Midterm 2 Solutions

Choose 6 out of 8 problems below.

1. Consider the sets described in (a)-(c) below. For each set, either prove it is an $(n-1)$-manifold in $\mathbb{R}^n$ or give an example to show that it is not.

(a) The graph of a differentiable function $f : U \to \mathbb{R}$ defined on an open set $U \subset \mathbb{R}^{n-1}$

(b) The zero set of a continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$

(c) The (nonempty) intersection of an $(n-1)$-dimensional manifold $M \subset \mathbb{R}^n$ and an open set $U \subset \mathbb{R}^n$

Solution. (a) The graph is the set

$$G = \{(x_1, \ldots, x_n) \,:\, x_n = f(x_1, \ldots, x_{n-1}),\, (x_1, \ldots, x_{n-1}) \in U\}.$$

Observe that G is a patch. A patch is itself a manifold, so G is a manifold.

(b) This is not a manifold in general; see problem 2.

(c) Let $a \in U \cap M$. Since $a \in M$ and M is a manifold, there is a neighborhood V of a such that $V \cap M$ is a patch. Then $V \cap (U \cap M)$ is also a patch, and M is a manifold.

2. Define $G : \mathbb{R}^2 \to \mathbb{R}$ by $G(x, y) = x^2 - y^2$.

(i) At which points (x, y) is $\nabla G(x, y) \neq 0$?

Solution. $\nabla G(x, y) \neq 0$ if and only if $(x, y) \neq (0, 0)$.

(ii) For each such point, what does the implicit function theorem say about the set

$$S = \{(x, y) \,:\, G(x, y) = 0\}?$$

Solution. Given (a, b) such that $D_2G(a, b) \neq 0$, there is a rectangle $Q = U \times V$ and a function $f : U \to V$ such that the graph of f is equal to S inside Q.

Given (a, b) such that $D_1G(a, b) \neq 0$, there is a rectangle $Q = U \times V$ and a function $h : V \to U$ such that the graph of h is equal to S inside Q.

(iii) Is S a manifold?

Solution. No. Here $S = \{(x, y) \,:\, y = \pm x\}$ fails to be a manifold at the origin. (In any ball around $(0, 0)$, it is impossible to write one coordinate of S as a function of the other coordinate.)

3. Let $M \subset \mathbb{R}^n$ be an $(n-1)$-dimensional manifold, let $a \in M$, and define

$$C = \{\phi : \mathbb{R} \to M \,:\, \phi \text{ is differentiable and } \phi(0) = a\}$$

and

$$T_a = \{\phi'(0) \,:\, \phi \in C\}.$$

Then $a + T_a$ is the tangent plane to M at a, and T_a is an $(n-1)$-dimensional vector space. Choose a patch around a and the corresponding chart h, and give a basis for T_a in terms of h.

Solution. Choose a chart $h : U \to \mathbb{R}$ and the corresponding patch P around a. Assume for simplicity that P has the form

$$P = \{(x_1, \ldots, x_n) \,:\, x_n = h(x_1, \ldots, x_{n-1}),\, (x_1, \ldots, x_{n-1}) \in U\}.$$

(Recall the other cases differ only by a permutation of the coordinates.) Write $\phi(t) = (\psi(t), \phi_n(t))$ where $\psi(t) = (\phi_1(t), \ldots, \phi_{n-1}(t))$, and write $b = (a_1, \ldots, a_{n-1})$. Observe that $\phi_n(t) = h(\psi(t))$ in a neighborhood of $t = 0$ (specifically for those t such that $\psi(t) \in U$). Now:

$$\begin{aligned}
\phi'(0) &= \big(\psi'(0), \nabla h(b) \cdot \psi'(0)\big) \\
&= \left(\phi_1'(0), \ldots, \phi_{n-1}'(0), \sum_{i=1}^{n-1} D_i h(b) \phi_i'(0)\right) \\
&= \sum_{i=1}^{n-1} \phi_i'(0)\, (e_i + D_i h(b) e_n)
\end{aligned}$$

where $e_1, \ldots, e_n$ are the standard basis vectors in $\mathbb{R}^n$. A basis for T_a is then

$$e_1 + D_1 h(b) e_n,\; e_2 + D_2 h(b) e_n,\; \ldots,\; e_{n-1} + D_{n-1} h(b) e_n$$

since it is clear these vectors are linearly independent, and we have taken for granted the fact that T_a is an $(n-1)$-dimensional vector space.

4. Find the point of the unit sphere $S = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ which is closest to the plane $P = \{(x, y, z) : x + y + z = 3\}$.

