Solutions for MATH 4603 (Advanced Calculus I)
Fall 2022

Homework 13: Due on Tuesday 14 December

13-1. Show: Let $S, T \subseteq \mathbb{R}$. Then $\mathrm{JO}_{S\cup T} \leqslant \mathrm{JO}_S + \mathrm{JO}_T$.

Proof:
Let $Q := \{ \mathrm{TL}_Z \,|\, (Z \in \mathcal{OI}^{\mathbb{N}}) \,\&\, (\bigcup \mathbb{I}_Z \supseteq S \bigcup T) \,\&\, (\#\mathrm{supp}\, Z < \infty) \}$.
Let $A := \{ \mathrm{TL}_U \,|\, (U \in \mathcal{OI}^{\mathbb{N}}) \,\&\, (\bigcup \mathbb{I}_U \supseteq S) \,\&\, (\#\mathrm{supp}\, U < \infty) \}$.
Let $B := \{ \mathrm{TL}_V \,|\, (V \in \mathcal{OI}^{\mathbb{N}}) \,\&\, (\bigcup \mathbb{I}_V \supseteq T) \,\&\, (\#\mathrm{supp}\, V < \infty) \}$.
Then $\mathrm{JO}_{S\cup T} = \inf_Q$ and $\mathrm{JO}_S = \inf_A$ and $\mathrm{JO}_T = \inf_B$.
Want: $\inf_Q \leqslant \inf_A + \inf_B$.
Exactly one of the following is true:
(1) $\inf_A + \inf_B = \infty$ or (2) $\inf_A + \inf_B \neq \infty$.

Case (1):
We have $\inf_Q \in \mathbb{R}^* \leqslant \infty = \inf_A + \inf_B$.
End of Case (1).

Case (2):
We have $\inf_A = \mathrm{JO}_S \geqslant 0$ and $\inf_B = \mathrm{JO}_T \geqslant 0$,
so $\inf_A \geqslant 0$ and $\inf_B \geqslant 0$.
Then $\inf_A \leqslant \inf_A + \inf_B$ and $\inf_B \leqslant \inf_A + \inf_B$.
Also, $\inf_A + \inf_B \geqslant 0$, so $\inf_A + \inf_B \in [0; \infty]$.
So, since $\inf_A + \inf_B \neq \infty$, we get: $\inf_A + \inf_B \in [0; \infty] \backslash \{\infty\}$.
Then $\inf_A + \inf_B \in [0; \infty] \backslash \{\infty\} = [0; \infty) < \infty$,
so $\inf_A + \inf_B < \infty$.
Then $\inf_A \leqslant \inf_A + \inf_B < \infty$ and $\inf_B \leqslant \inf_A + \inf_B < \infty$,
so $\inf_A < \infty$ and $\inf_B < \infty$.
Want: $\forall \varepsilon > 0$, $\inf_Q \leqslant \inf_A + \inf_B + \varepsilon$.
Given $\varepsilon > 0$. **Want:** $\inf_Q \leqslant \inf_A + \inf_B + \varepsilon$.
We have $\inf_A \geqslant 0 > -\infty$ and $\inf_A < \infty$.
Then $-\infty < \inf_A < \infty$, so $\inf_A + (\varepsilon/2) > \inf_A$.
Since $\neg(\inf_A + (\varepsilon/2) \leqslant \inf_A)$, we get $\neg(\inf_A + (\varepsilon/2) \leqslant A)$.
Since $\neg(\inf_A + (\varepsilon/2) \leqslant A)$, choose $a \in A$ s.t. $\inf_A + (\varepsilon/2) > a$.
We have $\inf_B \geqslant 0 > -\infty$ and $\inf_B < \infty$.
Then $-\infty < \inf_B < \infty$, so $\inf_B + (\varepsilon/2) > \inf_B$.
Since $\neg(\inf_B + (\varepsilon/2) \leqslant \inf_B)$, we get $\neg(\inf_B + (\varepsilon/2) \leqslant B)$.

Since $\neg(\inf_B + (\varepsilon/2) \leqslant B)$, choose $b \in B$ s.t. $\inf_B + (\varepsilon/2) > b$.
Since $a \in A = \{ \mathrm{TL}_U \,|\, (U \in \mathcal{OI}^{\mathbb{N}}) \,\&\, (\bigcup \mathbb{I}_U \supseteq S) \,\&\, (\#\mathrm{supp}\, U < \infty) \}$,
choose $U \in \mathcal{OI}^{\mathbb{N}}$ s.t. $\bigcup \mathbb{I}_U \supseteq S$ and $\#\mathrm{supp}\, U < \infty$ and $\mathrm{TL}_U = a$.
Since $b \in B = \{ \mathrm{TL}_V \,|\, (V \in \mathcal{OI}^{\mathbb{N}}) \,\&\, (\bigcup \mathbb{I}_V \supseteq T) \,\&\, (\#\mathrm{supp}\, V < \infty) \}$,
choose $V \in \mathcal{OI}^{\mathbb{N}}$ s.t. $\bigcup \mathbb{I}_V \supseteq T$ and $\#\mathrm{supp}\, V < \infty$ and $\mathrm{TL}_V = b$.
Let $Z := (U_1, V_1, U_2, V_2, U_3, V_3, \ldots)$. Then $Z \in \mathcal{OI}^{\mathbb{N}}$.
Then, by a class theorem, we have:

$$\mathbb{I}_Z = \mathbb{I}_U \bigcup \mathbb{I}_V \quad \text{and} \quad \#\mathrm{supp}\, Z = \#\mathrm{supp}\, U + \#\mathrm{supp}\, V$$
$$\text{and} \quad \mathrm{TL}_Z = \mathrm{TL}_U + \mathrm{TL}_V.$$

Since $\mathrm{supp}\, U < \infty$ and $\mathrm{supp}\, V < \infty$, we get: $\#\mathrm{supp}\, U + \#\mathrm{supp}\, V < \infty$.
Then $\#\mathrm{supp}\, Z = \#\mathrm{supp}\, U + \#\mathrm{supp}\, V < \infty$, so $\#\mathrm{supp}\, Z < \infty$.
Since $\mathbb{I}_U \supseteq S$ and $\mathbb{I}_V \supseteq T$, we get: $\mathbb{I}_U \bigcup \mathbb{I}_V \supseteq S \bigcup T$.
Then $\bigcup \mathbb{I}_Z = \bigcup \left(\mathbb{I}_U \bigcup \mathbb{I}_V \right) = \left(\bigcup \mathbb{I}_U \right) \bigcup \left(\bigcup \mathbb{I}_V \right) \supseteq S \bigcup T$,
so $\bigcup \mathbb{I}_Z \supseteq S \bigcup T$.
Since $Z \in \mathcal{OI}^{\mathbb{N}}$ and $\bigcup \mathbb{I}_Z \supseteq S \bigcup T$ and $\#\mathrm{supp}\, Z < \infty$, we get:
$\mathrm{TL}_Z \in \{ \mathrm{TL}_Z \,|\, (Z \in \mathcal{OI}^{\mathbb{N}}) \,\&\, (\bigcup \mathbb{I}_Z \supseteq S \bigcup T) \,\&\, (\#\mathrm{supp}\, Z < \infty) \}$.
So, by definition of Q, we get $\mathrm{TL}_Z \in Q$.
Then $a + b = \mathrm{TL}_U + \mathrm{TL}_V = \mathrm{TL}_Z \in Q \geqslant \inf_Q$, so $\inf_Q \leqslant a + b$.
Then $\inf_Q \leqslant a + b < (\inf_A + (\varepsilon/2)) + (\inf_B + (\varepsilon/2))$.
Then $\inf_Q < \inf_A + \inf_B + \varepsilon$, so $\inf_Q \leqslant \inf_A + \inf_B + \varepsilon$.
End of Case (2). QED

13-2. Show: $\forall k \in \mathbb{N}$,

$$\forall S_1, \ldots, S_k \subseteq \mathbb{R}, \quad \mathrm{JO}_{S_1 \cup \cdots \cup S_k} \leqslant \mathrm{JO}_{S_1} + \cdots + \mathrm{JO}_{S_k}.$$

Proof:
Let $A := \{ k \in \mathbb{N} \,|\, \forall S_1, \ldots, S_k \subseteq \mathbb{R},\ \mathrm{JO}_{S_1 \cup \cdots \cup S_k} \leqslant \mathrm{JO}_{S_1} + \cdots + \mathrm{JO}_{S_k} \}$.
We have: $\forall S_1 \subseteq \mathbb{R}$, $\mathrm{JO}_{S_1} \leqslant \mathrm{JO}_{S_1}$. Then $1 \in A$.
By the PMI, want: $\forall k \in A$, $k + 1 \in A$.
Given $k \in A$. **Want:** $k + 1 \in A$.
Know: $\forall S_1, \ldots, S_k \subseteq \mathbb{R}$, $\mathrm{JO}_{S_1 \cup \cdots \cup S_k} \leqslant \mathrm{JO}_{S_1} + \cdots + \mathrm{JO}_{S_k}$.
Want: $\forall S_1, \ldots, S_{k+1} \subseteq \mathbb{R}$, $\mathrm{JO}_{S_1 \cup \cdots \cup S_{k+1}} \leqslant \mathrm{JO}_{S_1} + \cdots + \mathrm{JO}_{S_{k+1}}$.
Given $S_1, \ldots, S_{k+1} \subseteq \mathbb{R}$. **Want:** $\mathrm{JO}_{S_1 \cup \cdots \cup S_{k+1}} \leqslant \mathrm{JO}_{S_1} + \cdots + \mathrm{JO}_{S_{k+1}}$.
Know: $\mathrm{JO}_{S_1 \cup \cdots \cup S_k} \leqslant \mathrm{JO}_{S_1} + \cdots + \mathrm{JO}_{S_k}$.
Then $\mathrm{JO}_{S_1 \cup \cdots \cup S_k} + \mathrm{JO}_{S_{k+1}} \leqslant (\mathrm{JO}_{S_1} + \cdots + \mathrm{JO}_{S_k}) + \mathrm{JO}_{S_{k+1}}$.
Then $\mathrm{JO}_{S_1 \cup \cdots \cup S_k} + \mathrm{JO}_{S_{k+1}} \leqslant \mathrm{JO}_{S_1} + \cdots + \mathrm{JO}_{S_{k+1}}$.
By a class theorem, $\mathrm{JO}_{(S_1 \cup \cdots \cup S_k) \cup S_{k+1}} = \mathrm{JO}_{S_1 \cup \cdots \cup S_k} + \mathrm{JO}_{S_{k+1}}$.
Then $\mathrm{JO}_{S_1 \cup \cdots \cup S_{k+1}} = \mathrm{JO}_{S_1 \cup \cdots \cup S_k} + \mathrm{JO}_{S_{k+1}}$.

Then $\mathrm{JO}_{S_1\cup\cdots\cup S_{k+1}} = \mathrm{JO}_{S_1\cup\cdots\cup S_k} + \mathrm{JO}_{S_{k+1}} \leqslant \mathrm{JO}_{S_1} + \cdots + \mathrm{JO}_{S_{k+1}}$. QED

13-3. Show: Let $k \in \mathbb{N}$. Assume: $\forall \alpha \in \mathbb{R}, \forall \beta \geqslant \alpha,\ \mathrm{JO}^k_{[\alpha;\beta]} \geqslant \beta - \alpha$. Let $a \in \mathbb{R}$. Let $b \geqslant a$. Let $I := [a;b]$. Then $\mathrm{JO}^{k+1}_I \geqslant b - a$.

Proof: We have: $\forall \alpha \in \mathbb{R}, \forall \beta < \alpha,\ \mathrm{JO}^k_{[\alpha;\beta]} = \mathrm{JO}^k_{\varnothing} = 0 > \beta - \alpha$.
Then: $\forall \alpha, \beta \in \mathbb{R},\ \mathrm{JO}^k_{[\alpha;\beta]} \geqslant \beta - \alpha$.
Want: $\inf(\mathrm{TJOC}^{k+1}_I) \geqslant b - a$. **Want:** $\mathrm{TJOC}^{k+1}_I \geqslant b - a$.
Assume: $\neg(\mathrm{TJOC}^{k+1}_I \geqslant b - a)$. **Want:** Contradiction.
Choose $v \in \mathrm{TJOC}^{k+1}_I$ s.t. $v < b - a$.
Choose $U \in \mathrm{JOC}^{k+1}_I$ s.t. $v = TL_U$.
Since $U \in \mathrm{JOC}^{k+1}_I$, we know:
$U \in \mathcal{OI}^{\mathbb{N}}$ and $\#\mathrm{supp}\, U \leqslant k + 1$ and $\bigcup \mathbb{I}_U \supseteq I$.
Since $b \in [a;b] = I \subseteq \bigcup \mathbb{I}_U$, choose $W \in \mathbb{I}_U$ s.t. (if you wish)$b \in W$.
Since $W \in \mathbb{I}_U$ and since $U \in \mathcal{OI}^{\mathbb{N}}$, choose $j \in \mathbb{N}$ s.t. $W = U_j$.
Since $b \in W = U_j$, we get: $b \in U_j$. Then $U_j \neq \varnothing$.
Since $U_j \in \mathcal{OI}$, choose $\beta, \gamma \in \mathbb{R}$ s.t. $U_j = (\beta;\gamma)$.
Then, since $U_j \neq \varnothing$, we get: $\beta < \gamma$. Then $L_{U_j} = \gamma - \beta$.
Since $b \in U_j = (\beta;\gamma)$, we get: $\beta < b < \gamma$.
Then $I \backslash U_j = [a;b] \backslash (\beta;\gamma) = [a;\beta]$.
Let $\alpha := a$. Then $\mathrm{JO}^k_{[\alpha;\beta]} \geqslant \beta - \alpha$. Also, $I \backslash U_j = [\alpha;\beta]$.
Let $J := [\alpha;\beta]$. Then $\mathrm{JO}^k_J \geqslant \beta - \alpha$. Also, $I \backslash U_j = J$.

Define $V \in \mathcal{OI}^{\mathbb{N}}$ by: $\forall i \in \mathbb{N},\ V_i = \begin{cases} U_i, & \text{if } i \neq j \\ \varnothing, & \text{if } i = j. \end{cases}$

Then: $V \in \mathcal{OI}^{\mathbb{N}}$ and $\#\mathrm{supp}\, V \leqslant k$ and
$\bigcup \mathbb{I}_V \supseteq (\bigcup \mathbb{I}_U) \backslash U_j$ and $\mathrm{TL}_V = \mathrm{TL}_U - L_{U_j}$.
We have: $\bigcup \mathbb{I}_V \supseteq (\bigcup \mathbb{I}_U) \backslash U_j \supseteq I \backslash U_j = J$.
Then $V \in \mathrm{JOC}^k_J$. Then $\mathrm{TL}_V \in \mathrm{TJOC}^k_J$.
Then $\mathrm{TL}_V \geqslant \inf(\mathrm{TJOC}^k_J) = \mathrm{JO}^k_J \geqslant \beta - \alpha$, so $\mathrm{TL}_V \geqslant \beta - \alpha$.
Then $v = \mathrm{TL}_U = \mathrm{TL}_V + L_{U_j} = \mathrm{TL}_V + (\gamma - \beta) \geqslant (\beta - \alpha) + (\gamma - \beta)$.
Then $v \geqslant \gamma - \alpha$. Recall: $b < \gamma$ and $a = \alpha$.
Then $v \geqslant \gamma - \alpha > b - \alpha = b - a$, so $v > b - a$.
However, by choice of v, we have: $v < b - a$. Contradiction. QED

13-4. Show: Let $T \subseteq \mathbb{R}$. Then $\mathrm{Int}\, T$ is open.

Proof: **Want:** $\forall a \in \operatorname{Int} T$, $\exists B \in \mathcal{B}(x)$ s.t. $B \subseteq \operatorname{Int} T$.
Given $a \in \operatorname{Int} T$. **Want:** $\exists B \in \mathcal{B}(x)$ s.t. $B \subseteq \operatorname{Int} T$.
Since $a \in \operatorname{Int} T$, choose $B \in \mathcal{B}(x)$ s.t. $B \subseteq T$.
Then $B \in \mathcal{B}(x)$. **Want:** $B \subseteq \operatorname{Int} T$.
Since $B \in \mathcal{B}(x) \subseteq \mathcal{B}_{\mathbb{R}}$, by a class theorem,
we conclude that B is open.
Then $B = \operatorname{Int} B$.
Since $B \subseteq T$, by a class theorem,
we conclude that $\operatorname{Int} B \subseteq \operatorname{Int} T$.
Then $B = \operatorname{Int} B \subseteq \operatorname{Int} T$. QED

13-5. Show: Let $S \subseteq \mathbb{R}$. Then $\operatorname{Cl} S$ is closed.

