta - \alpha$. Let $a \in \mathbb{R}$. Let $b \geqslant a$. Let $I := [a; b]$. Then $\mathrm{JO}^{k+1}_I \geqslant b - a$.

13-4. Show: Let $T \subseteq \mathbb{R}$. Then $\operatorname{Int} T$ is open.

13-5. Show: Let $S \subseteq \mathbb{R}$. Then $\operatorname{Cl} S$ is closed.

Homework 12: Due on Tuesday 6 December

12-1. Using the PMI, show: Let $a \in \mathbb{R}$. Then:

$$\forall j \in \mathbb{N}, \qquad ((\bullet)^j)^{\mathbb{T}}_a - j \cdot a^{j-1} \cdot (\bullet) \in \mathcal{O}(1).$$

12-2. Show: Let $j \in \mathbb{N}$. Then $((\bullet)^j)' = j \cdot (\bullet)^{j-1}$.

12-3. Show: Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $y \in \mathbb{R}^\times_0$.

Assume: as $x \to -\infty$, $f_x \to y$.

Then: as $x \to -\infty$, $(1/f)_x \to 1/y$.

12-4. Show: Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{D}_f$.

Assume: f has a local strict-maximum at a.

Then: $f^{\mathbb{T}}_a$ has a local strict-maximum at 0.

12-5. Show: Let $s, t \in \mathbb{R}^{\mathbb{N}}$.

Assume: $\mathbb{I}_s$ and $\mathbb{I}_t$ are both bounded.

Then: $\exists$strictly-increasing $\ell \in \mathbb{N}^{\mathbb{N}}$ s.t.

$s \circ \ell$ and $t \circ \ell$ are both convergent.

Homework 11: Due on Tuesday 29 November

11-1. Show: Define $r : \mathbb{R}^\times_0 \to \mathbb{R}$ by: $\forall x \in \mathbb{R}^\times_0,\ r_x = 1/x$.
Then: $\forall a \in \mathbb{R}^\times_0,\ r'_a = -\dfrac{1}{a^2}$.

11-2. Show: Let $g : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{D}'_g$.
Assume $g_a \neq 0$. Then: $\left(\dfrac{1}{g}\right)'_a = -\dfrac{g'_a}{g^2_a}$.

11-3. Show: Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{D}'_f \cap \mathbb{D}'_g$.
Assume $g_a \neq 0$. Then: $\left(\dfrac{f}{g}\right)'_a = \dfrac{g_a \cdot f'_a - f_a \cdot g'_a}{g_a^2}$.

11-4. Show: Let $U, V \subseteq \mathbb{R}$ both be open. Then:
(i) $U \cup V$ is open and (ii) $U \cap V$ is open.

11-5. Show: Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$. Assume: $\forall$open $U \subseteq \mathbb{R}$, f^*U is open.
Then: f is continuous.

Homework 10: Due on Tuesday 22 November

10-1. Show: Let $w, x \in \mathbb{R}^{\mathbb{N}}$, $\ell \in \mathbb{N}^{\mathbb{N}}$, $q \in \mathbb{R}$.
Assume: (ℓ is strictly-increasing) & ($w \circ \ell \to q$).
Assume: $\forall j \in \mathbb{N}$, $|w_j - x_j| < 1/j$.
Then: $x \circ \ell \to q$.

10-2. Show: Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$.
Assume: as $x \to \infty$, $f_x \to -\infty$.
Assume: as $x \to \infty$, $g_x \to -\infty$.
Then: as $x \to \infty$, $(f \cdot g)_x \to \infty$.

10-3. Show: Let $f := (\bullet)^3$. Then $D_2 f = 3 \cdot 2^2 \cdot (\bullet)$.

10-4. Show: Let $f := (\bullet)^3$. Then $\forall x \in \mathbb{R}$, $f'_x = 3x^2$.

10-5. Show: Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a, c \in \mathbb{R}$.
Then $D_a(c \cdot f) \;=^*\; c \cdot (D_a f)$.

Homework 9: Due on Tuesday 15 November

9-1. Show: Define $f : \mathbb{R} \dashrightarrow \mathbb{R}$ by: $\forall x \in \mathbb{R}$, $f_x = 1/x$.
Then: as $x \to -\infty$, $f_x \to 0$.

9-2. Show: Let $D \subseteq \mathbb{R}$, $s \in (\mathbb{R}^D)^{\mathbb{N}}$, $f \in \mathbb{R}^D$.
Assume: $\forall j \in \mathbb{N}$, s_j is continuous.
Assume: $s \to f$ uniformly.
Then: f is continuous.

9-3. Show: Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $k \in \mathbb{N}_0$.
Assume: ($f \in \mathcal{O}(k)$) & (near 0, $f = g$).
Then: $g \in \mathcal{O}(k)$.