Solution. Note that $(1, 1, 1)$ is orthogonal to P. Let $(x, y, z) \in \mathbb{R}^3$ and let (x_0, y_0, z_0) be the point on P closest to (x, y, z). Then $(x_0, y_0, z_0) = (x, y, z) + t(1, 1, 1)$ for some $t \in \mathbb{R}$. Using $x_0 + y_0 + z_0 = 3$ we get

$$t = \frac{3 - x - y - z}{3}.$$

Thus, the (square) distance from (x, y, z) to P is

$$|(x_0, y_0, z_0) - (x, y, z)|^2 = |t(1, 1, 1)|^2 = 3t^2 = \frac{(3 - x - y - z)^2}{3}.$$

So we must minimize the function $(3 - x - y - z)^2$ on S. Using Lagrange multipliers we get

$$x + y + z - 3 = \lambda x = \lambda y = \lambda z.$$

If $\lambda = 0$ then $(x, y, z) \in P$, but P is disjoint from S. So $\lambda \neq 0$ and $x = y = z$. On S we obtain $x = y = z = \pm\sqrt{3}/3$. One verifies by hand that the minimum distance is attained when $x = y = z = \sqrt{3}/3$.

5. Let $f(x) = (\tan^{-1} x)(\sin^2 x)$.

(a) Assuming that the Taylor expansions of $\tan^{-1} x$ and $\sin^2 x$ at 0 are

$$x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots \qquad \text{and} \qquad x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 + \dots$$

respectively, prove that the 5th order Taylor polynomial of f at 0 is

$$x^3 - \frac{2}{3}x^5.$$

Solution. Write

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \bar{R}_3(x), \qquad \lim_{x\to 0} \frac{\bar{R}_3(x)}{x^3} = 0$$
$$\sin^2 x = x^2 - \frac{1}{3}x^4 + \tilde{R}_4(x), \qquad \lim_{x\to 0} \frac{\tilde{R}_4(x)}{x^4} = 0.$$

Then

$$f(x) = x^3 - \frac{2}{3}x^5 + R(x)$$

where

$$R(x) = \bar{R}_3(x)\left(x^2 - \frac{1}{3}x^4\right) + \tilde{R}_4(x)\left(x - \frac{1}{3}x^3\right) + \bar{R}_3(x)\tilde{R}_4(x).$$

One checks that $\lim_{x\to 0} R(x)/x^5 = 0$, which proves the 5th order Taylor polynomial is as claimed.

(b) Use this to show that 0 is a critical point of

$$g(x) = \begin{cases} \frac{f(x)}{x^3}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

What kind of critical point is 0?

Solution. Note that

$$g(x) = 1 - \frac{2}{3}x^2 + \hat{R}(x)$$

where $\hat{R}(x) = R(x)/x^3$ if $x \neq 0$ and $\hat{R}(0) = 0$. Thus $\lim_{x\to 0} \hat{R}(x)/x^2 = 0$, which shows that $1 - (2/3)x^2$ is the 2nd order Taylor polynomial of g at 0. It follows that $g'(0) = 0$. One can see that g has a local maximum at 0; see the argument of problem 6(b).

6. Assume that $f : \mathbb{R} \to \mathbb{R}$ is of class C^5 and can be written in the form

$$f(x) = 1 + x^4 + R(x),$$

where $\lim_{x\to 0} R(x) = 0$.

(a) Must it be true that f has a local minimum at 0? Prove it or give a counterexample.

Solution. No. For example if $R(x) = x$ then f has neither a local max nor min at zero.

(b) If $1 + x^4$ is the 4th degree Taylor polynomial of f at 0, must it be true that f has a local minimum at 0? Prove it or give a counterexample.

Solution. Yes. We have

$$f(x) = 1 + x^4 + R_4(x), \qquad \lim_{x\to 0} \frac{R_4(x)}{x^4} = 0.$$

Thus

$$\lim_{x\to 0} \frac{f(x) - 1}{x^4} = 1 > 0$$

which shows that $f(x) > 1$ for $x \neq 0$ in a neighborhood of 0.

7. Find the 3rd degree Taylor polynomial of $f(x, y, z) = xyz$ at $a = (1, 0, 1)$.

Solution. Note that

$$\begin{aligned} xyz &= (x-1)y(z-1) + xy + yz - y \\ &= (x-1)y(z-1) + (x-1)y + y + y(z-1) + y - y \\ &= y + (x-1)y + y(z-1) + (x-1)y(z-1) + R_3(x, y, z) \end{aligned}$$

where $R_3(x, y, z) \equiv 0$. It follows that $y + (x-1)y + y(z-1) + (x-1)y(z-1)$ is the 3rd degree Taylor polynomial of f at a.