Proof: Let $T := \mathbb{R} \backslash S$.
By a class theorem $\partial T = \partial S$.
Then $\operatorname{Int} T = T \backslash \partial T = T \backslash \partial S \subseteq (\mathbb{R} \backslash S) \backslash \partial S \subseteq \mathbb{R} \backslash (S \cup \partial S) = \mathbb{R} \backslash (\operatorname{Cl} S)$,
so $\operatorname{Int} T = \mathbb{R} \backslash (\operatorname{Cl} S)$, so $\mathbb{R} \backslash (\operatorname{Int} T) = \operatorname{Cl} S$.
Want: $\mathbb{R} \backslash (\operatorname{Int} T)$ is closed.
By HW#13-4, we get: $\operatorname{Int} T$ is open.
Then, by a class theorem, $\mathbb{R} \backslash (\operatorname{Int} T)$ is closed. QED

Homework 12: Due on Tuesday 6 December

12-1. Using the PMI, show: Let $a \in \mathbb{R}$. Then:

$$\forall j \in \mathbb{N}, \qquad ((\bullet)^j)_a^{\mathbb{T}} - j \cdot a^{j-1} \cdot (\bullet) \in \mathcal{O}(1).$$

Proof: Let $I := \{ j \in \mathbb{N} \,|\, ((\bullet)^j)_a^{\mathbb{T}} - j \cdot a^{j-1} \cdot (\bullet) \in \mathcal{O}(1) \}$. **Want:** $I = \mathbb{N}$.
We have: $((\bullet)^1)_a^{\mathbb{T}} - 1 \cdot a^{1-1} \cdot (\bullet) = ((\bullet))_a^{\mathbb{T}} - 1 \cdot 1 \cdot (\bullet)$
$= (\bullet) - (\bullet) = \mathbf{0} \in \mathcal{O}(1)$. Then $1 \in I$.
By the PMI, want: $\forall j \in I$, $j+1 \in I$.
Given $j \in I$. **Want:** $j+1 \in I$.
Know: $((\bullet)^j)_a^{\mathbb{T}} - j \cdot a^{j-1} \cdot (\bullet) \in \mathcal{O}(1)$.
Want: $((\bullet)^{j+1})_a^{\mathbb{T}} - (j+1) \cdot a^{(j+1)-1} \cdot (\bullet) \in \mathcal{O}(1)$.
Let $R := ((\bullet)^j)_a^{\mathbb{T}} - j \cdot a^{j-1} \cdot (\bullet)$.
Let $S := (\bullet)^{j+1})_a^{\mathbb{T}} - (j+1) \cdot a^{(j+1)-1} \cdot (\bullet)$.
Know: $R \in \mathcal{O}(1)$. **Want:** $S \in \mathcal{O}(1)$.
We have: $\forall h \in \mathbb{R}$,

$$R_h = (a+h)^j - a^j - j \cdot a^{j-1} \cdot h,$$

and so $$(a+h)^j = a^j + j \cdot a^{j-1} \cdot h + R_h.$$

We have: $\forall h \in \mathbb{R}$,

$$\begin{aligned}
S_h &= (a+h)^{j+1} - a^{j+1} - (j+1) \cdot a^{(j+1)-1} \cdot h \\
&= (a+h) \cdot (a+h)^j - a^{j+1} - (j+1) \cdot a^j \cdot h \\
&= (a+h) \cdot (a^j + j \cdot a^{j-1} \cdot h + R_h) - a^{j+1} - (j+1) \cdot a^j \cdot h \\
&= \quad a \cdot a^j + a \cdot j \cdot a^{j-1} \cdot h + a \cdot R_h \\
&\quad + h \cdot a^j + h \cdot j \cdot a^{j-1} \cdot h + h \cdot R_h \\
&\quad\quad - a^{j+1} - (j+1) \cdot a^j \cdot h \\
&= \quad a^{j+1} + \quad j \cdot a^j \cdot h + a \cdot R_h \\
&\quad\quad + \quad a^j \cdot h + \quad j \cdot a^{j-1} \cdot h^2 + h \cdot R_h \\
&\quad - a^{j+1} - (j+1) \cdot a^j \cdot h \\
&= \quad a \cdot R_h + j \cdot a^{j-1} \cdot h^2 + h \cdot R_h \\
&= (a \cdot R + j \cdot a^{j-1} \cdot (\bullet)^2 + (\bullet) \cdot R)_h.
\end{aligned}$$

Then $S = a \cdot R + j \cdot a^{j-1} \cdot (\bullet)^2 + (\bullet) \cdot R$.

By algebraic linearity of $\mathcal{O}(1)$, want: $a \cdot R\,,\ j \cdot a^{j-1} \cdot (\bullet)^2\,,\ (\bullet) \cdot R \in \mathcal{O}(1)$.

Since $R \in \mathcal{O}(1)$, by algebraic linearaity of $\mathcal{O}(1)$,

we conclude: $a \cdot R \in \mathcal{O}(1)$.

We have $j \cdot a^{j-1} \cdot (\bullet)^2 \in \mathcal{Q} = \mathcal{H}_2 \subseteq \widehat{\mathcal{O}}(2) \subseteq \mathcal{O}(1)$.

Want: $(\bullet) \cdot R \in \mathcal{O}(1)$.

Since $(\bullet) \in \mathcal{L} = \mathcal{H}_1 \subseteq \widehat{\mathcal{O}}(1)$ and $R \in \mathcal{O}(1)$,

we conclude: $(\bullet) \cdot R \in \mathcal{O}(1+1)$.

Then $(\bullet) \cdot R \in \mathcal{O}(1+1) = \mathcal{O}(2) \subseteq \mathcal{O}(1)$. QED

12-2. Show: Let $j \in \mathbb{N}$. Then $((\bullet)^j)' = j \cdot (\bullet)^{j-1}$.

Proof: **Want:** $\forall a \in \mathbb{R}$, $((\bullet)^j)'_a = (j \cdot (\bullet)^{j-1})_a$.

Given $a \in \mathbb{R}$. **Want:** $((\bullet)^j)'_a = (j \cdot (\bullet)^{j-1})_a$.

Want: $((\bullet)^j)'_a = j \cdot a^{j-1}$. By HW#12-1, $((\bullet)^j)^{\mathbb{T}}_a - j \cdot a^{j-1} \cdot (\bullet) \in \mathcal{O}(1)$.

Then $j \cdot a^{j-1} \cdot (\bullet) \in \mathrm{LINS}_a(\,(\bullet)^j\,)$, so, by uniqueness of linearization,

we get: $\mathrm{LINS}_a(\,(\bullet)^j\,) = \{\, j \cdot a^{j-1} \cdot (\bullet) \,\}$.

Then $D_a(\,(\bullet)^j\,) = \mathrm{UE}(\mathrm{LINS}_a(\,(\bullet)^j\,) = j \cdot a^{j-1} \cdot (\bullet)$.

Then $((\bullet)^j)'_a = [\,D_a(\,(\bullet)^j\,)\,] = [\,j \cdot a^{j-1} \cdot (\bullet)\,]$

$= (\,j \cdot a^{j-1} \cdot (\bullet)\,)_1 = j \cdot a^{j-1} \cdot 1 = j \cdot a^{j-1}$. QED

12-3. Show: Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $y \in \mathbb{R}_0^\times$.

Assume: as $x \to -\infty$, $f_x \to y$.

Then: as $x \to -\infty$, $(1/f)_x \to 1/y$.

Proof: **Want:** $\forall \varepsilon > 0$, $\exists N \in \mathbb{R}$ s.t., $\forall x \in \mathbb{D}_{1/f}$,

$(\,x < N\,) \Rightarrow (\,|(1/f)_x - (1/y)| < \varepsilon\,)$.

Given $\varepsilon > 0$. **Want:** $\exists N \in \mathbb{R}$ s.t., $\forall x \in \mathbb{D}_{1/f}$,

$$(\, x < N \,) \Rightarrow (\, |(1/f)_x - (1/y)| < \varepsilon \,).$$

Since $y \in \mathbb{R}_0^\times$, we get $|y| > 0$.
Then $|y|/2 > 0$ and $(|y|/2) \cdot |y| > 0$ and $1/|y| > 0$ and $2/|y| > 0$.
Let $\rho \;:=\; \min\{\, |y|/2 \,,\, \varepsilon \cdot (|y|/2) \cdot |y| \,\}$.
Then $\rho > 0$ and $\rho \leqslant |y|/2$ and $\rho \leqslant \varepsilon \cdot (|y|/2) \cdot |y|$.
Since $\rho \leqslant |y|/2$, we get $-\rho \geqslant -|y|/2$, and so $|y| - \rho \geqslant |y| - (|y|/2)$.
Since $\rho \leqslant \varepsilon \cdot (|y|/2) \cdot |y|$, we get: $\rho \cdot (2/|y|) \cdot (1/|y|) \leqslant \varepsilon$.
Since as $x \to -\infty$, $f_x \to y$,
choose $N \in \mathbb{R}$ s.t., $\forall x \in \mathbb{D}_f$, $(\, x < N \,) \Rightarrow (\, |f_x - y| < \rho \,)$.
Then $N \in \mathbb{R}$. **Want:** $\forall x \in \mathbb{D}_{1/f}$, $(\, x < N \,) \Rightarrow (\, |(1/f)_x - (1/y)| < \varepsilon \,)$.
Given $x \in \mathbb{D}_{1/f}$. Assume: $x < N$. **Want:** $|(1/f)_x - (1/y)| < \varepsilon$.
We have $x \in \mathbb{D}_{1/f} \subseteq \mathbb{D}_f$ and $x < N$,
so, by choice of N, we get: $|f_x - y| < \rho$.
Since $|\bullet|$ is Lipschitz-1, we get: $|\,|f_x| - |y|\,| \leqslant |f_x - y|$.
Then $|\,|f_x| - |y|\,| \leqslant |f_x - y| < \rho$.
Then $|\,|f_x| - |y|\,| < \rho$, so $|y| - \rho < |f_x| < |y| + \rho$.
Then $|f_x| > |y| - \rho \geqslant |y| - (|y|/2) = |y|/2$.
Since $|f_x| > |y|/2 > 0$, it follows that $1/|f_x| < 2/|y|$.
We compute: $|(1/f)_x - (1/y)| = \left| \dfrac{1}{f_x} - \dfrac{1}{y} \right| = \left| \dfrac{y - f_x}{f_x \cdot y} \right| = \dfrac{|y - f_x|}{|f_x| \cdot |y|}$
Then $|(1/f)_x - (1/y)| = |y - f_x| \cdot (1/|f_x|) \cdot (1/|y|)$.
So, since $|y - f_x| = |f_x - y| < \rho$ and $1/|f_x| < 2/|y|$ and $1/|y| > 0$,
we get: $|(1/f)_x - (1/y)| < \rho \cdot (2/|y|) \cdot (1/|y|)$.
So, since $\rho \cdot (2/|y|) \cdot (1/|y|) \leqslant \varepsilon$, we get $|(1/f)_x - (1/y)| < \varepsilon$. QED

12-4. Show: Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{D}_f$.
Assume: f has a local strict-maximum at a.
Then: $f_a^{\mathbb{T}}$ has a local strict-maximum at 0.

Proof: **Want:** $\exists B \in \mathcal{B}(0)$ s.t., $\forall h \in B_0^\times$, $(f_a^{\mathbb{T}})_h < (f_a^{\mathbb{T}})_0$.
Since f has a local strict-maximum at a,
choose $C \in \mathcal{B}(a)$ s.t., $\forall x \in B_a^\times$, $f_x < f_a$.
Let $B := C - a$. Then $B \in \mathcal{B}(a - a) = \mathcal{B}(0)$.
Want: $\forall h \in B_0^\times$, $(f_a^{\mathbb{T}})_h < (f_a^{\mathbb{T}})_0$.
Given $h \in B_0^\times$. **Want:** $(f_a^{\mathbb{T}})_h < (f_a^{\mathbb{T}})_0$.
Let $x := a + h$. Then $x \in B_{a+0}^\times = B_a^\times$.
So, by choice of B, we get: $f_x < f_a$.
So, since $x = a + h$, we get: $f_{a+h} < f_a$. Then $f_{a+h} - f_a < 0$.

Since $a \in \mathbb{D}_f$, we get $f_a - f_a = 0$.
Then $(f_a^{\mathbb{T}})_h = f_{a+h} - f_a < 0 = f_a - f_a = f_{a+0} - f_a = (f_a^{\mathbb{T}})_0$. QED

12-5. Show: Let $s, t \in \mathbb{R}^{\mathbb{N}}$.
Assume: $\mathbb{I}_s$ and $\mathbb{I}_t$ are both bounded.
Then: $\exists$strictly-increasing $\ell \in \mathbb{N}^{\mathbb{N}}$ s.t.
$s \circ \ell$ and $t \circ \ell$ are both convergent.

Proof: By properness of $\mathbb{R}$, since $\mathbb{I}_s$ is bounded, s is subconvergent.
Choose a subsequence σ of s s.t. σ is convergent.
Since σ is a subsequence of s,
choose a strictly-increasing $p \in \mathbb{N}^{\mathbb{N}}$ s.t. $\sigma = s \circ p$.
Since $\mathbb{I}_{t \circ p} \subseteq \mathbb{I}_t$ and since $\mathbb{I}_t$ is bounded,
it follows that $\mathbb{I}_{t \circ p}$ is bounded.
Then, by properness of $\mathbb{R}$, we get: $t \circ p$ is subconvergent.
Choose a subsequence τ of $t \circ p$ s.t. τ is convergent.
Since τ is a subsequence of $t \circ p$,
choose a strictly-increasing $q \in \mathbb{N}^{\mathbb{N}}$ s.t. $\tau = t \circ p \circ q$.
We have $q : \mathbb{N} \to \mathbb{N}$ and $p : \mathbb{N} \to \mathbb{N}$, so $p \circ q : \mathbb{N} \to \mathbb{N}$, so $p \circ q \in \mathbb{N}^{\mathbb{N}}$.
Since q and p are both strictly-increasing, $p \circ q$ is strcitly-increasing.
Let $\ell := p \circ q$. Then $\ell \in \mathbb{N}^{\mathbb{N}}$ and ℓ is strictly-increasing.
Want: $s \circ \ell$ and $t \circ \ell$ are both convergent.
We have $\tau = t \circ p \circ q = t \circ \ell$.
So, since τ is convergent, we get: $t \circ \ell$ is convergent.
Want: $s \circ \ell$ is convergent.
Since $q \in \mathbb{N}^{\mathbb{N}}$ and since q is strictly-increasing,
it follows that $\sigma \circ q$ is a subsequence of σ.
So, since σ is convergent, by a class theorem, $\sigma \circ q$ is convergent.
So, since $s \circ \ell = s \circ p \circ q = \sigma \circ q$, we concude: $s \circ \ell$ is convergent. QED

Homework 11: Due on Tuesday 29 November

11-1. Show: Define $r : \mathbb{R}_0^{\times} \to \mathbb{R}$ by: $\forall x \in \mathbb{R}_0^{\times}$, $r_x = 1/x$.
Then: $\forall a \in \mathbb{R}_0^{\times}$, $r'_a = -\dfrac{1}{a^2}$.

Proof: Given $a \in \mathbb{R}_0^{\times}$. **Want:** $r'_a = -\dfrac{1}{a^2}$.
Let $L := -\dfrac{1}{a^2} \cdot (\bullet)$. Then $[L] = L_1 = -\dfrac{1}{a^2}$. **Want:** $r'_a = [L]$.