9-4. Show: Let $s, t \in \mathbb{R}^{\mathbb{N}}$.
Assume: $(s \to \infty) \;\&\; (\forall j \in \mathbb{N},\; t_j \geqslant s_j)$.
Then: $t \to \infty$.

9-5. Show:
Let $a := (1 , 1/2 , 1/2 , 1/4 , 1/4 , 1/4 , 1/4 , 1/8 , \ldots)$.
NOTE: The terms of a are, in order:
a single "1", then two "1/2"s,
then four "1/4"s, then eight "1/8"s,
then sixteen "1/16"s, then thirty-two "1/32"s, *etc.*
Let $b := (1 , 1/2 , 1/3 , 1/4 , 1/5 , 1/6 , 1/7 , 1/8 , \ldots)$.
Define $s, t \in \mathbb{R}^{\mathbb{N}}$ by: $\forall j \in \mathbb{N}$,
$s_j = a_1 + \cdots + a_j$ and $t_j = b_1 + \cdots + b_j$.
Then: (i) $\mathbb{I}_s$ is unbounded and
(ii) $\forall j \in \mathbb{N},\; t_j > s_j/2$ and
(iii) $t \to \infty$.

Homework 8: Due on Wednesday 9 November

8-1. Show: Let $f \in \mathcal{O}(3)$, $g \in \mathcal{O}(4)$. Then: $f \cdot g \in \mathcal{O}(7)$.

8-2. Show: Let $s \in \mathbb{R}^{\mathbb{N}}$.
Assume $\mathbb{I}_s$ is bounded. Then s is subconvergent.

8-3. Show: Let $s \in \mathbb{R}^{\mathbb{N}}$.
Assume s is convergent. Then s is Cauchy.

8-4. Show: Let $s \in \mathbb{R}^{\mathbb{N}}$.
Assume s is Cauchy and subconvergent. Then s is convergent.

8-5. Show: Let $s \in \mathbb{R}^{\mathbb{N}}$.
Assume s is Cauchy. Then s is convergent.

Homework 7: Due on Tuesday 1 November

7-1. Show: Let $k \in \mathbb{N}^{\mathbb{N}}$. Assume k is strictly-increasing.
Then: $\forall j \in \mathbb{N},\; k_j \geqslant j$.

7-2. Show: Let $s, t \in \mathbb{R}$. Assume $s < t$.
Then: $\exists x \in \mathbb{Q}$ s.t. $s < x < t$.

7-3. Show: Let $s, t, u \in \mathbb{R}^{\mathbb{N}}$, $a \in \mathbb{R}$.
Assume: $\forall j \in \mathbb{N}$, $s_j \leqslant t_j \leqslant u_j$.

Assume: $s \to a$ and $u \to a$.
Then: $t \to a$.

7-4. Show: Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$.
Assume: as $x \to a$, $f_x \to -\infty$.
Assume: as $x \to a$, $g_x \to -\infty$.
Then: as $x \to a$, $(f \cdot g)_x \to \infty$.

7-5. Show: Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $q \in \mathbb{D}_{f \cdot g}$.
Then: $(f \cdot g)_q^{\mathbb{T}} \;=\; f_q^{\mathbb{T}} \cdot g_q \;+\; f_q \cdot g_q^{\mathbb{T}} \;+\; f_q^{\mathbb{T}} \cdot g_q^{\mathbb{T}}$.

Homework 6: Due on Tuesday 25 October

6-1. Show: $\forall a \in \mathbb{R}, \exists x \in \mathbb{R}$ s.t. $x^5 + x^3 + x = a$.

6-2. Show: Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $a, b \in \mathbb{R}$.
Assume: as $x \to a$, $f_x \to b$. Assume: g is continuous at b.
Then: as $x \to a$, $(g \circ f)_x \to g_b$.

6-3. Show: Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a, b \in \mathbb{D}_f$.
Assume $a \neq b$. Let $m := \mathrm{DQ}_f(a, b)$.
Define $g : \mathbb{R} \dashrightarrow \mathbb{R}$ by: $\forall x \in \mathbb{R}, \quad g_x = f_x - mx$.
Then: $g_a = g_b$.

6-4. Show: Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{D}_f$.
Assume $f_a^{\mathbb{T}}$ is continuous at 0. Then f is continuous at a.

6-5. Show: Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{D}_{g \circ f}$. Then $(g \circ f)_a^{\mathbb{T}} \;=\; (g_{f_a}^{\mathbb{T}}) \circ (f_a^{\mathbb{T}})$.

Homework 5: Due on Wednesday 19 October

5-1. Show: Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$.
Assume f and g are both continuous at a.
Then $f + g$ is continuous at a.

