Midterm 1 Solutions

Choose 6 out of 9 problems below.

1. Assume $L : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear map such that $L(e_1) = (1, 1, 0)$, $L(e_2) = (0, 0, 1)$, and $L(e_3) = (1, 1, 1)$. Give bases for $\operatorname{Im} L$ and $\operatorname{Ker} L$.

Solution. The image of L is spanned by $L(e_1)$, $L(e_2)$, and $L(e_3)$. As

$$L(e_3) = L(e_1 + e_2) = L(e_1) + L(e_2), \tag{1}$$

in fact $\operatorname{Im} L$ is spanned by $L(e_1)$ and $L(e_2)$. Equation (1) and linearity imply $e_1+e_2-e_3 \in \operatorname{Ker} L$. Since $\dim \operatorname{Im} L + \dim \operatorname{Ker} L = 3$, we can conclude $\{e_1 + e_2 - e_3\}$ is a basis of $\operatorname{Ker} L$ and $\{L(e_1),\, L(e_2)\}$ is a basis for $\operatorname{Im} L$.

2. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 2x^2y/(x^4+y^2), & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Show that f is not continuous at 0.

Hint: Consider the values of f along the function $\psi(t) = (t, t^2)$.

Solution. Let $(x_n, y_n) = (n^{-1}, n^{-2})$. Then $\lim_{n\to\infty}(x_n, y_n) = (0,0)$ but

$$\lim_{n\to\infty} f(x_n, y_n) = \lim_{n\to\infty} \frac{2n^{-2}n^{-2}}{n^{-4}+n^{-4}} = 1 \neq f(0,0).$$

3. Let $f, g : \mathbb{R} \to \mathbb{R}^n$ be differentiable, with $f'(t) \neq 0$ and $g'(t) \neq 0$ for all $t \in \mathbb{R}$. Assume the distance between the (images of) the two curves is minimized at the points $p = f(s_0)$ and $q = g(t_0)$. Prove that then $p - q$ is orthogonal to both $f'(s_0)$ and $g'(t_0)$.

Hint: Consider the function $\rho(s,t) = |f(s) - g(t)|^2$.

Solution. Using the chain rule, compute

$$D_1\rho(s_0, t_0) = 2(f(s_0) - g(t_0)) \cdot f'(s_0) = 2(p - q) \cdot f'(s_0)$$
$$D_2\rho(s_0, t_0) = 2(f(s_0) - g(t_0)) \cdot g'(t_0) = 2(p - q) \cdot g'(t_0).$$

By assumption (s_0, t_0) is a critical point of ρ, so both these partial derivatives are zero.

4. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \left(x^{1/3} + y^{1/3}\right)^3.$$

Find the partial derivaties of f at 0. Then show that f is not differentiable at 0.

Solution. Compute

$$D_1 f(0,0) = \lim_{t \to 0} \frac{f(t,0) - f(0,0)}{t} = 1$$
$$D_2 f(0,0) = \lim_{t \to 0} \frac{f(0,t) - f(0,0)}{t} = 1$$

On the other hand, let $v = (1, 1)$ and notice that

$$D_v f(0,0) = \lim_{t \to 0} \frac{f(t,t) - f(0,0)}{t} = 8.$$

Thus,

$$D_v f(0,0) \neq D_1 f(0,0) + D_2 f(0,0),$$

showing f is not differentiable at $(0, 0)$.

5. Define $f : \mathbb{R}^2 \to \mathbb{R}^3$ and $g : \mathbb{R}^3 \to \mathbb{R}$ by

$$\begin{aligned} f(x,y) &= (xy,\, x+y,\, x^2\cos y) \\ g(x,y,z) &= 3xy + y^2 + z, \end{aligned}$$

and let $h \equiv g \circ f$. Compute the derivative matrix $h'(a)$ at $a = (1,0)$.

Solution. Note that

$$\begin{aligned} f'(x,y) &= \begin{pmatrix} y & x \\ 1 & 1 \\ 2x\cos y & -x^2\sin y \end{pmatrix} \\ g'(x,y,z) &= \begin{pmatrix} 3y & 3x+2y & 1 \end{pmatrix} \end{aligned}$$

Writing $b \equiv f(a) = (0,1,1)$ we have

$$h'(a) = g'(b)f'(a) = \begin{pmatrix} 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 5 \end{pmatrix}.$$

6. Let $f : U \to \mathbb{R}$ be a differentiable function, and $U \subset \mathbb{R}^2$ a connected open set. We say f is independent of y if there exists $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x)$ for $(x, y) \in U$.