Want: $[D_a r] = [L]$. **Want:** $D_a r = L$.
Want: $\mathrm{UE}(\mathrm{LINS}_a r) = L$. **Want:** $\mathrm{LINS}_a r = \{L\}$.
By uniqueness of linearization, **want:** $L \in \mathrm{LINS}_a r$.
Want: $r_a^{\mathbb{T}} - L \in o(1)$. Know: $\widehat{\mathcal{O}}(2) \subseteq o(1)$. **Want:** $r_a^{\mathbb{T}} - L \in \widehat{\mathcal{O}}(2)$.
Want: $\exists C \geqslant 0$, $\exists \delta > 0$ s.t., $\forall h \in \mathbb{R}$, $(|h| < \delta) \Rightarrow (|(r_a^{\mathbb{T}} - L)_h| \leqslant C \cdot |h|^2)$.
Since $a \in \mathbb{R}_0^\times$, we get: $|a| > 0$.
Let $C := \dfrac{2}{|a|^3}$ Then $C > 0$, so $C \geqslant 0$.
Let $\delta := \dfrac{|a|}{2}$. Then $\delta > 0$.
Want: $\forall h \in \mathbb{R}$, $(\,|h| < \delta\,) \Rightarrow (\,|(r_a^{\mathbb{T}} - L)_h| \leqslant C \cdot |h|^2\,)$.
Given $h \in \mathbb{R}$. Assume $|h| < \delta$. **Want:** $|(r_a^{\mathbb{T}} - L)_h| \leqslant C \cdot |h|^2$.
Since $|\,|a + h| - |a|\,| \leqslant |\,(a + h) - a\,| = |h| < \delta$,
we get $|a| - \delta < |a + h| < |a| + \delta$.
Then $|a + h| \geqslant |a| - \delta = |a| - \dfrac{|a|}{2} = \dfrac{|a|}{2}$.
Since $|a + h| > \dfrac{|a|}{2} > 0$, we get: $\dfrac{1}{|a + h|} < \dfrac{1}{|a|/2}$.
Then $\dfrac{|h|^2}{|a + h| \cdot |a|^2} \leqslant \dfrac{|h|^2}{(|a|/2) \cdot |a|^2}$.
We have

$$(r_a^T - L)_h = r_{a+h} - r_a - L_h = \frac{1}{a+h} - \frac{1}{a} - \left(-\frac{1}{a^2} \cdot h\right)$$
$$= \frac{a^2 - (a + h) \cdot a + (a + h) \cdot h}{(a + h) \cdot a^2}$$
$$= \frac{a^2 - a^2 - h \cdot a + a \cdot h + h^2}{(a + h) \cdot a^2} = \frac{h^2}{(a + h) \cdot a^2}.$$

Then $|(r_a^{\mathbb{T}} - L)_h| = \dfrac{|h|^2}{|a + h| \cdot |a|^2} \leqslant \dfrac{|h|^2}{(|a|/2) \cdot |a|^2} = \dfrac{2 \cdot |h|^2}{|a|^3} = C \cdot |h|^2$.
QED

11-2. Show: Let $g : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{D}'_g$.
Assume $g_a \neq 0$. Then: $\left(\dfrac{1}{g}\right)'_a = -\dfrac{g'_a}{g_a^2}$.

Proof: Define $r : \mathbb{R}_0^\times \to \mathbb{R}$ by: $\forall x \in \mathbb{R}_0^\times$, $r_x = 1/x$.
Then $1/g = r \circ g$. **Want:** $(r \circ g)'_a = -\dfrac{g'_a}{g_a^2}$.

Since $g_a \neq 0$, by HW#11-1, we get: $r'_{g_a} = -\dfrac{1}{g_a^2}$.

Then $r'_{g_a} \cdot g'_a = -\dfrac{g'_a}{g_a^2}$. Then $(r \circ g)'_a \;=^*\; r'_{g_a} \cdot g'_a \;=\; -\dfrac{g'_a}{g_a^2}$. QED

11-3. Show: Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{D}'_f \cap \mathbb{D}'_g$.

Assume $g_a \neq 0$. Then: $\left(\dfrac{f}{g}\right)'_a = \dfrac{g_a \cdot f'_a - f_a \cdot g'_a}{g_a^2}$.

Proof: Since $a \in \mathbb{D}'_g$ and $g_a \neq 0$, by HW#11-2, we get: $\left(\dfrac{1}{g}\right)'_a = -\dfrac{g'_a}{g_a^2}$.

Let $h := \dfrac{1}{g}$. Then $h'_a = -\dfrac{g'_a}{g_a^2}$. Also, $h_a = \dfrac{1}{g_a}$.

Then $\left(\dfrac{f}{g}\right)'_a \;=\; (f \cdot h)'_a \;=^*\; f'_a \cdot h_a + f_a \cdot h'_a$

$= \dfrac{f'_a}{g_a} - \dfrac{f_a \cdot g'_a}{g_a^2} \;=\; \dfrac{g_a \cdot f'_a - f_a \cdot g'_a}{g_a^2}$. QED

11-4. Show: Let $U, V \subseteq \mathbb{R}$ both be open. Then:

(i) $U \cup V$ is open and (ii) $U \cap V$ is open.

Proof: Proof of (i):

Want: $\forall a \in U \cup V$, $\exists B \in \mathcal{B}(a)$ s.t. $B \subseteq U \cup V$.

Given $a \in U \cup V$. **Want:** $\exists B \in \mathcal{B}(a)$ s.t. $B \subseteq U \cup V$.

Since $a \in U \cup V$, choose $T \in \{U, V\}$ s.t. $a \in T$.

Since $T \in \{U, V\}$, it follows that T is open and that $T \subseteq U \cup V$.

Since T is open and $a \in T$, choose $B \in \mathcal{B}(a)$ s.t. $B \subseteq T$.

Then $B \in \mathcal{B}(a)$. **Want:** $B \subseteq U \cup V$. We have $B \subseteq T \subseteq U \cup V$.

End of proof of (i).

Proof of (ii):

Want: $\forall a \in U \cap V$, $\exists B \in \mathcal{B}(a)$ s.t. $B \subseteq U \cap V$.

Given $a \in U \cap V$. **Want:** $\exists B \in \mathcal{B}(a)$ s.t. $B \subseteq U \cap V$.

Since $a \in U \cap V \subseteq U$ and since U is open,

choose $\lambda > 0$ s.t. $B(a, \lambda) \subseteq U$.

Since $a \in U \cap V \subseteq V$ and since V is open,

choose $\mu > 0$ s.t. $B(a, \mu) \subseteq V$.

Let $\delta := \min\{\lambda, \mu\}$. Then $\delta > 0$ and $\delta \leqslant \lambda$ and $\delta \leqslant \mu$.

Let $B := B(a, \delta)$. Then $B \in \mathcal{B}(a)$. **Want:** $B \subseteq U \cap V$.

Since $\delta \leqslant \lambda$, we get $B(a, \delta) \subseteq B(a, \lambda)$.

Since $\delta \leqslant \mu$, we get $B(a,\delta) \subseteq B(a,\mu)$.
Since $B = B(a,\delta) \subseteq B(a,\lambda) \subseteq U$ and $B = B(a,\delta) \subseteq B(a,\mu) \subseteq V$,
we get: $B \subseteq U \cap V$.
End of proof of (ii). QED

11-5. Show: Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$. Assume: $\forall$open $U \subseteq \mathbb{R}$, f^*U is open.
Then: f is continuous.

Proof: **Want:** $\forall a \in \mathbb{D}_f$, f is continuous at a.
Given $a \in \mathbb{D}_f$. **Want:** f is continuous at a.
Want: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall x \in \mathbb{D}_f$,
$$(\,|x-a| < \delta\,) \Rightarrow (\,|f_x - f_a| < \varepsilon\,).$$
Given $\varepsilon > 0$. **Want:** $\exists \delta > 0$ s.t., $\forall x \in \mathbb{D}_f$,
$$(\,|x-a| < \delta\,) \Rightarrow (\,|f_x - f_a| < \varepsilon\,).$$
Let $U := B(f_a, \varepsilon)$. By a class theorem, U is open.
Then, by assumption, f^*U is open.
Since $|f_a - f_a| = 0 < \varepsilon$, we get: $f_a \in B(f_a, \varepsilon)$.
Since $f_a \in B(f_a,\varepsilon) = U$, we get: $a \in f^*U$.
Since f^*U is open and $a \in f^*U$, choose $B \in \mathcal{B}(a)$ s.t. $B \subseteq f^*U$.
Since $B \in \mathcal{B}(a)$, choose $\delta > 0$ s.t. $B = B(a,\delta)$. Then $\delta > 0$.
Want: $\forall x \in \mathbb{D}_f$, $(\,|x-a| < \delta\,) \Rightarrow (\,|f_x - f_a| < \varepsilon\,)$.
Given $x \in \mathbb{D}_f$. Assume: $|x-a| < \delta$. **Want:** $|f_x - f_a| < \varepsilon$.
Since $|x-a| < \delta$, we get: $x \in B(a,\delta)$.
Since $x \in B(a,\delta) = B \subseteq f^*U$, we get: $f_x \in U$.
Since $f_x \in U = B(f_a,\varepsilon)$, we get: $|f_x - f_a| < \varepsilon$. QED

Homework 10: Due on Tuesday 22 November

10-1. Show: Let $w, x \in \mathbb{R}^{\mathbb{N}}$, $\ell \in \mathbb{N}^{\mathbb{N}}$, $q \in \mathbb{R}$.
Assume: (ℓ is strictly-increasing) & ($w \circ \ell \to q$).
Assume: $\forall j \in \mathbb{N}$, $|w_j - x_j| < 1/j$.
Then: $x \circ \ell \to q$.

Proof: **Want:** $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\,j \geqslant K\,) \Rightarrow (\,|(x \circ \ell)_j - q| < \varepsilon\,).$$
Given $\varepsilon > 0$. **Want:** $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\,j \geqslant K\,) \Rightarrow (\,|(x \circ \ell)_j - q| < \varepsilon\,).$$
Since $w \circ \ell \to q$, choose $L \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\,j \geqslant K\,) \Rightarrow (\,|(w \circ \ell)_j - q| < \varepsilon/2\,).$$

By the AP, choose $M \in \mathbb{N}$ s.t. $M > 2/\varepsilon$.
Let $K := \max\{L, M\}$. Then $K \geqslant L$ and $K \geqslant M$ and $K \in \mathbb{N}$.
Want: $\forall j \in \mathbb{N}$, $(j \geqslant K) \Rightarrow (|(x \circ \ell)_j - q| < \varepsilon)$.
Given $j \in \mathbb{N}$. Assume $j \geqslant K$. **Want:** $|(x \circ \ell)_j - q| < \varepsilon$.
Since $j \geqslant K \geqslant L$, by choice of L, we get: $|(w \circ \ell)_j - q| < \varepsilon/2$.
By assumption, $|x_{\ell_j} - w_{\ell_j}| < 1/\ell_j$.
By a class theorem, $\ell_j \geqslant j$. Then $\ell_j \geqslant j \geqslant K \geqslant M > 2/\varepsilon$.
Since $\ell_j \geqslant 2/\varepsilon > 0$, we get: $1/\ell_j < \varepsilon/2$.
Then $|x_{\ell_j} - w_{\ell_j}| < 1/\ell_j < \varepsilon/2$, so $|x_{\ell_j} - w_{\ell_j}| < \varepsilon/2$.
Then
$$\begin{aligned} |(x \circ \ell)_j - q| &\leqslant |(x \circ \ell)_j - (w \circ \ell)_j| + |(w \circ \ell)_j - q| \\ &= |x_{\ell_j} - w_{\ell_j}| + |(w \circ \ell)_j - q| \\ &< (\varepsilon/2) + (\varepsilon/2) \quad = \quad \varepsilon. \end{aligned}$$
QED

10-2. Show: Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$.
Assume: as $x \to \infty$, $f_x \to -\infty$.
Assume: as $x \to \infty$, $g_x \to -\infty$.
Then: as $x \to \infty$, $(f \cdot g)_x \to \infty$.

Proof: **Want:** $\forall M \in \mathbb{R}$, $\exists L \in \mathbb{R}$ s.t., $\forall x \in \mathbb{R}$,
$$(x > L) \Rightarrow ((f \cdot g)_x > M).$$
Given $M \in \mathbb{R}$. **Want:** $\exists L \in \mathbb{R}$ s.t., $\forall x \in \mathbb{R}$,
$$(x > L) \Rightarrow ((f \cdot g)_x > M).$$
Let $N := -\sqrt{\max\{M, 0\}}$. Then $N \in \mathbb{R}$ and $N \leqslant 0$.
Also, $N^2 = \max\{M, 0\}$, so $N^2 \geqslant M$.
Since as $x \to \infty$, $f_x \to -\infty$, choose $A \in \mathbb{R}$ s.t., $\forall x \in \mathbb{R}$,
$$(x > A) \Rightarrow (f_x < N).$$
Since as $x \to \infty$, $g_x \to -\infty$, choose $B \in \mathbb{R}$ s.t., $\forall x \in \mathbb{R}$,
$$(x > B) \Rightarrow (g_x < N).$$
Let $L := \max\{A, B\}$. Then $L \geqslant A$ and $L \geqslant B$ and $L \in \mathbb{R}$.
Want: $\forall x \in \mathbb{R}$, $(x > L) \Rightarrow ((f \cdot g)_x > M)$.
Given $x \in \mathbb{R}$. Assume $x > L$. **Want:** $(f \cdot g)_x > M$.
Since $x > L \geqslant A$, by choice of A, we have: $f_x < N$.
Then $f_x < N \leqslant 0$, so $-f_x > -N \geqslant 0$.
Since $x > L \geqslant B$, by choice of B, we have: $g_x < N$.
Then $g_x < N \leqslant 0$, so $-g_x > -N \geqslant 0$.
Since $-f_x > -N \geqslant 0$ and $-g_x > -N \geqslant 0$,
we get $(-f_x) \cdot (-g_x) > (-N) \cdot (-N)$, so $f_x \cdot g_x > N^2$.
Then $(f \cdot g)_x = f_x \cdot g_x > N^2 \geqslant M$. QED

10-3. Show: Let $f := (\bullet)^3$. Then $D_2 f = 3 \cdot 2^2 \cdot (\bullet)$.

Proof: Let $x := 2$. **Want:** $D_x f = 3 \cdot x^2 \cdot (\bullet)$.
Since $\mathbb{D}_f = \mathbb{R}$, we see that $\mathbb{D}_{f_x^{\mathbb{T}}} = \mathbb{R}$.
We have: $\forall h \in \mathbb{R}$,

$$\begin{aligned}(f_x^{\mathbb{T}} - 3 \cdot x^2 \cdot (\bullet))_h &= f_{x+h} - f_x - 3x^2 h \\ &= (x+h)^3 - x^3 - 3x^2 h \\ &= x^3 + 3x^2 h + 3xh^2 + h^3 - x^3 - 3x^2 h \\ &= 3xh^2 + h^3 \ = \ ((3 \cdot x \cdot (\bullet)^2 + (\bullet)^3)_h,\end{aligned}$$

and so $f_x^{\mathbb{T}} - 3 \cdot x^2 \cdot (\bullet) = 3 \cdot x \cdot (\bullet)^2 + (\bullet)^3$.
We have $3 \cdot x \cdot (\bullet)^2 \in \mathcal{H}_2 \subseteq \widehat{\mathcal{O}}(2) \subseteq \mathcal{o}(1)$.
Also, $(\bullet)^3 \in \mathcal{H}_3 \subseteq \widehat{\mathcal{O}}(3) \subseteq \mathcal{o}(1)$.
Since $3 \cdot x \cdot (\bullet)^2 \in \mathcal{o}(1)$ and $(\bullet)^3 \in \mathcal{o}(1)$,
and since $\mathcal{o}(1)$ is linearly closed, we get:

$$3 \cdot x \cdot (\bullet)^2 + (\bullet)^3 \in \mathcal{o}(1).$$

Since $f_x^{\mathbb{T}} - 3 \cdot x^2 \cdot (\bullet) = 3 \cdot x \cdot (\bullet)^2 + (\bullet)^3 \in \mathcal{o}(1)$,
we get $3 \cdot x^2 \cdot (\bullet) \in \mathrm{LINS}_x f$.
Then, by uniqueness of linearization, $\mathrm{LINS}_x f = \{3 \cdot x^2 \cdot (\bullet)\}$.
Then $D_x f \ = \ \mathrm{UE}(\mathrm{LINS}_x f) \ = \ \mathrm{UE}\{3 \cdot x^2 \cdot (\bullet)\} \ = \ 3 \cdot x^2 \cdot (\bullet)$. QED

10-4. Show: Let $f := (\bullet)^3$. Then $\forall x \in \mathbb{R}$, $f'_x = 3x^2$.