5-2. Show: Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$.
Assume f is continuous at a and g is continuous at f_a.
Then $g \circ f$ is continuous at a.

5-3. Show: Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $s \in (\mathbb{D}_f)^{\mathbb{N}}$, $a \in \mathbb{R}$.
Assume f is continuous at a and $s \to a$. Then $f \circ s \to f_a$.

5-4. Show: let $f : \mathbb{R} \dashrightarrow \mathbb{R}$.
Assume f is Lipschitz. Then f is uniformly continuous.

5-5. Show: let $f : \mathbb{R} \dashrightarrow \mathbb{R}$.
Assume f is uniformly continuous. Then f is continuous.

Homework 4: Due on Tuesday 11 October

4-1. Show: $(1, 2, 3, \ldots) \to \infty$.

4-2. Show: Let $s \in \mathbb{R}^{\mathbb{N}}$, $c < 0$.
Assume $s \to \infty$. Then $c \cdot s \to -\infty$.

4-3. Show: Let $s, t \in \mathbb{R}^{\mathbb{N}}$.
Assume $s \to \infty$ and $t \to \infty$. Then $s + t \to \infty$.

4-4. Show: Let $s \in (\mathbb{R}_0^\times)^{\mathbb{N}}$.
Assume $s \to \infty$. Then $1/s \to 0$.

4-5. Show: Let $s \in (\mathbb{R}_0^\times)^{\mathbb{N}}$, $c \in \mathbb{R}$.
Assume $s \to -\infty$. Then $c/s \to 0$.

Homework 3: Due on Tuesday 4 October

3-1. Using the 0-PMI, show: Let $x \in \mathbb{R}$. Then: $\forall j \in \mathbb{N}_0$, $|x^j| = |x|^j$.

3-2. Show: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall x \in \mathbb{R}$,
$$(\ |x - 4| < \delta \ ) \ \Rightarrow \ (\ |x^3 - 5x^2 + 4x| < \varepsilon \ ).$$

3-3. Show: Define $f : \mathbb{R} \to \mathbb{R}$ by: $\forall x \in \mathbb{R}$, $f_x = x^2$.
Then: $\forall a \in \mathbb{R}$, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall x \in \mathbb{R}$,
$$(\ |x - a| < \delta \ ) \ \Rightarrow \ (\ |f_x - f_a| < \varepsilon \ ).$$

3-4. Show: $\forall M \in \mathbb{R}$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\ j \geqslant K \ ) \ \Rightarrow \ (\ j^4 - 7j - 9 > M \ ).$$

3-5. Show: Let $x, y \in \mathbb{R}$. Then: $| \, |y| - |x| \, | \leqslant | \, y - x \, |$.

Homework 2: Due on Tuesday 27 September

2-1. Show: Let $S \subseteq \mathbb{R}^*$. Then: $\sup_S =^* \max_S$.

2-2. Show: $\forall M \in \mathbb{R}$, $\exists N \in \mathbb{R}$ s.t., $\forall x \in \mathbb{R}$,
$$(\ x < N \ ) \ \Rightarrow \ (\ x^2 > M \ ).$$

2-3. Show: $\forall M \in \mathbb{R}$, $\exists K \in \mathbb{N}_0$ s.t., $\forall j \in \mathbb{N}_0$,
$$(\ j \geqslant K \ ) \ \Rightarrow \ (\ 2^j > M \ ).$$

2-4. Show: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}_0$ s.t., $\forall j \in \mathbb{N}_0$,
$$(\ j \geqslant K\ ) \ \Rightarrow\ \left(\ \frac{1}{2^j} < \varepsilon\ \right).$$

2-5. Using the 0-PMI, show: $\forall k \in \mathbb{N}_0$,
$$1 + 2 + 4 + 8 + \cdots + 2^k \ =\ 2^{k+1} - 1.$$

Homework 1: Due on Tuesday 20 September

1-1. Show: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $3\delta^8 + 7\delta^6 + 2\delta^4 \leqslant 6\varepsilon$.

1-2. Show: Let $\varepsilon > 0$. Then $\exists \delta > 0$ s.t. $3\delta^8 + 7\delta^6 + 2\delta^4 \leqslant 6\varepsilon$.

1-3. Show: $\forall N \in \mathbb{R}$, $\exists \delta > 0$ s.t. $\forall x \in \mathbb{R}$,
$$(\,-\delta < x < 0\,) \ \Rightarrow\ (\,1/x < N\,).$$

1-4. Show: Let $N \in \mathbb{R}$. Then $\exists \delta > 0$ s.t. $\forall x \in \mathbb{R}$,
$$(\,-\delta < x < 0\,) \ \Rightarrow\ (\,1/x < N\,).$$

1-5. Using the PMI, show:
$$\forall k \in \mathbb{N}, \qquad 1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$