Let $f : (a, b) \times (c, d) \to \mathbb{R}$ be differentiable with $D_2 f(x, y) = 0$ for all $(x, y) \in (a, b) \times (c, d)$. Show that then f is independent of y.

Solution. Let $u = (x, y_1)$ and $v = (x, y_2)$ be in $(a, b) \times (c, d)$. It suffices to show $f(u) = f(v)$. By the Mean Value Theorem there is w on the line segment between u and v such that

$$f(v) - f(u) = f'(w) \cdot (v - u).$$

But,

$$f'(w) \cdot (v - u) = D_{v-u} f(w) = D_{(y_2 - y_1)e_2} f(w) = (y_2 - y_1) D_2 f(w) = 0.$$

7. Show that $xy \leq 1/4$ if (x, y) is on the line $x + y = 1$ and $x \geq 0$, $y \geq 0$. Use this to prove the algebraic/geometric mean inequality:

$$\sqrt{ab} \leq \frac{a+b}{2}, \qquad a, b \geq 0.$$

Hint: Let $x = a/(a + b)$, $y = b/(a + b)$.

Solution. Using Lagrange multipliers to maximize xy on the line $x+y = 1$, we obtain $x = y = \lambda$. Since $x + y = 1$ this implies $x = y = 1/2$. The only other possible maximizers of f in the first quadrant are on its boundary, but xy is zero there. Thus $xy \leq 1/4$ for $x \geq 0$, $y \geq 0$. With x, y as in the hint we have $x + y = 1$ and so

$$xy = \frac{ab}{(a+b)^2} \leq \frac{1}{4}.$$

Multiplying both sides by $(a + b)^2$ and then taking square roots gives the result.

8. Recall that an isosceles triangle has two edges of the same length, and an equilateral triangle's edges all have the same length. Prove that the isosceles triangle of maximum area that can be inscribed within a unit circle is an equilateral triangle.

Solution. Without loss of generality we can consider a unit circle centered at the origin and an isosceles triangle with vertices $(0,1)$, (x,y), and $(-x,y)$. So we want to maximize the area $A = x(1-y)$ of this triangle on the unit circle $x^2+y^2=1$. Using Lagrange multipliers this gives $1-y = 2\lambda x$, $-x = 2\lambda y$ and so

$$\frac{y-1}{x} = \frac{x}{y}. \tag{2}$$

Thus $x^2 = y(y-1)$. Using $x^2+y^2=1$ we get $1-y^2 = y(y-1)$. Solving this equation for y we get $y = 1, -1/2$. Putting this back into the area formula we see it must be maximized when $y = -1/2$. So the coordinates of our triangle are $(0,1)$, $(-\sqrt{3}/2, -1/2)$ and $(\sqrt{3}/2, -1/2)$, and it is straightforward to check this is an equilateral triangle.

9. Find and classify the critical points of the function

$$f(x, y) = (x^2 + y^2)e^{x^2-y^2}.$$

Solution. Compute

$$D_1 f(x, y) = 2x(1 + x^2 + y^2)e^{x^2-y^2}$$
$$D_2 f(x, y) = 2y(1 - x^2 - y^2)e^{x^2-y^2}$$

Observe that $e^{x^2-y^2}$ and $1 + x^2 + y^2$ are never zero. So if (x, y) is a critical point then $x = 0$; our critical points are therefore $(0, 0)$ and $(0, \pm 1)$. Letting $H(x, y)$ be the Hessian matrix of f at (x, y), we find

$$H(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad \text{(positive definite)}$$
$$H(0,1) = \begin{pmatrix} 4/e & 0 \\ 0 & -4/e \end{pmatrix} \qquad \text{(indefinite)}$$
$$H(0,-1) = \begin{pmatrix} 4/e & 0 \\ 0 & -4/e \end{pmatrix} \qquad \text{(indefinite).}$$

Thus f has a local minimum at $(0, 0)$ and a saddle points at $(0, \pm 1)$.