Proof: Since $\mathbb{D}_f = \mathbb{R}$, we see that $\mathbb{D}_{f_x^{\mathbb{T}}} = \mathbb{R}$.
We have: $\forall h \in \mathbb{R}$,

$$\begin{aligned}(f_x^{\mathbb{T}} - 3 \cdot x^2 \cdot (\bullet))_h &= f_{x+h} - f_x - 3x^2 h \\ &= (x+h)^3 - x^3 - 3x^2 h \\ &= x^3 + 3x^2 h + 3xh^2 + h^3 - x^3 - 3x^2 h \\ &= 3xh^2 + h^3 \ = \ ((3 \cdot x \cdot (\bullet)^2 + (\bullet)^3)_h,\end{aligned}$$

and so $f_x^{\mathbb{T}} - 3 \cdot x^2 \cdot (\bullet) = 3 \cdot x \cdot (\bullet)^2 + (\bullet)^3$.
We have $3 \cdot x \cdot (\bullet)^2 \in \mathcal{H}_2 \subseteq \widehat{\mathcal{O}}(2) \subseteq \mathcal{o}(1)$.
Also, $(\bullet)^3 \in \mathcal{H}_3 \subseteq \widehat{\mathcal{O}}(3) \subseteq \mathcal{o}(1)$.
Since $3 \cdot x \cdot (\bullet)^2 \in \mathcal{o}(1)$ and $(\bullet)^3 \in \mathcal{o}(1)$,
and since $\mathcal{o}(1)$ is linearly closed, we get:

$$3 \cdot x \cdot (\bullet)^2 + (\bullet)^3 \in \mathcal{o}(1).$$

Since $f_x^{\mathbb{T}} - 3 \cdot x^2 \cdot (\bullet) = 3 \cdot x \cdot (\bullet)^2 + (\bullet)^3 \in \mathcal{o}(1)$,
we get $3 \cdot x^2 \cdot (\bullet) \in \mathrm{LINS}_x f$.
Then, by uniqueness of linearization, $\mathrm{LINS}_x f = \{3 \cdot x^2 \cdot (\bullet)\}$.
Then $D_x f \ = \ \mathrm{UE}(\mathrm{LINS}_x f) \ = \ \mathrm{UE}\{3 \cdot x^2 \cdot (\bullet)\} \ = \ 3 \cdot x^2 \cdot (\bullet)$.

Then $f'_x = [D_x f] = [3 \cdot x^2 \cdot (\bullet)]$
$\stackrel{*}{=} (3 \cdot x^2 \cdot (\bullet))_1 = 3 \cdot x^2 \cdot 1 = 3x^2$. QED

10-5. Show: Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a, c \in \mathbb{R}$.
Then $D_a(c \cdot f) =^* c \cdot (D_a f)$.

Proof: **Want:** $(c \cdot (D_a f) \neq ☹) \Rightarrow (D_a(c \cdot f) = c \cdot (D_a f))$.
Assume: $c \cdot (D_a f) \neq ☹$. **Want:** $D_a(c \cdot f) = c \cdot (D_a f)$.
Since $c \cdot (D_a f)$☹, we get: $D_a f \neq ☹$. Then $D_a f \in \mathrm{LINS}_a f$.
Let $L := D_a f$. Then $L \in \mathrm{LINS}_a f$, so $f_a^{\mathbb{T}} - L \in \mathcal{O}(1)$.
So, since $\mathcal{O}(1)$ is linearly closed, we get: $c \cdot (f_a^{\mathbb{T}} - L) \in \mathcal{O}(1)$.
Then $(c \cdot f)_a^{\mathbb{T}} - c \cdot L = c \cdot f_a^{\mathbb{T}} - c \cdot L = c \cdot (f_a^{\mathbb{T}} - L) \in \mathcal{O}(1)$,
and so $c \cdot L \in \mathrm{LINS}_a(c \cdot f)$.
Then, by uniqueness of linearization, we get $\mathrm{LINS}_a(c \cdot f) = \{c \cdot L\}$.
Then $D_a(c \cdot f) = \mathrm{UE}(\mathrm{LINS}_a(c \cdot f)) = \mathrm{UE}\{c \cdot L\} = c \cdot L$.
Then $D_a(c \cdot f) = c \cdot L = c \cdot (D_a f)$. QED

Homework 9: Due on Tuesday 15 November

9-1. Show: Define $f : \mathbb{R} \dashrightarrow \mathbb{R}$ by: $\forall x \in \mathbb{R}$, $f_x = 1/x$.
Then: as $x \to -\infty$, $f_x \to 0$.

Proof: **Want:** $\forall \varepsilon > 0$, $\exists N \in \mathbb{R}$ s.t., $\forall x \in \mathbb{D}_f$,
$(x < N) \Rightarrow (|f_x - 0| < \varepsilon)$.
Given $\varepsilon > 0$. **Want:** $\exists N \in \mathbb{R}$ s.t., $\forall x \in \mathbb{D}_f$,
$(x < N) \Rightarrow (|f_x - 0| < \varepsilon)$.
Let $N := -1/\varepsilon$. Then $-1/N = \varepsilon$.
Since $\varepsilon > 0$, we get: $N \in \mathbb{R}$ and $N < 0$.
Want: $\forall x \in \mathbb{D}_f$, $(x < N) \Rightarrow (|f_x - 0| < \varepsilon)$.
Given $x \in \mathbb{D}_f$. Assume: $x < N$. **Want:** $|f_x - 0| < \varepsilon$.
We have $x < N < 0$, so $x < 0$, so $1/x < 0$, so $|1/x| = -1/x$.
Then $|f_x - 0| = |f_x| = |1/x| = -1/x$. Also, $\varepsilon = -1/N$.
Want: $-1/x < -1/N$.
We have $x < N < 0$, so $-x > -N > 0$, so $1/(-x) < 1/(-N)$.
Then $-1/x = 1/(-x) < 1/(-N) = -1/N$. QED

9-2. Show: Let $D \subseteq \mathbb{R}$, $s \in (\mathbb{R}^D)^{\mathbb{N}}$, $f \in \mathbb{R}^D$.
Assume: $\forall j \in \mathbb{N}$, s_j is continuous.

Assume: $s \to f$ uniformly.
Then: f is continuous.

Proof: **Want:** $\forall a \in \mathbb{D}_f$, f is continuous at a.
Given $a \in \mathbb{D}_f$. **Want:** f is continuous at a.
Want: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall x \in \mathbb{D}_f$,
$$(\,|x - a| < \delta\,) \Rightarrow (\,|[f(x)] - [f(a)]| < \varepsilon\,).$$
Given $\varepsilon > 0$. **Want:** $\exists \delta > 0$ s.t., $\forall x \in \mathbb{D}_f$,
$$(\,|x - a| < \delta\,) \Rightarrow (\,|[f(x)] - [f(a)]| < \varepsilon\,).$$
Since $s \to f$ uniformly, choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$, $\forall x \in D$
$$(\,j \geqslant K\,) \Rightarrow (\,|[s_j(x)] - [f(x)]| < \varepsilon/3\,).$$
By assumption, $s \in (\mathbb{R}^D)^{\mathbb{N}}$, so $s_K \in \mathbb{R}^D$, so $\mathbb{D}_{s_K} = D$.
Also, $f \in \mathbb{R}^D$, so $\mathbb{D}_f = D$.
By assumption, $\forall j \in \mathbb{N}$, s_j is continuous. Then s_K is continuous.
So, since $a \in \mathbb{D}_f = D = \mathbb{D}_{s_K}$, we see that s_K is continuous at a.
Choose $\delta > 0$ s.t., $\forall x \in \mathbb{D}_{s_K}$,
$$(\,|x - a| < \delta\,) \Rightarrow (\,|[s_K(x)] - [s_K(a)]| < \varepsilon/3\,).$$
Then $\delta > 0$. **Want:** $\forall x \in \mathbb{D}_f$,
$$(\,|x - a| < \delta\,) \Rightarrow (\,|[f(x)] - [f(a)]| < \varepsilon\,).$$
Given $x \in \mathbb{D}_f$. Assume: $|x - a| < \delta$. **Want:** $|[f(x)] - [f(a)]| < \varepsilon$.
We have $x \in D_f = D$, so $x \in D$. Then $x \in D = \mathbb{D}_{s_K}$, so $x \in \mathbb{D}_{s_K}$.
Since $K \geqslant K$ and $x \in D$, by choice of K, $|[s_K(x)] - [f(x)]| < \varepsilon/3$.
Since $|x - a| < \delta$, by choice of δ, $|[s_K(x)] - [s_K(a)]| < \varepsilon/3$.
Since $K \geqslant K$ and $a \in D$, by choice of K, $|[s_K(a)] - [f(a)]| < \varepsilon/3$.
Then $|[f(x)] - [f(a)]|$
$$\leqslant |[f(x)] - [s_K(x)]| + |[s_K(x)] - [s_K(a)]||[s_K(a)] - [f(a)]|$$
$$= |[s_K(x)] - [f(x)]| + |[s_K(x)] - [s_K(a)]||[s_K(a)] - [f(a)]|$$
$$< (\varepsilon/3) + (\varepsilon/3) + (\varepsilon/3) = \varepsilon.$$ QED

9-3. Show: Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $k \in \mathbb{N}_0$.
Assume: $(\ f \in \mathcal{O}(k)\ )\ \&\ (\text{ near } 0,\ f = g\ )$.
Then: $g \in \mathcal{O}(k)$.

Proof: **Want:** $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall x \in \mathbb{R}$,
$$(\,|x| < \delta\,) \Rightarrow (\,|g_x| < \varepsilon \cdot |x|^k\,).$$
Given $\varepsilon > 0$. **Want:** $\exists \delta > 0$ s.t., $\forall x \in \mathbb{R}$,
$$(\,|x| < \delta\,) \Rightarrow (\,|g_x| < \varepsilon \cdot |x|^k\,).$$
Since $f \in \mathcal{O}(k)$, choose $\alpha > 0$ s.t., $\forall x \in \mathbb{R}$,
$$(\,|x| < \alpha\,) \Rightarrow (\,|f_x| < \varepsilon \cdot |x|^k\,).$$

Since near 0, $f = g$, choose $\beta > 0$ s.t.,
on $B(0, \delta)$, $f = g$.
Then $\forall x \in B(0, \delta)$, we have $f_x = g_x$.
Let $\delta := \min\{\alpha, \beta\}$. Then $\delta > 0$.
Want: $\forall x \in \mathbb{R}$, $(\, |x| < \delta \,) \Rightarrow (\, |g_x| < \varepsilon \cdot |x|^k \,)$.
Given $x \in \mathbb{R}$. Assume: $|x| < \delta$. **Want:** $|g_x| < \varepsilon \cdot |x|^k$.
Since $|x| < \delta \leqslant \alpha$, by choice of α, we get: $|f_x| < \varepsilon \cdot |x|^k$.
Since $|x - 0| < \delta \leqslant \beta$, we get: $x \in B(0, \beta)$, and so $f_x = g_x$.
Then $|g_x| = |f_x| < \varepsilon \cdot |x|^k$. QED

9-4. Show: Let $s, t \in \mathbb{R}^{\mathbb{N}}$.
Assume: $(\ s \to \infty\ )\ \&\ (\ \forall j \in \mathbb{N},\ t_j \geqslant s_j\ )$.
Then: $t \to \infty$.

Proof: **Want:** $\forall M \in \mathbb{R}$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$(\, j \geqslant K \,) \Rightarrow (\, t_j > M \,)$.
Given $M \in \mathbb{R}$. **Want:** $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$(\, j \geqslant K \,) \Rightarrow (\, t_j > M \,)$.
Since $s \to \infty$, choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$(\, j \geqslant K \,) \Rightarrow (\, s_j > M \,)$.
Then $K \in \mathbb{N}$.
Want: $\forall j \in \mathbb{N}$, $(\, j \geqslant K \,) \Rightarrow (\, t_j > M \,)$.
Given $j \in \mathbb{N}$. Assume: $j \geqslant K$. **Want:** $t_j > M$.
Since $j \in \mathbb{N}$, by assumption, we get: $t_j \geqslant s_j$.
Since $j \geqslant K$, by choice of K, we get: $s_j > M$.
Then $t_j \geqslant s_j > M$. QED

9-5. Show:
Let $a := (\, 1\, ,\, 1/2\, ,\, 1/2\, ,\, 1/4\, ,\, 1/4\, ,\, 1/4\, ,\, 1/4\, ,\, 1/8\, ,\, \ldots)$.
NOTE: The terms of a are, in order:
a single "1", then two "1/2"s,
then four "1/4"s, then eight "1/8"s,
then sixteen "1/16"s, then thirty-two "1/32"s, *etc.*
Let $b := (\, 1\, ,\, 1/2\, ,\, 1/3\, ,\, 1/4\, ,\, 1/5\, ,\, 1/6\, ,\, 1/7\, ,\, 1/8\, ,\, \ldots)$.
Define $s, t \in \mathbb{R}^{\mathbb{N}}$ by: $\forall j \in \mathbb{N}$,
$s_j = a_1 + \cdots + a_j$ and $t_j = b_1 + \cdots + b_j$.
Then: (i) $\mathbb{I}_s$ is unbounded and
(ii) $\forall j \in \mathbb{N}$, $t_j > s_j/2$ and
(iii) $t \to \infty$.

Proof: Proof of (i):
Define $m \in \mathbb{N}^{\mathbb{N}}$ by: $\forall j \in \mathbb{N}$, $m_j = 1 + 2 + 4 + 8 + 16 + \cdots + 2^{j-1}$.
By a class theorem, $\forall j \in \mathbb{N}$, $s_{m_j} = j$.
Then: $\forall j \in \mathbb{N}$, $j = s_{m_j} \in \mathbb{I}_s$.
Since, $\forall j \in \mathbb{N}$, $j \in \mathbb{I}_s$, it follows that $\mathbb{N} \subseteq \mathbb{I}_s$.
Assume $\mathbb{I}_s$ is bounded. **Want:** Contradiction.
Choose $B \in \mathcal{B}_{\mathbb{R}}$ s.t. $\mathbb{I}_s \subseteq B$.
Choose $a \in \mathbb{R}$ and $r > 0$ s.t. $B = B(a, r)$.
By the AP, choose $j \in \mathbb{N}$ s.t. $j > a + r$.
Since $j \in \mathbb{N} \subseteq \mathbb{I}_s \subseteq B = B(a, r) = (a - r; a + r) < a + r < j$,
we conclude that $j < j$. Contradiction.
End of proof of (i).

Proof of (ii):
Given $j \in \mathbb{N}$. **Want:** $t_j > s_j/2$.
By a class theorem, $\forall i \in \mathbb{N}$, $b_i > s_i/2$.
Then all of the following are true:
$b_1 > a_1/2, \quad b_2 > a_2/2, \quad b_3 > a_3/2, \quad \cdots \quad b_j > a_j/2.$
Then $b_1 + \cdots + b_j > (a_1/2) + \cdots + (a_j/2)$.
Then $t_j = b_1 + \cdots + b_j > (a_1/2) + \cdots + (a_j/2) = (a_1 + \cdots + a_j)/2 = s_j/2$.
End of proof of (ii).

Proof of (iii):
Since $\forall j \in \mathbb{N}$, $a_j > 0$, it follows that s is strictly-increasing.
Then s is semi-increasing. By (i), $\mathbb{I}_s$ is unbounded.
Then, by a class theorem, $s \to \infty$.
By (ii), we have: $\forall j \in \mathbb{N}$, $t_j > (s/2)_j$.
So, by HW#9-4, it suffices to show: $s/2 \to \infty$.
Want: $\forall M \in \mathbb{R}$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\, j \geqslant K \,) \Rightarrow (\, (s/2)_j > M \,).$$
Given $M \in \mathbb{R}$. **Want:** $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\, j \geqslant K \,) \Rightarrow (\, (s/2)_j > M \,).$$
Since $s \to \infty$, choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\, j \geqslant K \,) \Rightarrow (\, s_j > 2 \cdot M \,).$$
Then $K \in \mathbb{N}$. **Want:** $\forall j \in \mathbb{N}$, $(\, j \geqslant K \,) \Rightarrow (\, (s/2)_j > M \,)$.
Given $j \in \mathbb{N}$. Assume: $j \geqslant K$. **Want:** $(s/2)_j > M$.
Since $j \geqslant K$, by choice of K, we get: $s_j > 2 \cdot M$.
Then $s_j/2 > (2 \cdot M)/2$.

Then $(s/2)_j = s_j/2 > (2 \cdot M)/2 = M$.
End of proof of (iii). QED

Homework 8: Due on Wednesday 9 November

8-1. Show: Let $f \in \mathcal{O}(3)$, $g \in \mathcal{O}(4)$. Then: $f \cdot g \in \mathcal{O}(7)$.