See the last page for a calculation using the Taylor expansion formula.

8. Assume that the function $f : \mathbb{R}^n \to \mathbb{R}$ is of class C^3 in a neighborhood of the critical point a. Let $H(a)$ be Hessian matrix of f at a. Define $q : \mathbb{R}^n \to \mathbb{R}$ and $L : \mathbb{R}^n \to \mathbb{R}^n$ by

$$q(x) = x^t H(a)x$$
$$L(x) = H(a)x$$

Which of the following statements are true? (Circle "T" for true and "F" for false. No justification is required.)

T/F If $H(a)$ has only nonnegative eigenvalues, then f has a local minimum at a

T/F If the maximum value of q on S^{n-1} is attained at x, then $L(x) = \lambda x$ for some $\lambda \in \mathbb{R}$

T/F $\nabla q(x) = 2L(x)$

T/F If q is positive on S^{n-1}, then f has a local minimum at a

Solution. The correct responses are, in order: F, T, T, T. The answers were obtained during the analysis of f near a, using the 2nd order Taylor expansion

$$f(x) = f(a) + \frac{1}{2}q(x) + R_2(x),$$

in particular the expression

$$\frac{f(x) - f(a)}{|x|^2} = \frac{1}{2}q\left(\frac{x}{|x|}\right) + \frac{R_2(x)}{|x|^2}.$$

Computing the 3rd order Taylor polynomial of $f(x,y,z) = xyz$ at $a = (1,0,1)$: For $h = (x-1, y, z-1) = (h_1, h_2, h_3)$ we have

$$f(x,y,z) = f(a+h) = f(a) + D_h f(a) + \frac{1}{2} D_h^2 f(a) + \frac{1}{6} D_h^3 f(a) + R_3(h).$$

Compute

$$\begin{aligned} D_h f(x,y,z) &= h_1 D_1 f(x,y,z) + h_2 D_2 f(x,y,z) + h_3 D_3 f(x,y,z) \\ &= h_1 yz + h_2 xz + h_3 xy \end{aligned}$$

and

$$\begin{aligned} D_h^2 f(x,y,z) &= D_h(D_h f(x,y,z)) \\ &= D_h(h_1 yz + h_2 xz + h_3 xy) \\ &= h_1 D_1(h_1 yz + h_2 xz + h_3 xy) + h_2 D_2(h_1 yz + h_2 xz + h_3 xy) \\ &\quad + h_3 D_3(h_1 yz + h_2 xz + h_3 xy) \\ &= h_1(h_2 z + h_3 y) + h_2(h_1 z + h_3 x) + h_3(h_1 y + h_2 x) \\ &= 2h_1 h_2 z + 2h_1 h_3 y + 2h_2 h_3 x \end{aligned}$$

and

$$\begin{aligned} D_h^3 f(x,y,z) &= D_h(D_h^2 f(x,y,z)) \\ &= D_h(2h_1 h_2 z + 2h_1 h_3 y + 2h_2 h_3 x) \\ &= h_1 D_1(2h_1 h_2 z + 2h_1 h_3 y + 2h_2 h_3 x) + h_2 D_2(2h_1 h_2 z + 2h_1 h_3 y + 2h_2 h_3 x) \\ &\quad + h_3 D_3(2h_1 h_2 z + 2h_1 h_3 y + 2h_2 h_3 x) \\ &= 2h_1 h_2 h_3 + 2h_1 h_2 h_3 + 2h_1 h_2 h_3 \\ &= 6h_1 h_2 h_3 \end{aligned}$$

At $a = (1,0,1)$ we have

$$\begin{aligned} f(a) &= 0 \\ D_h f(a) &= h_2 \\ D_h^2 f(a) &= 2h_1 h_2 + 2h_2 h_3 \\ D_h^3 f(a) &= 6h_1 h_2 h_3 \end{aligned}$$

and so

$$\begin{aligned} f(x,y,z) &= h_2 + h_1 h_2 + h_2 h_3 + h_1 h_2 h_3 + R_3(h) \\ &= y + (x-1)y + y(z-1) + (x-1)y(z-1) + R_3(h). \end{aligned}$$

Remark: When taking the derivatives $D_h f(x,y,z)$, we abuse notation by using the same letters x, y, z to write both the function formula for f and the coordinates of h. This is an abuse because $h = (x-1, y, z-1)$ is considered constant when the derivatives are taken. If this bothers you, you can write $h = (x_0 - 1, y_0, z_0 - 1)$ in the above, and save the letters x, y, z for the formula for f.