Proof: **Want:** $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall x \in \mathbb{R}$,
$$(\,|x| < \delta\,) \Rightarrow (\,|(f \cdot g)_x| \leqslant \varepsilon \cdot |x|^7\,).$$
Given $\varepsilon > 0$. **Want:** $\exists \delta > 0$ s.t., $\forall x \in \mathbb{R}$,
$$(\,|x| < \delta\,) \Rightarrow (\,|(f \cdot g)_x| \leqslant \varepsilon \cdot |x|^7\,).$$
Since $f \in \mathcal{O}(3)$, choose $\alpha > 0$ s.t., $\forall x \in \mathbb{R}$,
$$(\,|x| < \alpha\,) \Rightarrow (\,|f_x| \leqslant \sqrt{\varepsilon} \cdot |x|^3\,).$$
Since $g \in \mathcal{O}(4)$, choose $\beta > 0$ s.t., $\forall x \in \mathbb{R}$,
$$(\,|x| < \beta\,) \Rightarrow (\,|g_x| \leqslant \sqrt{\varepsilon} \cdot |x|^4\,).$$
Let $\delta := \min\{\alpha, \beta\}$. Then $\delta > 0$ and $\delta \leqslant \alpha$ and $\delta \leqslant \beta$.
Want: $\forall x \in \mathbb{R}$, $(\,|x| < \delta\,) \Rightarrow (\,|(f \cdot g)_x| \leqslant \varepsilon \cdot |x|^7\,)$.
Given $x \in \mathbb{R}$. Assume $|x| < \delta$. **Want:** $|(f \cdot g)_x| \leqslant \varepsilon \cdot |x|^7$.
Since $|x| < \delta \leqslant \alpha$, by choice of α, we get: $|f_x| \leqslant \sqrt{\varepsilon} \cdot |x|^3$.
Since $|x| < \delta \leqslant \beta$, by choice of β, we get: $|g_x| \leqslant \sqrt{\varepsilon} \cdot |x|^4$.
Since $0 \leqslant |f_x| \leqslant \sqrt{\varepsilon} \cdot |x|^3$ and $0 \leqslant |g_x| \leqslant \sqrt{\varepsilon} \cdot |x|^4$,
we get: $|f_x| \cdot |g_x| \leqslant \sqrt{\varepsilon} \cdot |x|^3 \cdot \sqrt{\varepsilon} \cdot |x|^4$.
Then $|(f \cdot g)_x| = |f_x \cdot g_x|$
$$= |f_x| \cdot |g_x| \leqslant \sqrt{\varepsilon} \cdot |x|^3 \cdot \sqrt{\varepsilon} \cdot |x|^4 = \varepsilon \cdot |x|^7.$$
QED

8-2. Show: Let $s \in \mathbb{R}^{\mathbb{N}}$.
Assume $\mathbb{I}_s$ is bounded. Then s is subconvergent.

Proof: **Want:** $\exists$subsequence t of s s.t. t is convergent.
By a class theorem, choose a subsequence t of s s.t. t is semi-monotone.
Then t is a subsequence of s. **Want:** t is convergent.
Since t is a subsequence of s, we get: $\mathbb{I}_t \subseteq \mathbb{I}_s$.
So, since $\mathbb{I}_s$ is bounded, we get: $\mathbb{I}_t$ is bounded.
Since t is semi-monontone and $\mathbb{I}_t$ is bounded,
by a class theorem, we get: t is convergent. QED

8-3. Show: Let $s \in \mathbb{R}^{\mathbb{N}}$.
Assume s is convergent. Then s is Cauchy.

Proof: **Want:** $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall i, j \in \mathbb{N}$,

$$(\, i, j \geqslant K \,) \;\Rightarrow\; (\, |s_i - s_j| < \varepsilon \,).$$

Given $\varepsilon > 0$. **Want:** $\exists K \in \mathbb{N}$ s.t., $\forall i, j \in \mathbb{N}$,

$$(\, i, j \geqslant K \,) \;\Rightarrow\; (\, |s_i - s_j| < \varepsilon \,).$$

Since s is convergent, choose $z \in \mathbb{R}$ s.t. $s \to z$.
Since $s \to z$, choose $K \in \mathbb{N}$ s.t., $\forall k \in \mathbb{N}$,

$$(\, k \geqslant K \,) \;\Rightarrow\; (\, |s_k - z| < \varepsilon/2 \,).$$

Then $K \in \mathbb{N}$. **Want:** $\forall i, j \in \mathbb{N}$, $(\, i, j \geqslant K \,) \;\Rightarrow\; (\, |s_i - s_j| < \varepsilon \,)$.
Given $i, j \in \mathbb{N}$. Assume $i, j \geqslant K$. **Want:** $|s_i - s_j| < \varepsilon$.
Since $i \geqslant K$, by choice of K, we get: $|s_i - z| < \varepsilon/2$.
Since $j \geqslant K$, by choice of K, we get: $|s_j - z| < \varepsilon/2$, so $|z - s_j| < \varepsilon/2$.
Then $|s_i - s_j| \leqslant |s_i - z| + |z - s_j| < (\varepsilon/2) + (\varepsilon/2) = \varepsilon$. QED

8-4. Show: Let $s \in \mathbb{R}^{\mathbb{N}}$.
Assume s is Cauchy and subconvergent. Then s is convergent.

Proof: Since s is subconvergent,
choose a subsequence t of s s.t. t is convergent.
Since t is convergent, choose $z \in \mathbb{R}$ s.t. $t \to z$. **Want:** $s \to z$.
Want: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$(\, j \geqslant K \,) \;\Rightarrow\; (\, |s_j - z| < \varepsilon \,).$$

Given $\varepsilon > 0$. **Want:** $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$(\, j \geqslant K \,) \;\Rightarrow\; (\, |s_j - z| < \varepsilon \,).$$

Since s is Cauchy, choose $K \in \mathbb{N}$ s.t., $\forall i, j \in \mathbb{N}$,

$$(\, i, j \geqslant K \,) \;\Rightarrow\; (\, |s_i - s_j| < \varepsilon/2 \,).$$

Then $K \in \mathbb{N}$. **Want:** $\forall j \in \mathbb{N}$, $(\, j \geqslant K \,) \;\Rightarrow\; (\, |s_j - z| < \varepsilon \,)$.
Given $j \in \mathbb{N}$. Assume $j \geqslant K$. **Want:** $|s_j - z| < \varepsilon$.
Since $t \to z$, choose $L \in \mathbb{N}$ s.t., $\forall i \in \mathbb{N}$,

$$(\, i \geqslant L \,) \;\Rightarrow\; (\, |t_i - z| < \varepsilon/2 \,).$$

Let $i := \max\{K, L\}$. Then $i \in \mathbb{N}$ and $i \geqslant K$ and $i \geqslant L$.
Since t is a subsequence of s,
choose a strictly-increasing $\ell \in \mathbb{N}^{\mathbb{N}}$ s.t. $t = s \circ \ell$.
Since $\ell \in \mathbb{N}^{\mathbb{N}}$, and $i \in \mathbb{N}$, we get: $\ell_i \in \mathbb{N}$.
By a homework problem, we get: $\ell_i \geqslant i$.
Let $m := \ell_i$. Then $m \in \mathbb{N}$ and $m \geqslant i$.
Since $m \geqslant i \geqslant K$, we get: $m \geqslant K$.
Since $s_m = s_{\ell_i} = (s \circ \ell)_i = t_i$, we get $s_m = t_i$.
Since $i \geqslant L$, by choice of L, we get: $|t_i - z| < \varepsilon/2$.

Since $m, j \geqslant K$, by choice of K, we get: $|s_m - s_j| < \varepsilon/2$.
Then $|s_j - t_i| = |s_j - s_m| = |s_m - s_j| < \varepsilon/2$, so $|s_j - t_i| < \varepsilon/2$.
Then $|s_j - z| \leqslant |s_j - t_i| + |t_i - z| < (\varepsilon/2) + (\varepsilon/2) = \varepsilon$. QED

8-5. Show: Let $s \in \mathbb{R}^{\mathbb{N}}$.
Assume s is Cauchy. Then s is convergent.

Proof: By a class theorem, since s is Cauchy, $\mathbb{I}_s$ is bounded.
Then, by HW#8-2, we get: s is subconvergent.
Since s is Cauchy and subconvergent,
by HW#8-4, we get: s is convergent. QED

Homework 7: Due on Tuesday 1 November

7-1. Show: Let $k \in \mathbb{N}^{\mathbb{N}}$. Assume k is strictly-increasing.
Then: $\forall j \in \mathbb{N}$, $k_j \geqslant j$.

Proof: Let $S := \{j \in \mathbb{N} \,|\, k_j \geqslant j\}$. **Want:** $S = \mathbb{N}$.
Since $k_1 \in \mathbb{N} \geqslant 1$, we get $1 \in S$.
By the PMI, it suffice to show: $\forall k \in S$, $k + 1 \in S$.
Given $k \in S$. **Want:** $k + 1 \in S$.
Know: $k_j \geqslant j$. **Want:** $k_{j+1} \geqslant j + 1$.
Since k is strictly increasing, we have: $k_{j+1} > k_j$.
So, since $k_j, k_{j+1} \in \mathbb{Z}$, we get: $k_{j+1} \geqslant k_j + 1$.
Since $k_j \geqslant j$, by adding 1 to both sides, we get: $k_j + 1 \geqslant j + 1$.
Then $k_{j+1} \geqslant k_j + 1 \geqslant j + 1$. QED

7-2. Show: Let $s, t \in \mathbb{R}$. Assume $s < t$.
Then: $\exists x \in \mathbb{Q}$ s.t. $s < x < t$.

Proof: We have $t - s > 0$. By the AP, choose $j \in \mathbb{N}$ s.t. $j > 1/(t - s)$.
Then $j \cdot (t - s) > 1$, so $j \cdot t - j \cdot s > 1$.
Let $a := j \cdot s$ and $b := j \cdot t$. Then $b - a > 1$.
Then, by a class theorem, choose $k \in \mathbb{Z}$ s.t. $a < k < b$.
Since $k \in \mathbb{Z}$ and $j \in \mathbb{N}$, we get $k/j \in \mathbb{Q}$.
Let $x := k/j$. Then $x \in \mathbb{Q}$. **Want:** $s < x < t$.
We have $j \cdot x = k$.
Since $a < k < b$ and since $j \in \mathbb{N} > 0$, we get $a/j < k/j < b/j$.
Then $(j \cdot s)/j < (j \cdot x)/j < (j \cdot t)/j$. Then $s < x < t$. QED

7-3. Show: Let $s, t, u \in \mathbb{R}^{\mathbb{N}}$, $a \in \mathbb{R}$.
Assume: $\forall j \in \mathbb{N}$, $s_j \leqslant t_j \leqslant u_j$.
Assume: $s \to a$ and $u \to a$.
Then: $t \to a$.

Proof: **Want:** $\forall \varepsilon > 0, \exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$, $(\, j \geqslant K\,) \Rightarrow (\, |t_j - a| < \varepsilon\,)$.
Given $\varepsilon > 0$. **Want:** $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$, $(\, j \geqslant K\,) \Rightarrow (\, |t_j - a| < \varepsilon\,)$.
Since $s \to a$, choose $L \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$, $(\, j \geqslant L\,) \Rightarrow (\, |s_j - a| < \varepsilon\,)$.
Since $u \to a$, choose $M \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$, $(\, j \geqslant M\,) \Rightarrow (\, |u_j - a| < \varepsilon\,)$.
Let $K := \max\{L, M\}$. Then $K \geqslant L$ and $K \geqslant M$ and $K \in \mathbb{N}$.
Want: $\forall j \in \mathbb{N}$, $(\, j \geqslant K\,) \Rightarrow (\, |t_j - a| < \varepsilon\,)$.
Given $j \in \mathbb{N}$. Assume $j \geqslant K$. **Want:** $|t_j - a| < \varepsilon$.
Since $j \geqslant K \geqslant L$, by choice of L, we get $|s_j - a| < \varepsilon$.
Then $a - \varepsilon < s_j < a + \varepsilon$.
Since $j \geqslant K \geqslant M$, by choice of M, we get $|u_j - a| < \varepsilon$.
Then $a - \varepsilon < u_j < a + \varepsilon$.
By assumption, $s_j \leqslant t_j \leqslant u_j$.
We have $a - \varepsilon < s_j \leqslant t_j$, so $a - \varepsilon < t_j$.
We have $t_j \leqslant u_j < a + \varepsilon$, so $t_j < a + \varepsilon$.
Then $a - \varepsilon < t_j < a + \varepsilon$, so $|t_j - a| < \varepsilon$. QED

7-4. Show: Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$.
Assume: as $x \to a$, $f_x \to -\infty$.
Assume: as $x \to a$, $g_x \to -\infty$.
Then: as $x \to a$, $(f \cdot g)_x \to \infty$.

Proof: **Want:** $\forall M \in \mathbb{R}, \exists \delta > 0$ s.t., $\forall x \in \mathbb{D}_{f \cdot g}$,
$(\, 0 < |x - a| < \delta\,) \Rightarrow (\, (f \cdot g)_x > M\,)$.
Given $M \in \mathbb{R}$. **Want:** $\exists \delta > 0$ s.t., $\forall x \in \mathbb{D}_{f \cdot g}$,
$(\, 0 < |x - a| < \delta\,) \Rightarrow (\, (f \cdot g)_x > M\,)$.
Let $N := -\sqrt{\max\{M, 0\}}$.
Then $N \in \mathbb{R}$ and $N \leqslant 0$ and $N^2 = \max\{M, 0\}$.
Since as $x \to a$, $f_x \to -\infty$, choose $\alpha > 0$ s.t., $\forall x \in \mathbb{D}_f$,
$(\, 0 < |x - a| < \alpha\,) \Rightarrow (\, f_x < N\,)$.
Since as $x \to a$, $g_x \to -\infty$, choose $\beta > 0$ s.t., $\forall x \in \mathbb{D}_g$,
$(\, 0 < |x - a| < \beta\,) \Rightarrow (\, g_x < N\,)$.
Let $\delta := \min\{\alpha, \beta\}$. Then $\delta \leqslant \alpha$ and $\delta \leqslant \beta$ and $\delta > 0$.
Want: $\forall x \in \mathbb{D}_{f \cdot g}$, $(\, 0 < |x - a| < \delta\,) \Rightarrow (\, (f \cdot g)_x > M\,)$.
Given $x \in \mathbb{D}_{f \cdot g}$. Assume $0 < |x - a| < \delta$. **Want:** $(f \cdot g)_x > M$.

Since $x \in \mathbb{D}_{f \cdot g}$, we get: $\quad x \in \mathbb{D}_f \quad$ and $\quad x \in D_g$.
Since $0 < |x - a|$ and $|x - a| < \delta \leqslant \alpha$, we get $0 < |x - a| < \alpha$.
Then, by choice of α, we get $f_x < N$. $\quad$ Then $-f_x > -N$.
Since $0 < |x - a|$ and $|x - a| < \delta \leqslant \beta$, we get $0 < |x - a| < \beta$.
Then, by choice of β, we get $g_x < N$. $\quad$ Then $-g_x > -N$.
Since $N \leqslant 0$, we get $-N \geqslant 0$.
Since $-f_x > -N \geqslant 0$ and $-g_x \geqslant -N \geqslant 0$,
we get $(-f_x) \cdot (-g_x) > (-N) \cdot (-N)$. $\quad$ Then $f_x \cdot g_x > N^2$.
Then $(f \cdot g)_x = f_x \cdot g_x > N^2 = \max\{M, 0\} \geqslant M$. $\quad$ QED

7-5. Show: Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $q \in \mathbb{D}_{f \cdot g}$.
Then: $(f \cdot g)_q^{\mathbb{T}} \;=\; f_q^{\mathbb{T}} \cdot g_q \;+\; f_q \cdot g_q^{\mathbb{T}} \;+\; f_q^{\mathbb{T}} \cdot g_q^{\mathbb{T}}$.

Proof: **Want:** $\forall h \in \mathbb{R}, \quad ((f \cdot g)_q^{\mathbb{T}})_h \;=\; (f_q^{\mathbb{T}} \cdot g_q \;+\; f_q \cdot g_q^{\mathbb{T}} \;+\; f_q^{\mathbb{T}} \cdot g_q^{\mathbb{T}})_h$.
Given $h \in \mathbb{R}$. $\quad$ **Want:** $((f \cdot g)_q^{\mathbb{T}})_h \;=\; (f_q^{\mathbb{T}} \cdot g_q \;+\; f_q \cdot g_q^{\mathbb{T}} \;+\; f_q^{\mathbb{T}} \cdot g_q^{\mathbb{T}})_h$.
We have $((f \cdot g)_q^{\mathbb{T}})_h \;=\; (f \cdot g)_{q+h} - (f \cdot g)_q \;=\; f_{q+h} \cdot g_{q+h} - f_q \cdot g_q$.
Want: $f_{q+h} \cdot g_{q+h} - f_q \cdot g_q \;=\; (f_q^{\mathbb{T}} \cdot g_q \;+\; f_q \cdot g_q^{\mathbb{T}} \;+\; f_q^{\mathbb{T}} \cdot g_q^{\mathbb{T}})_h$.
Let $\quad a := f_q, \quad b := g_q, \quad A := f_{q+h}, \quad B := g_{q+h}$.
Want: $A \cdot B - a \cdot b \;=\; (f_q^{\mathbb{T}} \cdot g_q \;+\; f_q \cdot g_q^{\mathbb{T}} \;+\; f_q^{\mathbb{T}} \cdot g_q^{\mathbb{T}})_h$.
We have $A - a \;=\; f_{q+h} - f_q \;=\; (f_q^{\mathbb{T}})_h$.
We have $B - b \;=\; g_{q+h} - g_q \;=\; (g_q^{\mathbb{T}})_h$.
By the Naive Product Rule, we have

$$A \cdot B - a \cdot b \;=\; (A - a) \cdot b + a \cdot (B - b) + (A - a) \cdot (B - b).$$

Then

$$\begin{aligned} A \cdot B - a \cdot b \;&=\; (f_q^{\mathbb{T}})_h \cdot g_q + f_q \cdot (g_q^{\mathbb{T}})_h + (f_q^{\mathbb{T}})_h \cdot (g_q^{\mathbb{T}})_h \\ &=\; (f_q^{\mathbb{T}} \cdot g_q)_h + (f_q \cdot g_q^{\mathbb{T}})_h + (f_q^{\mathbb{T}} \cdot g_q^{\mathbb{T}})_h \\ &=\; (f_q^{\mathbb{T}} \cdot g_q + f_q \cdot g_q^{\mathbb{T}} + f_q^{\mathbb{T}} \cdot g_q^{\mathbb{T}})_h. \end{aligned}$$

QED

Homework 6: Due on Tuesday 25 October

6-1. Show: $\forall a \in \mathbb{R}, \exists x \in \mathbb{R}$ s.t. $x^5 + x^3 + x = a$.

Proof: Given $a \in \mathbb{R}$. $\quad$ **Want:** $\exists x \in \mathbb{R}$ s.t. $x^5 + x^3 + x = a$.
We have $-|a| \leqslant a$ and $|a| \geqslant a$. $\quad$ Let $b := |a|$.
Then $-b \leqslant a$ and $b \geqslant a$.
Also, $b \geqslant 0$, so $b^5 + b^3 \geqslant 0$.
Then $-(b^5 + b^3) - b \leqslant -b$ and $(b^5 + b^3) + b \geqslant b$.
Define $f : \mathbb{R} \to \mathbb{R}$ by: $\forall x \in \mathbb{R}$, $f_x = x^5 + x^3 + x$. $\quad$ Then f is continuous.
We have $f_{-b} = -(b^5 + b^3) - b \leqslant -b \leqslant a$ and $f_b = (b^5 + b^3) + b \geqslant b \geqslant a$.
Then $f_{-b} \leqslant a$ and $f_b \geqslant a$, and so $f_{-b} \leqslant a \leqslant f_b$, and so $a \in [f_{-b}|f_b]$.

Since f is continuous on $[-b|b]$, by the IVT, we have $[f_{-b}|f_b] \subseteq f_*[-b|b]$.
Since $a \in [f_{-b}|f_b] \subseteq f_*[-b|b]$ choose $x \in [-b|b]$ s.t. $f_x = a$.
Then $x \in [-b|b] \subseteq \mathbb{R}$. Want $x^5 + x^3 + x = a$.
We have $x^5 + x^3 + x = f_x = a$. QED

6-2. Show: Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $a, b \in \mathbb{R}$.
Assume: as $x \to a$, $f_x \to b$. Assume: g is continuous at b.
Then: as $x \to a$, $(g \circ f)_x \to g_b$.

Proof: **Want:** $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall x \in \mathbb{D}_{g\circ f}$,
$$(\,0 < |x-a| < \delta\,) \Rightarrow (\,|(g\circ f)_x - g_b| < \varepsilon\,).$$
Given $\varepsilon > 0$. **Want:** $\exists \delta > 0$ s.t., $\forall x \in \mathbb{D}_{g\circ f}$,
$$(\,0 < |x-a| < \delta\,) \Rightarrow (\,|(g\circ f)_x - g_b| < \varepsilon\,).$$
Since g is continuous at b, choose $\eta > 0$ s.t., $\forall y \in \mathbb{D}_g$,
$$(\,|y - b| < \eta\,) \Rightarrow (\,|g_y - g_b| < \varepsilon\,).$$
Since as $x \to a$, $f_x \to b$, choose $\delta > 0$ s.t., $\forall x \in \mathbb{D}_f$,
$$(\,0 < |x - a| < \delta\,) \Rightarrow (\,|f_x - b| < \eta\,).$$
Then $\delta > 0$. **Want:** $\forall x \in \mathbb{D}_{g\circ f}$,
$$(\,0 < |x-a| < \delta\,) \Rightarrow (\,|(g\circ f)_x - g_b| < \varepsilon\,).$$
Given $x \in \mathbb{D}_{g\circ f}$. Assume $0 < |x-a| < \delta$. **Want:** $|(g\circ f)_x - g_b| < \varepsilon$.
Since $x \in \mathbb{D}_{g\circ f}$, we get $f_x \in \mathbb{D}_g$. Then $f_x \neq ☹$, so $x \in \mathbb{D}_f$.
Since $0 < |x - a| < \delta$, by choice of δ, we get $|f_x - b| < \eta$.
Let $y := f_x$. Then $y = f_x \in \mathbb{D}_g$.
Since $|y - b| = |f_x - b| < \eta$, by choice of η, we get $|g_y - g_b| < \varepsilon$.
Then $|(g \circ f)_x - g_b| = |g_{f_x} - g_b| = |g_y - g_b| < \varepsilon$. QED

6-3. Show: Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a, b \in \mathbb{D}_f$.
Assume $a \neq b$. Let $m := \mathrm{DQ}_f(a, b)$.
Define $g : \mathbb{R} \dashrightarrow \mathbb{R}$ by: $\forall x \in \mathbb{R}$, $g_x \overset{*}{=} f_x - mx$.
Then: $g_a = g_b$.

Proof: We have $m = \mathrm{DQ}_f(a, b) = \dfrac{f_b - f_a}{b - a}$, so $m = \dfrac{f_b - f_a}{b - a}$.
Then $m \cdot (b - a) = f_b - f_a$.
Then $mb - ma = f_b - f_a$.
Then $f_a - ma = f_b - mb$.
Then $g_a = g_b$. QED

6-4. Show: Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{D}_f$.
Assume $f_a^{\mathbb{T}}$ is continuous at 0. Then f is continuous at a.

Proof: **Want:** $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall x \in \mathbb{D}_f$,

$$(\,|x-a| < \delta\,) \Rightarrow (\,|f_x - f_a| < \varepsilon\,).$$

Given $\varepsilon > 0$. **Want:** $\exists \delta > 0$ s.t., $\forall x \in \mathbb{D}_f$,

$$(\,|x-a| < \delta\,) \Rightarrow (\,|f_x - f_a| < \varepsilon\,).$$

Let $g := f_a^{\mathbb{T}}$. Then, by assumption, g is continuous at 0.

Since g is continuous at 0, choose $\delta > 0$ s.t., $\forall h \in \mathbb{D}_g$,

$$(\,|h-0| < \delta\,) \Rightarrow (\,|g_h - g_0| < \varepsilon\,).$$

Then $\delta > 0$. **Want:** $\forall x \in \mathbb{D}_f$, $(\,|x-a| < \delta\,) \Rightarrow (\,|f_x - f_a| < \varepsilon\,)$.

Given $x \in \mathbb{D}_f$. Assume $|x-a| < \delta$. **Want:** $|f_x - f_a| < \varepsilon$.

Since $x, a \in \mathbb{D}_f$, it follows that $f_x - f_a \neq$ ☹. Let $h := x - a$.

We have $g_h = (f_a^{\mathbb{T}})_h = f_{a+h} - f_a = f_x - f_a$.

So, since $f_x - f_a \neq$ ☹, we see that $g_h \neq$ ☹ and $g_h = f_x - f_a$.

Since $g_h \neq$ ☹, we get $h \in \mathbb{D}_g$.

So, since $|h-0| = |h| = |x-a| < \delta$, by choice of δ, we get $|g_h - g_0| < \varepsilon$.

Since $a \in \mathbb{D}_f$, we get $f_{a+0} - f_a = 0$.

Then $g_0 \stackrel{*}{=} (f_a^{\mathbb{T}})_0 = f_{a+0} - f_a = 0$. Then $g_0 = 0$.

Then $f_x - f_a = g_h = g_h - g_0$. Then $|f_x - f_a| = |g_h - g_0| < \varepsilon$. QED

6-5. Show: Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{D}_{g \circ f}$. Then $(g \circ f)_a^{\mathbb{T}} = (g_{f_a}^{\mathbb{T}}) \circ (f_a^{\mathbb{T}})$.

Proof: **Want:** $\forall h \in \mathbb{R}$, $((g \circ f)_a^{\mathbb{T}})_h = ((g_{f_a}^{\mathbb{T}}) \circ (f_a^{\mathbb{T}}))_h$.

Given $h \in \mathbb{R}$. **Want:** $((g \circ f)_a^{\mathbb{T}})_h = ((g_{f_a}^{\mathbb{T}}) \circ (f_a^{\mathbb{T}}))_h$.

We have: $((g \circ f)_a^{\mathbb{T}})_h = (g \circ f)_{a+h} - (g \circ f)_a$.

Want: $(g \circ f)_{a+h} - (g \circ f)_a = ((g_{f_a}^{\mathbb{T}}) \circ (f_a^{\mathbb{T}}))_h$.

Let $k := ((f_a)^{\mathbb{T}})_h$. Then: $((g_{f_a}^{\mathbb{T}}) \circ (f_a^{\mathbb{T}}))_h = (g_{f_a}^{\mathbb{T}})_k$.

Want: $(g \circ f)_{a+h} - (g \circ f)_a = (g_{f_a}^{\mathbb{T}})_k$.

We have: $(g_{f_a}^{\mathbb{T}})_k = g_{f_a+k} - g_{f_a}$.

Want: $(g \circ f)_{a+h} - (g \circ f)_a = g_{f_a+k} - g_{f_a}$.

We have: $k = (f_a^{\mathbb{T}})_h = f_{a+h} - f_a$.

Then $f_a + k = f_a + f_{a+h} - f_a$.

Since $a \in \mathbb{D}_f$, we get $f_a - f_a = 0$, so $f_a + f_{a+h} - f_a = f_{a+h}$.

Then $f_a + k = f_a + f_{a+h} - f_a = f_{a+h}$,

so $f_a + k = f_{a+h}$, so $g_{f_a+k} = g_{f_{a+h}}$.

Then $(g \circ f)_{a+h} - (g \circ f)_a = g_{f_{a+h}} - g_{f_a}$

$\stackrel{*}{=} g_{f_a+k} - g_{f_a}$. QED

Homework 5: Due on Wednesday 19 October

5-1. Show: Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$.
Assume f and g are both continuous at a.
Then $f + g$ is continuous at a.

Proof: **Want:** $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall x \in \mathbb{D}_{f+g}$,

$$(\,|x-a| < \delta\,) \Rightarrow (\,|(f+g)_x-(f+g)_a| < \varepsilon\,).$$

Given $\varepsilon > 0$. **Want:** $\exists \delta > 0$ s.t., $\forall x \in \mathbb{D}_{f+g}$,

$$(\,|x-a| < \delta\,) \Rightarrow (\,|(f+g)_x-(f+g)_a| < \varepsilon\,).$$

Since f is continuous at a, choose $\lambda > 0$ s.t., $\forall x \in \mathbb{D}_f$,

$$(\,|x - a| < \delta\,) \Rightarrow (\,|f_x - f_a| < \varepsilon/2\,).$$

Since g is continuous at a, choose $\mu > 0$ s.t., $\forall x \in \mathbb{D}_g$,

$$(\,|x - a| < \delta\,) \Rightarrow (\,|g_x - g_a| < \varepsilon/2\,).$$

Let $\delta := \min\{\lambda, \mu\}$. Then $\delta > 0$.
Want: $\forall x \in \mathbb{D}_{f+g}$, $(\,|x - a| < \delta\,) \Rightarrow (\,|(f + g)_x - (f + g)_a| < \varepsilon\,)$.
Given $x \in \mathbb{D}_{f+g}$. Assume $|x-a| < \delta$. **Want:** $|(f+g)_x-(f+g)_a| < \varepsilon$.
Since $x \in \mathbb{D}_{f+g}$, we get: $x \in \mathbb{D}_f$ and $x \in \mathbb{D}_g$.
Since $\delta = \min\{\lambda, \mu\}$, we get: $\delta \leqslant \lambda$ and $\lambda \leqslant \mu$.
Since $|x - a| < \delta \leqslant \lambda$, by choice of λ, we get: $|f_x - f_a| < \varepsilon/2$.
Since $|x - a| < \delta \leqslant \mu$, by choice of μ, we get: $|g_x - g_a| < \varepsilon/2$.
Then

$$\begin{aligned} |(f + g)_x - (f + g)_a| &= |(f_x + g_x) - (f_a + g_a)| \\ &= |f_x + g_x - f_a - g_a| \\ &= |f_x - f_a + g_x - g_a| \\ &\leqslant |f_x - f_a| + |g_x - g_a| \\ &< (\varepsilon/2) + (\varepsilon/2) = \varepsilon. \end{aligned}$$

QED

5-2. Show: Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$.
Assume f is continuous at a and g is continuous at f_a.
Then $g \circ f$ is continuous at a.

Proof: **Want:** $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall x \in \mathbb{D}_{g\circ f}$,

$$(\,|x-a| < \delta\,) \Rightarrow (\,|(g\circ f)_x-(g\circ f)_a| < \varepsilon\,).$$

Given $\varepsilon > 0$. **Want:** $\exists \delta > 0$ s.t., $\forall x \in \mathbb{D}_{g\circ f}$,

$$(\,|x-a| < \delta\,) \Rightarrow (\,|(g\circ f)_x-(g\circ f)_a| < \varepsilon\,).$$

Since g is continuous at f_a, choose $\eta > 0$ s.t., $\forall y \in \mathbb{D}_g$,

$$(\,|y - f_a| < \eta\,) \Rightarrow (\,|g_y - g_{f_a}| < \varepsilon\,).$$

Since f is continuous at a, choose $\delta > 0$ s.t., $\forall x \in \mathbb{D}_f$,

$$(\,|x - a| < \delta\,) \Rightarrow (\,|f_x - f_a| < \eta\,).$$

Then $\delta > 0$.
Want: $\forall x \in \mathbb{D}_{g\circ f}$, $(\,|x - a| < \delta\,) \Rightarrow (\,|(g \circ f)_x - (g \circ f)_a| < \varepsilon\,)$.

Given $x \in \mathbb{D}_{g\circ f}$. Assume $|x - a| < \delta$. **Want:** $|(g \circ f)_x - (g \circ f)_a| < \varepsilon$.
Since $x \in \mathbb{D}_{g\circ f}$, we get $f_x \in \mathbb{D}_g$, and so $x \in \mathbb{D}_f$.
Since $|x - a| < \delta$, by choice of δ, we get $|f_x - f_a| < \eta$.
Let $y := f_x$. Then $|y - f_a| < \eta$. Also, $y = f_x \in \mathbb{D}_g$, so $y \in \mathbb{D}_g$.
Since $|y - f_a| < \eta$, by choice of η, we get $|g_y - g_{f_a}| < \varepsilon$.
Then $|(g \circ f)_x - (g \circ f)_a| = |g_{f_x} - g_{f_a}| = |g_y - g_{f_a}| < \varepsilon$. QED

5-3. Show: Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $s \in (\mathbb{D}_f)^{\mathbb{N}}$, $a \in \mathbb{R}$.
Assume f is continuous at a and $s \to a$. Then $f \circ s \to f_a$.

Proof: **Want:** $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\, j \geqslant K\,) \Rightarrow (\, |(f \circ s)_j - f_a| < \varepsilon\,).$$
Given $\varepsilon > 0$. **Want:** $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\, j \geqslant K\,) \Rightarrow (\, |(f \circ s)_j - f_a| < \varepsilon\,).$$
Since f is continuous at a, choose $\delta > 0$ s.t., $\forall x \in \mathbb{D}_f$,
$$(\, |x - a| < \delta\,) \Rightarrow (\, |f_x - f_a| < \varepsilon\,).$$
Since $s \to a$, choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\, j \geqslant K\,) \Rightarrow (\, |s_j - a| < \delta\,).$$
Then $K \in \mathbb{N}$.
Want: $\forall j \in \mathbb{N}$, $(\, j \geqslant K\,) \Rightarrow (\, |(f \circ s)_j - f_a| < \varepsilon\,)$.
Given $j \in \mathbb{N}$. Assume $j \geqslant K$. **Want:** $|(f \circ s)_j - f_a| < \varepsilon$.
Since $j \geqslant K$, by choice of K, we get: $|s_j - a| < \delta$.
Since $s \in (\mathbb{D}_f)^{\mathbb{N}}$, we get: $s_j \in \mathbb{D}_f$. Let $x := s_j$. Then $x \in \mathbb{D}_f$.
Since $|x - a| = |s_j - a| < \delta$, by choice of δ, we get: $|f_x - f_a| < \varepsilon$.
Then $|(f \circ s)_j - f_a| = |f_{s_j} - f_a| = |f_x - f_a| < \varepsilon$. QED

5-4. Show: let $f : \mathbb{R} \dashrightarrow \mathbb{R}$.
Assume f is Lipschitz. Then f is uniformly continuous.

Proof: **Want:** $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall w, x \in \mathbb{D}_f$,
$$(\, |x - w| < \delta\,) \Rightarrow (\, |f_x - f_w| < \varepsilon\,).$$
Given $\varepsilon > 0$. **Want:** $\exists \delta > 0$ s.t., $\forall w, x \in \mathbb{D}_f$,
$$(\, |x - w| < \delta\,) \Rightarrow (\, |f_x - f_w| < \varepsilon\,).$$
Since f is Lipschitz, choose $L \geqslant 0$ s.t. f is L-Lipschitz.
Let $\delta := \varepsilon\,/\,(L + 1)$. Then $\delta > 0$.
Want: $\forall w, x \in \mathbb{D}_f$, $(\, |x - w| < \delta\,) \Rightarrow (\, |f_x - f_w| < \varepsilon\,)$.
Want: Given $w, x \in \mathbb{D}_f$. Assume $|x - w| < \delta$. **Want:** $|f_x - f_w| < \varepsilon$.
Since f is L-Lipschitz, we get: $|f_x - f_w| \leqslant L \cdot |w - x|$.

Since $L \geqslant 0$ and $|w - x| < \delta$, we get $L \cdot |x - w| \leqslant L \cdot \delta$.
Then $|f_x - f_w| \leqslant L \cdot |w - x| \leqslant L \cdot \delta = L \cdot \varepsilon / (L + 1) < \varepsilon$. QED

5-5. Show: let $f : \mathbb{R} \dashrightarrow \mathbb{R}$.
Assume f is uniformly continuous. Then f is continuous.

Proof: **Want:** $\forall a \in \mathbb{R}$, f is continuous at a.
Given $a \in \mathbb{R}$. **Want:** f is continuous at a.
Want: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall x \in \mathbb{D}_f$,

$$(\,|x - a| < \delta\,) \Rightarrow (\,|f_x - f_a| < \varepsilon\,).$$

Given $\varepsilon > 0$. **Want:** $\exists \delta > 0$ s.t., $\forall x \in \mathbb{D}_f$,

$$(\,|x - a| < \delta\,) \Rightarrow (\,|f_x - f_a| < \varepsilon\,).$$

Since f is uniformly continuous, choose $\delta > 0$ s.t., $\forall w, x \in \mathbb{D}_f$,

$$(\,|x - w| < \delta\,) \Rightarrow (\,|f_x - f_w| < \varepsilon\,).$$

Then $\delta > 0$.
Want: $\forall x \in \mathbb{D}_f$, $(\,|x - a| < \delta\,) \Rightarrow (\,|f_x - f_a| < \varepsilon\,)$.
Given $x \in \mathbb{D}_f$. Assume $|x - a| < \delta$. **Want:** $|f_x - f_a| < \varepsilon$.
Let $w := a$. Then $w = a \in \mathbb{D}_f$. Recall that $x \in \mathbb{D}_f$.
Also, $|x - w| = |x - a| < \delta$, so $|x - w| < \delta$.
Since $|x - w| < \delta$, by choice of δ, we get: $|f_x - f_w| < \varepsilon$.
Then $|f_x - f_a| = |f_x - f_w| < \varepsilon$. QED

Homework 4: Due on Tuesday 11 October

4-1. Show: $(1, 2, 3, \ldots) \to \infty$.

Proof: Let $s := (1, 2, 3, \ldots)$. **Want:** $s \to \infty$.
We have: $\forall j \in \mathbb{N}$, $s_j = j$.
Want: $\forall M \in \mathbb{R}$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$, $(\,j \geqslant K\,) \Rightarrow (\,s_j > M\,)$.
Given $M \in \mathbb{R}$. **Want:** $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$, $(\,j \geqslant K\,) \Rightarrow (\,s_j > M\,)$.
By the AP, choose $K \in \mathbb{N}$ s.t. $K > M$. Then $K \in \mathbb{N}$.
Want: $\forall j \in \mathbb{N}$, $(\,j \geqslant K\,) \Rightarrow (\,s_j > M\,)$.
Given $j \in \mathbb{N}$. Assume $j \geqslant K$. **Want:** $s_j > M$.
We have $s_j = j \geqslant K > M$. QED

4-2. Show: Let $s \in \mathbb{R}^{\mathbb{N}}$, $c < 0$.
Assume $s \to \infty$. Then $c \cdot s \to -\infty$.

Proof: **Want:** $\forall N \in \mathbb{R}$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$, $(\,j \geqslant K\,) \Rightarrow (\,(c \cdot s)_j < N\,)$.

Given $N \in \mathbb{R}$. **Want:** $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$, $(\, j \geqslant K\,) \Rightarrow (\,(c \cdot s)_j < N\,)$.
Let $M := N/c$. Then $M \in \mathbb{R}$ and $c \cdot M = N$.
Since $s \to \infty$, choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$, $(\, j \geqslant K\,) \Rightarrow (\, s_j > M\,)$.
Then $K \in \mathbb{N}$. **Want:** $\forall j \in \mathbb{N}$, $(\, j \geqslant K\,) \Rightarrow (\,(c \cdot s)_j < N\,)$.
Given $j \in \mathbb{N}$. Assume $j \geqslant K$. **Want:** $(c \cdot s)_j < N$.
Since $j \geqslant K$, by choice of K, we have $s_j > M$.
So, since $c < 0$, we get $c \cdot s_j < c \cdot M$.
Then $(c \cdot s)_j = c \cdot s_j < c \cdot M = N$. QED

4-3. Show: Let $s, t \in \mathbb{R}^{\mathbb{N}}$.
Assume $s \to \infty$ and $t \to \infty$. Then $s + t \to \infty$.

Proof: **Want:** $\forall M \in \mathbb{R}$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$(\, j \geqslant K\,) \Rightarrow (\,(s+t)_j > M\,)$.
Given $M \in \mathbb{R}$. **Want:** $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$(\, j \geqslant K\,) \Rightarrow (\,(s+t)_j > M\,)$.
Since $s \to \infty$, choose $A \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$, $(\, j \geqslant A\,) \Rightarrow (\, s_j > M/2\,)$.
Since $t \to \infty$, choose $B \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$, $(\, j \geqslant B\,) \Rightarrow (\, t_j > M/2\,)$.
Let $K := \max\{A, B\}$. Then $K \in \{A, B\} \subseteq \mathbb{N}$.
Want: $\forall j \in \mathbb{N}$, $(\, j \geqslant K\,) \Rightarrow (\,(s+t)_j > M\,)$.
Given $j \in \mathbb{N}$. Assume $j \geqslant K$. **Want:** $(s+t)_j > M$.
We have $K \geqslant A$ and $K \geqslant B$.
Since $j \geqslant K \geqslant A$, by choice of A, we get $s_j > M/2$.
Since $j \geqslant K \geqslant B$, by choice of B, we get $t_j > M/2$.
Then $(s+t)_j = s_j + t_j > (M/2) + (M/2) = M$. QED

4-4. Show: Let $s \in (\mathbb{R}_0^\times)^{\mathbb{N}}$.
Assume $s \to \infty$. Then $1/s \to 0$.

Proof: **Want:** $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$(\, j \geqslant K\,) \Rightarrow (\,|(1/s)_j - 0| < \varepsilon\,)$.
Given $\varepsilon > 0$. **Want:** $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$(\, j \geqslant K\,) \Rightarrow (\,|(1/s)_j - 0| < \varepsilon\,)$.
Let $M := 1/\varepsilon$. Since $\varepsilon > 0$, we get $M \in \mathbb{R}$ and $M > 0$ and $1/M = \varepsilon$.
Since $s \to \infty$, choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$, $(\, j \geqslant K\,) \Rightarrow (\, s_j > M\,)$
Then $K \in \mathbb{N}$. **Want:** $\forall j \in \mathbb{N}$, $(\, j \geqslant K\,) \Rightarrow (\,|(1/s)_j - 0| < \varepsilon\,)$.
Given $j \in \mathbb{N}$. Assume $j \geqslant K$. **Want:** $|(1/s)_j - 0| < \varepsilon$.
Since $j \geqslant K$, by choice of K, we have $s_j > M$.
We have $s_j > M > 0$, so $s_j > 0$, so $1/s_j > 0$, so $|1/s_j| = 1/s_j$.

Since $s_j > M > 0$, we also get $1/s_j < 1/M$.
Then $|(1/s)_j - 0| = |(1/s)_j| = |1/s_j| = 1/s_j < 1/M = \varepsilon$. QED

4-5. Show: Let $s \in (\mathbb{R}_0^\times)^\mathbb{N}$, $c \in \mathbb{R}$.
Assume $s \to -\infty$. Then $c/s \to 0$.

Proof: **Want:** $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(j \geqslant K) \Rightarrow (|(c/s)_j - 0| < \varepsilon).$$
Given $\varepsilon > 0$. **Want:** $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(j \geqslant K) \Rightarrow (|(c/s)_j - 0| < \varepsilon).$$
Let $M := (|c| + 1)/\varepsilon$. Then $M \in \mathbb{R}$ and $M > 0$.
Since $s \to -\infty$, choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$, $(j \geqslant K) \Rightarrow (s_j < -M)$.
Then $K \in \mathbb{N}$. **Want:** $\forall j \in \mathbb{N}$, $(j \geqslant K) \Rightarrow (|(c/s)_j - 0| < \varepsilon)$.
Given $j \in \mathbb{N}$. Assume $j \geqslant K$. **Want:** $|(c/s)_j - 0| < \varepsilon$.
Since $j \geqslant K$, by choice of K, we get $s_j < -M$.
Since $s_j < -M < 0$, it follows that $s_j < 0$, so $|s_j| = -s_j$.
Since $-s_j > M > 0$, we get $1/(-s_j) < 1/M$. Then $|c|/(-s_j) \leqslant |c|/M$.
We have $1/M = \varepsilon/(|c| + 1)$, and so $|c| \cdot (1/M) < \varepsilon$.
Then $|c|/|s_j| = |c|/(-s_j) \leqslant |c|/M = |c| \cdot (1/M) < \varepsilon$, so $|c|/|s_j| < \varepsilon$.
Then $|(c/s)_j - 0| = |(c/s)_j| = |c/s_j| = |c|/|s_j| < \varepsilon$. QED

Below is an alternate proof.

Proof:

Claim: $-s \to \infty$.
Proof of Claim: **Want:** $\forall M \in \mathbb{R}$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(j \geqslant K) \Rightarrow ((-s)_j > M).$$
Given $M \in \mathbb{R}$. **Want:** $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(j \geqslant K) \Rightarrow ((-s)_j > M).$$
Since $s \to -\infty$, choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(j \geqslant K) \Rightarrow (s_j < -M).$$
Then $K \in \mathbb{N}$. **Want:** $\forall j \in \mathbb{N}$, $(j \geqslant K) \Rightarrow ((-s)_j > M)$.
Given $j \in \mathbb{N}$. Assume $j \geqslant K$. **Want:** $(-s)_j > M$.
Since $j \geqslant K$, by choice of K, we get: $s_j < -M$. Then $-s_j > M$.
Then $(-s)_j = -s_j > M$.
End of proof of Claim.

Let $b := -c$, $r := -s$. By the Claim, $r \to \infty$.
So, by HW 4-4, $1/r \to 0$. So, by HW 4-2, $b \cdot (1/r) \to b \cdot 0$.

So, since $b \cdot 0 = 0$, we get: $b \cdot (1/r) \to 0$. Want: $c/s \to 0$.
Want: $b \cdot (1/r) = c/s$. **Want:** $\forall j \in \mathbb{N}$, $(b \cdot (1/r))_j = (c/s)_j$.
Given $j \in \mathbb{N}$. **Want:** $(b \cdot (1/r))_j = (c/s)_j$.
We have $r_j = (-s)_j = -s_j$, so $r_j = -s_j$. Also, $(c/s)_j = c/s_j$.
Then $b/r_j = (-c)/(-s_j) = c/s_j = (c/s)_j$, so $b/r_j = (c/s)_j$.
Then $(b \cdot (1/r))_j = b \cdot (1/r)_j = b \cdot (1/r_j) = b/r_j = (c/s)_j$. QED

Homework 3: Due on Tuesday 4 October

3-1. Using the 0-PMI, show: Let $x \in \mathbb{R}$. Then: $\forall j \in \mathbb{N}_0$, $|x^j| = |x|^j$.

Proof: Let $S := \{j \in \mathbb{N} \,|\, |x^j| = |x|^j\}$. **Want:** $S = \mathbb{N}$.
$|x^0| = |1| = 1 = |x|^0$, so $0 \in S$.
By PMI, want: $\forall j \in S$, $j+1 \in S$.
Given $j \in S$. **Want:** $j+1 \in S$.
Know: $|x^j| = |x|^j$. **Want:** $|x^{j+1}| = |x|^{j+1}$.
We have: $|x^{j+1}| = |x^j \cdot x| = |x^j| \cdot |x| = |x|^j \cdot |x| = |x|^{j+1}$. QED

3-2. Show: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall x \in \mathbb{R}$,

$$(\ |x-4| < \delta\ ) \Rightarrow (\ |x^3 - 5x^2 + 4x| < \varepsilon\ ).$$

Proof:
Given $\varepsilon > 0$. **Want:** $\exists \delta > 0$ s.t., $\forall x \in \mathbb{R}$,

$$(\ |x-4| < \delta\ ) \Rightarrow (\ |x^3 - 5x^2 + 4x| < \varepsilon\ ).$$

Let $\delta := \min\{1, \varepsilon/31, \varepsilon/30\}$. Then $\delta > 0$.
Want: $\forall x \in \mathbb{R}$, $(\ |x-4| < \delta\ ) \Rightarrow (\ |x^3 - 5x^2 + 4x| < \varepsilon\ )$.
Given $x \in \mathbb{R}$. Assume $|x-4| < \delta$. **Want:** $|x^3 - 5x^2 + 4x| < \varepsilon$.
We have $\delta \leqslant 1$ and $\delta \leqslant \varepsilon/31$.
Since $|x| = |x-4+4| \leqslant |x-4| + |4| < \delta + 4 \leqslant 5$, we get $|x|^2 + |x| < 30$.
Then $|x^2 - x| \leqslant |x|^2 + |x| < 30$.
Then $|(x-4)\cdot(x^2-x)| = |x-4| \cdot |x^2-x| \leqslant \delta \cdot 30 \leqslant (\varepsilon/31)\cdot 30 < \varepsilon$.
Then $|x^3 - 5x^2 + 4x| = |(x-4)\cdot(x^2-x)| < \varepsilon$. QED

3-3. Show: Define $f : \mathbb{R} \to \mathbb{R}$ by: $\forall x \in \mathbb{R}$, $f_x = x^2$.
Then: $\forall a \in \mathbb{R}$, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall x \in \mathbb{R}$,

$$(\ |x-a| < \delta\ ) \Rightarrow (\ |f_x - f_a| < \varepsilon\ ).$$

Proof: Given $a \in \mathbb{R}$, $\varepsilon > 0$.
Want: $\exists \delta > 0$ s.t., $\forall x \in \mathbb{R}$, $(\ |x-a| < \delta\ ) \Rightarrow (\ |f_x - f_a| < \varepsilon\ )$.
Let $\delta := \varepsilon/(1 + 2 \cdot |a|)$. Then $\delta > 0$.

Want: $\forall x \in \mathbb{R}, \quad (\ |x-a| < \delta \ ) \ \Rightarrow \ (\ |f_x - f_a| < \varepsilon \ )$.
Given $x \in \mathbb{R}$. Assume $|x-a| < \delta$. **Want:** $|f_x - f_a| < \varepsilon$.
We have $|x| = |x-a+a| \leqslant |x-a| + |a| < \delta + |a|$, so $|x| < \delta + |a|$.
Then $|x| + |a| < \delta + 2 \cdot |a| \leqslant 1 + 2 \cdot |a|$.
Then $\delta \cdot (|x| + |a|) = (\varepsilon/(1 + 2 \cdot |a|)) \cdot (|x| + |a|)$
$\leqslant (\varepsilon/(1 + 2 \cdot |a|)) \cdot (1 + 2 \cdot |a|) < \varepsilon$.
Then $|(x-a) \cdot (x+a)| = |x-a| \cdot |x+a| \leqslant \delta \cdot |x+a| \leqslant \delta \cdot (|x| + |a|) < \varepsilon$.
Then $|f_x - f_a| = |x^2 - a^2| = |(x-a) \cdot (x+a)| < \varepsilon$. QED

3-4. Show: $\forall M \in \mathbb{R}$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\ j \geqslant K \ ) \ \Rightarrow \ (\ j^4 - 7j - 9 > M \ ).$$

Proof: Given $M \in \mathbb{R}$. **Want:** $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\ j \geqslant K \ ) \ \Rightarrow \ (\ j^4 - 7j - 9 > M \ ).$$
Let $L := \max\{2, M+9\}$. Then $L \leqslant 2$ and $L \leqslant M+9$.
By the Archimedean Principle,
choose $K \in \mathbb{N}$ such that $K > L$. Then $K \in \mathbb{N}$.
Want: $\forall j \in \mathbb{N}, \quad (\ j \geqslant K \ ) \ \Rightarrow \ (\ j^4 - 7j - 9 > M \ )$.
Given $j \in \mathbb{N}$. Assume $j \geqslant K$. **Want:** $j^4 - 7j - 9 > M$.
We have $j - 9 \geqslant K - 9 > L - 9 \geqslant (M+9) - 9 = M$, so $j - 9 > M$.
We have $j \geqslant K > L \geqslant 2$. Then $j^3 > 8$.
So, since $j \in \mathbb{N} > 0$, we get $j^3 \cdot j > 8j$.
Then $j^4 - 7j - 9 = j^3 \cdot j - 7j - 9 > 8j - 7j - 9 = j - 9 > M$. QED

3-5. Show: Let $x, y \in \mathbb{R}$. Then: $| \ |y| \ - \ |x| \ | \ \leqslant \ | \ y \ - \ x \ |$.

Proof: Let $a := |x|$ and $b := |y|$ and $\varepsilon := |y - x|$. **Want:** $|b - a| \leqslant \varepsilon$.
Want: $a - \varepsilon \leqslant b \leqslant a + \varepsilon$.
Want: (1) $a - \varepsilon \leqslant b$ and (2) $b \leqslant a + \varepsilon$.

Proof of (1):
We have $a = |x| = |y + (x-y)| \leqslant |y| + |x-y| = |y| + |y-x| = b + \varepsilon$.
Then $a \leqslant b + \varepsilon$. Then $a - \varepsilon \leqslant b$.
End of proof of (1).

Proof of (2):
We have $b = |y| = |x + (y-x)| \leqslant |x| + |y-x| = a + \varepsilon$.
Then $b \leqslant a + \varepsilon$.
End of proof of (2). QED

Homework 2: Due on Tuesday 27 September

2-1. Show: Let $S \subseteq \mathbb{R}^*$. Then: $\sup_S \;\; =^* \;\; \max_S$.

Proof: **Want:** $(\ \max_S \neq ☹\ ) \;\Rightarrow\; (\ \sup_S = \max_S\ )$.
Assume $\max_S \neq ☹$. **Want:** $\sup_S = \max_S$.
We have: $\max_S \;=\; \mathrm{UE}(S \cap \mathrm{UB}_S) \;\; {}^*\!\in\;\; S \cap \mathrm{UB}_S$, so $\max_S \in S \cap \mathrm{UB}_S$.
Since $\max_S \in S \leqslant \sup_S$, we get $\max_S \leqslant \sup_S$.
Since $\max_S \in \mathrm{UB}_S \geqslant \sup_S$, we get $\max_S \geqslant \sup_S$.
Since $\max_S \leqslant \sup_S$ and $\max_S \geqslant \sup_S$,
we get $\max_S = \sup_S$, so $\sup_S = \max_S$. QED

2-2. Show: $\forall M \in \mathbb{R}$, $\exists N \in \mathbb{R}$ s.t., $\forall x \in \mathbb{R}$,
$$(\ x < N\ ) \;\Rightarrow\; (\ x^2 > M\ ).$$

Proof: Given $M \in \mathbb{R}$. **Want:** $\exists N \in \mathbb{R}$ s.t., $\forall x \in \mathbb{R}$,
$$(\ x < N\ ) \;\Rightarrow\; (\ x^2 > M\ ).$$
Let $N := -\sqrt{\max\{M, 0\}}$. Then $N \in \mathbb{R}$.
Want: $\forall x \in \mathbb{R}$, $(\ x < N\ ) \;\Rightarrow\; (\ x^2 > M\ )$.
Given $x \in \mathbb{R}$. Assume $x < N$. **Want:** $x^2 > M$.
Since $N = -\sqrt{\max\{M, 0\}}$, we get $N \leqslant 0$ and $N^2 = \max\{M, 0\}$.
Since $x < N \leqslant 0$, we get $x^2 > N^2$.
Then $x^2 > N^2 = \max\{M, 0\} \geqslant M$. QED

2-3. Show: $\forall M \in \mathbb{R}$, $\exists K \in \mathbb{N}_0$ s.t., $\forall j \in \mathbb{N}_0$,
$$(\ j \geqslant K\ ) \;\Rightarrow\; (\ 2^j > M\ ).$$

Proof: Given $M \in \mathbb{R}$. **Want:** $\exists K \in \mathbb{N}_0$ s.t., $\forall j \in \mathbb{N}_0$,
$$(\ j \geqslant K\ ) \;\Rightarrow\; (\ 2^j > M\ ).$$
By the Archimedean Principle, choose $K \in \mathbb{N}$ such that $K > M$.
Then $K \in \mathbb{N} \subseteq \mathbb{N}_0$, so $K \in \mathbb{N}_0$.
Want: $\forall j \in \mathbb{N}_0$, $(\ j \geqslant K\ ) \;\Rightarrow\; (\ 2^j > M\ )$.
Given $j \in \mathbb{N}_0$. Assume $j \geqslant K$. **Want:** $2^j > M$.
By an in class theorem, $2^j \geqslant j + 1$.
Then $2^j \geqslant j + 1 > j \geqslant K > M$. QED

2-4. Show: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}_0$ s.t., $\forall j \in \mathbb{N}_0$,
$$(\ j \geqslant K\ ) \;\Rightarrow\; \left(\ \frac{1}{2^j} < \varepsilon\ \right).$$

Proof: Given $\varepsilon > 0$. **Want:** $\exists K \in \mathbb{N}_0$ s.t., $\forall j \in \mathbb{N}_0$,

$$(\ j \geqslant K\ ) \ \Rightarrow\ \left(\ \frac{1}{2^j} < \varepsilon\ \right).$$

Let $M := 1/\varepsilon$. Then $M \in \mathbb{R}$ and $M > 0$ and $\dfrac{1}{M} = \varepsilon$.
By HW 2-3, choose $K \in \mathbb{N}_0$ s.t. $\forall j \in \mathbb{N}_0$,

$$(\ j \geqslant K\ ) \ \Rightarrow\ (\ 2^j > M\ ).$$

Then $K \in \mathbb{N}_0$. **Want:** $\forall j \in \mathbb{N}_0$, $(\ j \geqslant K\ ) \ \Rightarrow\ \left(\ \dfrac{1}{2^j} < \varepsilon\ \right)$.
Given $j \in \mathbb{N}_0$. Assume $j \geqslant K$. **Want:** $\dfrac{1}{2^j} < \varepsilon$.
By the choice of K, since $j \geqslant K$, we get $2^j > M$.
Since $2^j > M > 0$, we get $\dfrac{1}{2^j} < \dfrac{1}{M}$.
Then $\dfrac{1}{2^j} < \dfrac{1}{M} = \varepsilon$. QED

2-5. Using the 0-PMI, show: $\forall k \in \mathbb{N}_0$,

$$1 + 2 + 4 + 8 + \cdots + 2^k \ = \ 2^{k+1} - 1.$$

Proof: Let $S := \{j \in \mathbb{N}_0 \,|\, 1 + 2 + 4 + 8 + \cdots + 2^k \ = \ 2^{k+1} - 1\}$.
Want: $S = \mathbb{N}_0$. Since $1 = 2^{0+1} - 1$, we get $0 \in S$.
By the 0-PMI, want: $\forall j \in S$, $j + 1 \in S$.
Given $j \in S$. **Want:** $j + 1 \in S$.
Know: $1 + 2 + 4 + 8 + \cdots + 2^k \ = \ 2^{k+1} - 1$.
Want: $1 + 2 + 4 + 8 + \cdots + 2^k + 2^{k+1} \ = \ 2^{(k+1)+1} - 1$.
We have $1 + 2 + 4 + 8 + \cdots + 2^k + 2^{k+1}$

$$\begin{aligned} &= 2^{k+1} - 1 + 2^{k+1} \\ &= 2^{k+1} + 2^{k+1} - 1 \\ &= 2^{k+1} \cdot 1 + 2^{k+1} \cdot 1 - 1 \\ &= 2^{k+1} \cdot (1 + 1) - 1 \\ &= 2^{k+1} \cdot 2 - 1 \\ &= 2^{k+1} \cdot 2^1 - 1 \\ &= 2^{(k+1)+1} - 1. \end{aligned}$$

QED

Homework 1: Due on Tuesday 20 September

1-1. Show: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $3\delta^8 + 7\delta^6 + 2\delta^4 \leqslant 6\varepsilon$.

Proof: Given $\varepsilon > 0$. **Want:** $\exists \delta > 0$ s.t. $3\delta^8 + 7\delta^6 + 2\delta^4 \leqslant 6\varepsilon$.
Let $\delta := \min\{ \sqrt[8]{2\varepsilon/3}\ ,\ \sqrt[6]{2\varepsilon/7}\ ,\ \sqrt[4]{\varepsilon}\ \}$. Then $\delta > 0$.
Want: $3\delta^8 + 7\delta^6 + 2\delta^4 \leqslant 6\varepsilon$.
Since $0 \leqslant \delta \leqslant \sqrt[8]{2\varepsilon/3}$, we get $\delta^8 \leqslant (\sqrt[8]{2\varepsilon/3})^8$.
Since $0 \leqslant \delta \leqslant \sqrt[6]{2\varepsilon/7}$, we get $\delta^6 \leqslant (\sqrt[6]{2\varepsilon/7})^6$.
Since $0 \leqslant \delta \leqslant \sqrt[4]{\varepsilon}$, we get $\delta^4 \leqslant (\sqrt[4]{\varepsilon})^4$.
Then $\delta^8 \leqslant 2\varepsilon/3$ and $\delta^6 \leqslant 2\varepsilon/7$ and $\delta^4 \leqslant \varepsilon$.
Then $3\delta^8 \leqslant 2\varepsilon$ and $7\delta^6 \leqslant 2\varepsilon$ and $2\delta^4 \leqslant 2\varepsilon$.
Then $3\delta^8 + 7\delta^6 + 2\delta^4 \leqslant 6\varepsilon$. QED

1-2. Show: Let $\varepsilon > 0$. Then $\exists \delta > 0$ s.t. $3\delta^8 + 7\delta^6 + 2\delta^4 \leqslant 6\varepsilon$.

Proof: Let $\delta := \min\{ \sqrt[8]{2\varepsilon/3}\ ,\ \sqrt[6]{2\varepsilon/7}\ ,\ \sqrt[4]{\varepsilon}\ \}$. Then $\delta > 0$.
Want: $3\delta^8 + 7\delta^6 + 2\delta^4 \leqslant 6\varepsilon$.
Since $0 \leqslant \delta \leqslant \sqrt[8]{2\varepsilon/3}$, we get $\delta^8 \leqslant (\sqrt[8]{2\varepsilon/3})^8$.
Since $0 \leqslant \delta \leqslant \sqrt[6]{2\varepsilon/7}$, we get $\delta^6 \leqslant (\sqrt[6]{2\varepsilon/7})^6$.
Since $0 \leqslant \delta \leqslant \sqrt[4]{\varepsilon}$, we get $\delta^4 \leqslant (\sqrt[4]{\varepsilon})^4$.
Then $\delta^8 \leqslant 2\varepsilon/3$ and $\delta^6 \leqslant 2\varepsilon/7$ and $\delta^4 \leqslant \varepsilon$.
Then $3\delta^8 \leqslant 2\varepsilon$ and $7\delta^6 \leqslant 2\varepsilon$ and $2\delta^4 \leqslant 2\varepsilon$.
Then $3\delta^8 + 7\delta^6 + 2\delta^4 \leqslant 6\varepsilon$. QED

1-3. Show: $\forall N \in \mathbb{R}$, $\exists \delta > 0$ s.t. $\forall x \in \mathbb{R}$,
$$(\,-\delta < x < 0\,) \Rightarrow (\,1/x < N\,).$$

Proof: Given $N \in \mathbb{R}$. **Want:** $\exists \delta > 0$ s.t. $\forall x \in \mathbb{R}$,
$$(\,-\delta < x < 0\,) \Rightarrow (\,1/x < N\,).$$
Let $A := \max\{-N, 1\}$. Then $A \geqslant -N$ and $A \geqslant 1$.
We have $A \geqslant 1 > 0$, so $A > 0$.
Let $\delta := 1/A$. Then $1/\delta = A$. Also, $\delta > 0$.
Want: $\forall x \in \mathbb{R}$, $(\,-\delta < x < 0\,) \Rightarrow (\,1/x < N\,)$.
Given $x \in \mathbb{R}$. Assume $-\delta < x < 0$. **Want:** $1/x < N$.
Since $-\delta < x < 0$, we get $\delta > -x > 0$, so $1/\delta < 1/(-x)$.
Then $-1/\delta > -1/(-x) = 1/x$, so $1/x < -1/\delta$.
Since $1/\delta = A \geqslant -N$, we get: $-1/\delta \leqslant N$.
Then $1/x < -1/\delta \leqslant N$. QED

1-4. Show: Let $N \in \mathbb{R}$. Then $\exists \delta > 0$ s.t. $\forall x \in \mathbb{R}$,
$$(\,-\delta < x < 0\,) \Rightarrow (\,1/x < N\,).$$

Proof: Let $A := \max\{-N, 1\}$. Then $A \geqslant -N$ and $A \geqslant 1$.
We have $A \geqslant 1 > 0$, so $A > 0$.
Let $\delta := 1/A$. Then $1/\delta = A$. Also, $\delta > 0$.
Want: $\forall x \in \mathbb{R}$, $(-\delta < x < 0) \Rightarrow (1/x < N)$.
Given $x \in \mathbb{R}$. Assume $-\delta < x < 0$. **Want:** $1/x < N$.
Since $-\delta < x < 0$, we get $\delta > -x > 0$, so $1/\delta < 1/(-x)$.
Then $-1/\delta > -1/(-x) = 1/x$, so $1/x < -1/\delta$.
Since $1/\delta = A \geqslant -N$, we get: $-1/\delta \leqslant N$.
Then $1/x < -1/\delta \leqslant N$. QED

1-5. Using the PMI, show:

$$\forall k \in \mathbb{N}, \qquad 1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

Proof: Let $S := \left\{ k \in \mathbb{N} \,\middle|\, 1^2 + 2^2 + \cdots + k^2 = \dfrac{k(k+1)(2k+1)}{6} \right\}$.

Want: $S = \mathbb{N}$. Since $1^2 = \dfrac{1 \cdot (1+1) \cdot (2 \cdot 1 + 1}{6}$, we get $1 \in S$.

By the PMI, we wish to show: $\forall k \in S$, $k+1 \in S$.
Given $k \in S$. **Want:** $k+1 \in S$.

Know: $1^2 + 2^2 + \cdots + k^2 = \dfrac{k(k+1)(2k+1)}{6}$.

Want: $1^2 + 2^2 + \cdots + k^2 + (k+1)^2 =$

$$\frac{(k+1)((k+1)+1)(2 \cdot (k+1)+1)}{6}.$$

We have $1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = \dfrac{k(k+1)(2k+1)}{6} + (k+1)^2$

$$= \frac{k(k+1)(2k+1)}{6} + \frac{6 \cdot (k+1)^2}{6}$$

$$= \frac{k(k+1)(2k+1) + 6 \cdot (k+1)^2}{6}$$

$$= \frac{(k+1) \cdot (k(2k+1) + 6 \cdot (k+1))}{6}$$

$$= \frac{(k+1) \cdot (2k^2 + k + 6k + 6)}{6}$$

$$
\begin{aligned}
&= \frac{(k+1)\cdot(2k^2+7k+6)}{6} \\
&= \frac{(k+1)\cdot(k+2)\cdot(2k+3)}{6} \\
&= \frac{(k+1)\cdot((k+1)+1)\cdot(2\cdot(k+1)+1)}{6}. \qquad \text{QED}
\end{aligned}
$$